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MODULUS OF LATTICE-VALUED MEASURES

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Abstract. Let X be a completely regular Hausdorff space, E a Banach lattice, and μ an E-valued countably additive, regular Borel measure on X. Some results about the countable additivity and regularity of the modulus $|\mu|$ are proved. Also in special cases, it is proved that $L_1(\mu) = L_1(|\mu|)$.

1. Introduction and Notation

In ([**13, 8**]), some results are proved about the countable additivity of the modulus of countable additive measures taking values in Banach lattices. In this paper we prove some additional properties of the modulus.

All vector spaces are taken over reals R. For a completely regular Hausdorff space X, $C_b(X)$ is the space of all bounded real-valued continuous functions on X. For $f \in C_b(X)$, the set $f^{-1}(0)$ is called a zero-set of X and the elements of the σ-algebra generated by zero-sets are called Baire sets ([**16, 15**]). $\mathcal{B}(X)$ and $\mathcal{B}_0(X)$ are the classes of Borel and Baire subsets of X and $M_\sigma(X),\ M_\tau(X),\ M_t(X)$ denote the spaces of σ-additive, τ-smooth and tight Baire measures on X([**15, 16**]). The elements of $M_\sigma(X)$ are real-valued, countably additive measures on $\mathcal{B}_0(X)$. An element $\mu \in M_\sigma(X)$ is called τ-smooth if for any decreasing net $\{f_\alpha\} \subset C_b(X),\ f_\alpha \downarrow 0$, we have $\mu(f_\alpha) \to 0$. Every τ-smooth measure has a unique extension to a Borel measure which is inner regular by closed subsets and outer regular by open subsets of X; an element $\mu \in M_\sigma(X)$ is called tight if for any uniformly bounded net $\{f_\alpha\} \subset C_b(X)$, $f_\alpha \to 0$, uniformly on compact subsets of X, we have $\mu(f_\alpha) \to 0$. Every tight measure has a unique extension to a Borel measure which is inner regular by compact subsets and outer regular by open subsets of X ([**15, 16**]).

Now we come to vector-valued measures; integrability of scalar-valued functions is taken in the sense of [**7**] (see also [**9, 10**]). Suppose $(E, \|\cdot\|)$ is a Banach space. If $\mathcal{A}$ is a σ-algebra of subsets of a set X, $\mu\colon \mathcal{A} \to E$ a countably additive vector measure, we denote the semi-variation of μ by $\bar{\mu}\colon \mathcal{A} \to R^+$, $\bar{\mu}(A) = \sup\{|g \circ \mu|(A)\colon g \in E', \|g\| \le 1\}$; this semi-variation can be extended to functions in $L_1(\mu)$ ([**7**, p. 23]). Also we consider the submeasure $\dot{\mu}\colon \mathcal{A} \to R^+$,

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$\dot{\mu}(A) = \sup\{\|\mu(B)\| : B \in \mathcal{A}, B \subset A\}$ ([**5, 3**]). It is easily verified that $\dot{\mu}$ is countably sub-additive [**3**] and $\dot{\mu} \leq \bar{\mu} \leq 4\dot{\mu}$. A countably additive measure $\lambda : \mathcal{A} \to R^+$ is called a control measure for μ if, for an $A \in \mathcal{A}$, $\lambda(A) = 0$ implies $\mu(A) = 0$. This implies $\lim_{\lambda(A)\to 0} \bar{\mu}(A) = 0$. This control measure can be chosen such that $\lambda = |g \circ \mu|$ for some $g \in E'$ with $\|g\| \leq 1$ ([**7**]) and has the properties that (i) $|f \circ \mu| << \lambda$ for every $f \in E'$ with $\|f\| \leq 1$; (ii) if $\lambda(A) = 0$, then $\bar{\mu}(A) = 0$; (iii) $\lim_{\lambda(A)\to 0} \bar{\mu}(A) = 0$; (iv) $\lambda \leq \bar{\mu}$. We also have the result that if $f : Y \to R$ is a measurable function, $B \in \mathcal{A}$ and $|f| \leq c$ on B, then $\| \int_B f d\mu\| \leq c\bar{\mu}(B)$.

For locally convex spaces, notations and results of [**11**], [**12**] will be used. For a locally convex space E with dual E' and $f \in E'$, $x \in E$, $\langle f, x\rangle$ will stand for $f(x)$.

Suppose E is a Hausdorff locally convex space and X is a completely regular Hausdorff space. A countably additive measure $\mu : \mathcal{B}_0(X) \to E$ is called an E-valued Baire measure, a countably additive measure $\mu : \mathcal{B}(X) \to E$ is called an E-valued τ-smooth Borel measure if $f \circ \mu \in M_\tau(X)$ for every $f \in E'$, and a countably additive measure $\mu : \mathcal{B}(X) \to E$ is called an E-valued tight Borel measure if $f \circ \mu \in M_t(X)$ for every $f \in E'$. Many other characterizations of these measures are given in [**6**] (see also [**9**]).

Suppose E is a Dedekind complete Banach lattice, $\mathcal{A}$ is a σ-algebra of subsets of a set X and $\mu : \mathcal{A} \to E$ is a order-bounded, finitely additive measure. If $B(X)$ is the space of all real-valued, bounded, $\mathcal{A}$-measurable functions on X, then μ can also be considered as a linear, order-bounded mapping $\mu : B(X) \to E$. The evaluation of μ^+, μ^-, $|\mu|$ from $\mu : \mathcal{A} \to E$ or $\mu : B(X) \to E$ are the same ([**13**], [**14**]).

Now if E is a Dedekind complete Banach lattice and $\mu : \mathcal{A} \to E$ is a order-bounded, countably additive measure in the norm topology of E, it is proved in [**13**] that, under certain conditions, $|\mu|$ is also countably additive. Now if X is a completely regular Hausdorff space, then there are some regularity properties like τ-smooth or tight, associated with μ; the question arises whether these properties are carried also by $|\mu|$ and when $L_1(\mu) = L_1(|\mu|)$. We consider these questions in this paper.

2. Main Results

We first prove a basic result

Theorem 1. *Suppose E is a Banach lattice, $\mathcal{A}$ is a sigma-algebra of subsets of a set X and $\mu\colon \mathcal{A} \to E$ is a countably additve measure. Assume that there is a a countably additive measure $\lambda\colon \mathcal{A} \to \mathbb{R}^+$ such that $\mu << \lambda$. If $|\mu|$ exists and is countably additive, then $|\mu| << \lambda$.*

Proof. Fix a $c > 0$. We claim there is a λ-null set $A_c \in \mathcal{A}$ such that for any λ-null set $B \in \mathcal{A}$, B disjoint from A_c, $\||\mu|(B)\| < c$. Suppose this is not true. We start with a λ-null set $A \in \mathcal{A}$ and take a λ-null set $B_1 \in \mathcal{A}$, disjoint from A, such that $\||\mu|(B_1)\| > c$. Now $A_1 = A \cup B_1$ is λ-null and so there is λ-null $B_2 \in \mathcal{A}$, disjoint from A_1, such that $\||\mu|(B_2)\| > c$. Continuing this process we get a disjoint sequence $\{B_n\} \subset \mathcal{A}$ such that $\||\mu|(B_n)\| > c$, for all n, which, because of the countably additivity of $|\mu|$, is a contradiction.

Now $A = \cup A_{1/n}$ is a λ-null. Define $\nu : \mathcal{A} \to E,\ \nu(B) = |\mu|((X \setminus A) \cap B)$. Take any $C \in \mathcal{A}$ such that $\lambda(C) = 0$. From the construction of A, $|\mu|(C \setminus A) = 0$ which gives $\nu(C) = 0$. This means $\nu << \lambda$. Fix a $B \in \mathcal{A}$ and an $f \in E', f \geq 0$. Since A is λ-null, $\pm f \circ \mu(A \cap B) = 0$. Thus

$$\pm f \circ \mu(B) = \pm f \circ \mu(B \setminus A) \leq f \circ |\mu|((X \setminus A) \cap B) = f \circ \nu(B).$$

Thus $\pm\mu \leq \nu \leq |\mu|$ This means $\nu = |\mu|$. This proves the result. □

Corollary 2 ([**8**]). *Suppose E is a Banach lattice not containng a positive copy of ℓ_∞, $\mathcal{A}$ a sigma-algebra of subsets of a set X and $\mu\colon \mathcal{A} \to E$ a countably additve measure. Assume $\mu << \lambda$ for some countably additive measure $\lambda\colon \mathcal{A} \to R^+$ and that $|\mu|$ exists. Then $|\mu|$ is also countably additive and $|\mu| << \lambda$.*

Proof. The countably additivity is proved in ([**8**], see also [**3**]). The other part follow from Theorem 1. □

We first prove a simple lemma.

Lemma 3. *Let E be a Dedekind complete vector lattice, $\mathcal{A}$ is a σ-algebra of subsets of a set X, and $\mu : \mathcal{A} \to E$ a order-bounded, finitely additive measure. Then for any $A, B \in \mathcal{A}$,*

$$\mu^+(A) \leq \mu(A \cap B) + \mu^+(X) - \mu(B).$$

Proof.

$$\mu(B) - \mu(A \cap B) = \mu(A \cup B \setminus A) \leq \mu^+(A \cup B \setminus A) = \mu^+(A \cup B) - \mu^+(A).$$

Thus $\mu^+(A) \leq \mu(A \cap B) + \mu^+(X) - \mu(B)$. □

A part of the following result is known. We give a different proof.

Theorem 4. [**13**, Theorem 4, Theorem 5(1)] *Suppose E is a Dedekind complete Banach lattice which has either an order continuous norm or is an AM space with unit, $\mathcal{A}$ a sigma-algebra of subsets of a set X and $\mu\colon \mathcal{A} \to E$ an order bounded, countably additve measure in norm topology. Then the modulus $|\mu|$ is also countably additive. Also for a measurable and bounded f, $f \geq 0$,*

$$\bar{\mu}(f) \leq (\bar{|\mu|})(f) \leq 4\bar{\mu}(f).$$

Proof. Take a decreasing sequence $\{A_n\} \subset \mathcal{A}$ with $A_n \downarrow \emptyset$ and take a $B \in \mathcal{A}$. By Lemma 3,

$$\mu^+(A_n) \leq \mu(A_n \cap B) + \mu^+(X) - \mu(B). \tag{1}$$

We first consider the case that it is an AM space with unit. Thus $E = (C(S), \|\cdot\|)$, for some compact Stonian space S. Fix a $c > 0$. Since μ is countably additive, $\dot{\mu}(A_n) \downarrow 0$ and so $\dot{\mu}(A_n) \leq c$ for all $n \geq k$ for some $k \in \mathbb{N}$. Fix $n \geq k$. Now $\|\mu(A_n \cap B)\| \leq \dot{\mu}(A_n \cap B) \leq \dot{\mu}(A_n) \leq c$. Thus $\mu(A_n \cap B) \leq c \cdot 1$, $1 \in C(S)$ being the unit function. So we get $\mu^+(A_n) \leq c \cdot 1 + \mu^+(X) - \mu(B)$. Taking the inf on the right side as B varies in $\mathcal{A}$, we get $\mu^+(A_n) \leq \cdot 1$. Thus μ^+ and similarly μ^- are countably additive. This proves the result.

It is sufficient to prove the result for $0 \leq f \leq 1$. Take two measurable f, g, $0 \leq f \leq 1$, $0 \leq g \leq 1$. From $f + g = f \vee g + f \wedge g$ we get

$$\mu^+(f) \leq \mu(f \wedge g) + \mu^+(1) - \mu(g).$$

Since $\|\mu(f \wedge g)\| \leq \bar{\mu}(f)$, $\mu(f \wedge g) \leq \bar{\mu}(f) \cdot 1$. Thus we have

$$\mu^+(f) \leq \bar{\mu}(f) \cdot 1 + \mu^+(1) - \mu(g).$$

Taking inf on the right side as g, $0 \leq g \leq 1$, varies, we have $\mu^+(f) \leq \bar{\mu}(f) \cdot 1$. Thus $\|\mu^+(f)\| \leq \bar{\mu}(f)$. From this it is easy to prove that

$$\bar{\mu}(f) \leq (\bar{|\mu|})(f) \leq 2\bar{\mu}(f).$$

Now we consider the case E has an order continuous norm. In this case $E' = E'_n$, the order continous dual of E. Let $\inf(\mu^+(A_n)) = z \geq 0$; we have $\langle h, z \rangle = \lim \langle h, \mu^+(A_n) \rangle$, for all $h \in E'$. From (1)

$$z \leq \mu^+(A_n) \leq \mu(A_n \cap B) + \mu^+(X) - \mu(B).$$

Take $h \in (E'_n)_+$. We get

$$\langle h, z \rangle \leq \langle h, \mu^+(X) \rangle - \langle h, \mu(B) \rangle + \langle h, \mu(A_n \cap B) \rangle,$$

for all n. Since $A_n \cap B \downarrow \emptyset$, $\langle h, \mu(A_n \cap B) \rangle \to 0$. Thus

$$\langle h, z \rangle \leq \langle h, \mu^+(X) \rangle - \langle h, \mu(B) \rangle.$$

This gives $z \leq \mu^+(X) - \mu(B)$. Taking the inf on the right side as B varies in $\mathcal{A}$, we get $z = 0$. This proves that μ^+ is countably additive in the weak topology and so is countably additive in the norm topology. In a similar way μ^- is countably additive in the norm topology and thus $|\mu|$ is countably additive.

As in the case of AM space, we take two measurable f, g, $0 \leq f \leq 1$, $0 \leq g \leq 1$ and proceed as in that case. We get

$$\mu^+(f) \leq \mu(f \wedge g) + \mu^+(1) - \mu(g).$$

Taking $h \in E'_n$, $\|h\| \leq 1$, $h \geq 0$, we get

$$\langle h, \mu^+(f) \rangle \leq \langle h, \mu(f \wedge g) \rangle + \langle h, \mu^+(1) - \mu(g) \rangle.$$

Using the fact $\langle h, \mu(f \wedge g) \rangle \leq \bar{\mu}(f)$, this becomes

$$\langle h, \mu^+(f) \rangle \leq \bar{\mu}(f) + \langle h, \mu^+(1) - \mu(g) \rangle.$$

Taking inf on the right side as g, $0 \leq g \leq 1$ varies, we have $\langle h, \mu^+(f) \rangle \leq \bar{\mu}(f)$. Thus $\bar{\mu^+} \leq 2\bar{\mu}(f)$. So $\bar{\mu}(f) \leq (\bar{|\mu|})(f) \leq 4\bar{\mu}(f)$. □

We use this theorem to prove the following result.

Theorem 5. *Suppose E is a Dedekind complete Banach lattice which has either an order continuous norm or is an AM space with unit, $\mathcal{A}$ is a sigma-algebra of subsets of a set X and $\mu : \mathcal{A} \to E$ is an order bounded, countably additive measure in the norm topology. Then $L_1(\mu) = L_1(|\mu|)$.*

Proof. By Theorem 4, $|\mu|$ is countably additive. Choose $x' \in E'$ with $\|x'\| \leq 1$ such that $\lambda = |x' \circ \mu|$ has the property that $\mu << \lambda$. By Theorem 1, $|\mu| << \lambda$. Also it is a simple verification that $|x' \circ \mu| \leq |x'| \circ |\mu|$ and so $\lambda \leq |x'| \circ |\mu|$. Take $f \in L_1(\mu)$, $f \geq 0$ and put $f_n = f \wedge n$. By dominated convergence therorem, $\bar{\mu}(|f_n - f_m|) \to 0$. By Theorem 4, $|\bar{\mu}|(|f_n - f_m|) \to 0$ Since $L_1(|\mu|)$ is complete, $f_n \to \tilde{f} \in L_1(|\mu|)$. Since $\lambda \leq \bar{\mu}$ and also $\lambda \leq |x'| \circ |\mu| \leq (|\bar{\mu}|)$, we have $\int |f - f_n| \mathrm{d}\lambda \to 0$ and also $\int |\tilde{f} - f_n| \mathrm{d}\lambda \to 0$. Thus $f = \tilde{f}$. Thus $L_1(\mu) \subset L_1(|\mu|)$. In a very similar way $L_1(|\mu|) \subset L_1(\mu)$. This proves the result. □

Now we come to the regularity of $|\mu|$.

Theorem 6. *Suppose X is a completely regular Hausdorff space, E is a Banach lattice, $\mu\colon \mathcal{B}(X) \to E$ is an order bounded, countably additive, τ-smooth (resp. tight) Borel measure in the norm topology. If $|\mu|$ exists and is countably additive, then it is also τ-smooth (resp. tight).*

Proof. Choose measure λ, satisfying $\mu << \lambda$, to be τ-smooth if μ is τ-smooth and tight then μ is tight.

First we consider the case when μ is tight. Fix a Borel $B \subset X$ and let $\{C_\alpha\}$ be the increasing net of all compact subsets of B. Put $U_\alpha = (B \setminus C_\alpha)$. Since λ is tight, $\lambda(U_\alpha) \to 0$. Since, by Theorem 1, $|\mu| << \lambda$, we have $\lim_{\lambda(A) \to 0} |\bar{\mu}|(A) = 0$. This means $|\bar{\mu}|(U_\alpha) \to 0$. This proves that $|\mu|$ is tight.

Now we consider the case when μ is τ-smooth. In this case we take an increasing net $\{V_\alpha\}$ of open subsets of X with $V = \cup V_\alpha$ and $U_\alpha = \chi_{V \setminus V_\alpha}$. As in the case of tight measure above, we get $|\bar{\mu}|(U_\alpha) \to 0$. This proves $|\mu|$ is τ-smooth. □

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References

1. Aliprantis C. D. and Burkinshaw O., *Positive Operators*, Academic Press, 1985.
2. Diestel J.and Uhl J. J., *Vector Measures*, Amer. Math. Soc. Surveys, Vol. 15, Amer. Math. Soc., 1977.
3. Drewnowski L., *Topological rings of sets, continuous set functions, integration I, II,* Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys., **20** (1972), 269–286.
4. Grothendieck A., *Sur les applicationes lineares faiblement compactes d'espace du type $C(K)$*, Can. J. Math. **5** (1953), 129–173.
5. Khurana S. S., *Extension and regularity of group-valued Baire measures*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., **22** (1974), 891–895.
6. ______, *Vector measures on topological spaces*, Georgian Math. J., **14** (2007), 687–698
7. Kluvánek I., and Knowles G., *Vector measures and Control Systems*, North-Holand/ Americal Elsevier 1975.
8. Kronsbein G. and Meyer-Nieberg P., *Factorization of vector measures*, Arch. Math, **63** (1994), 541–548.
9. Lewis D. R., *Integration with respect to vector measures*, Pac. J. Math., **33** (1970), 157–165.
10. ______, *On integrability and summability in vector spaces*, Ill. J. Math., **16** (1972), 294–307.
11. Schaeffer H. H., *Topological Vector spaces*, Springer Verlag 1986.
12. ______, *Banach Lattices and positive operators*, Springer-Verlag 1974.

13. Schaeffer H. H. and Zhang, Xaio-Dong, *A note on order bounded vector measures*, Arch. Math, **63** (1994), 152–157.

14. Schmidt K. D., *On the Jordan decomposition for vector measures*, Probability in Banach spaces, IV (Oberwolfach, 1982), 198–203, Lecture Notes in Math., 990, Springer, Berlin-New York, 1983.

15. Varadarajan V. S., *Measures on topological spaces*, Amer. Math. Soc. Transl. **(2)48** (1965), 161–220.

16. Wheeler R. F., *Survey of Baire measures and strict topologies*, Expos. Math., **2** (1983), 97–190.

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