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# ON IDEALS AND CONGRUENCE RELATIONS IN TRELLISES

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Abstract. In this paper an attempt has been made to extend the ideal theory of lattices to trellises. Hashimoto's results concerning ideals and congruences in lattices are extended to trellises. Also a characterization of trellises in which every ideal is the kernel of at most one congruence relation is obtained by extending the corresponding well-known result in lattices due to Grätzer and Schmidt.

## 1. Introduction

A reflexive and antisymmetric binary relation $\trianglelefteq$ on a set $A$ is called a *pseudo-order* on $A$. A *pseudo-ordered set* or *a psoset* $(A; \trianglelefteq)$ consists of a nonempty set $A$ and a pseudo-order $\trianglelefteq$ on $A$. By a trellis we mean a psoset in which any two elements have a greatest lower bound and a least upper bound. The notion of a trellis, introduced by E. Fried [**2**] and H. L. Skala [**8**], is a natural generalization of that of a lattice. There is extensive literature on ideals and congruence relations in lattices (see Hashimoto [**6**], G. Grätzer and E. T. Schmidt [**4**]). These are motivated by the pioneering work of Hashimoto based on an open problem posed by G. Birkhoff ([**1**, Problem 73]). This paper extends some of these well-known results to trellises.

In the first section, some basic theorems on ideals in lattices are extended to trellises with some modifications. In the second section, by extending Hashimoto's theorem( see [**3**]) concerning lattices to trellises, it is proved that, if in a trellis $L$, every ideal is the kernel of at least one congruence relation, then its ideal lattice $I(L)$ is distributive. Also, some necessary and sufficient conditions for the existence of a one to one correspondence between ideals and congruence relations in trellises are obtained. Finally, a characterization of trellises in which every ideal is the kernel of at most one congruence relation is given.

For undefined notations and terminology, the reader may refer to H. L. Skala [**8**], G. Grätzer [**3**].

## 2. Ideals in trellises

An element $a$ of a trellis is said to be *left transitive* if $x \trianglelefteq y \trianglelefteq a$ implies $x \trianglelefteq a$.

For an element $a$ of a trellis $L$, $(a] := \{x \in L : x \trianglelefteq a\}$.

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A pseudo chain (p-chain) $C = \{a_i | i = 1, 2, \ldots\}$ of elements of a psoset $(A; \trianglelefteq)$ is said to be an *infinite ascending p-chain* in $A$ if $a_1 \triangleleft a_2 \triangleleft a_3, \ldots$, where C is infinite. A psoset $(A; \trianglelefteq)$ is said to satisfy the *ascending p-chain condition* if there is no infinite ascending p-chain in $A$.

**Definition 2.1.** A nonempty subset $I$ of a trellis $L$ is called an *ideal* if $I$ is a subtrellis of $L$ and for each element $a$ of $I$, $b \trianglelefteq a$ implies that $b$ is in $I$. For a nonempty subset $B$ of a trellis $L$, *the ideal generated by* $B$, denoted by $\langle B \rangle$, is the intersection of all ideals of $L$ containing $B$. If $B$ is a singleton $\{a\}$, then we write $\langle a \rangle$ for $\langle \{a\} \rangle$. An ideal $I$ is said to be *principal* if $I = \langle a \rangle$ for an element $a$ of $L$.

**Remark 2.2.** For an element $a$ of a trellis $L$, $\langle a \rangle = (a]$ if and only if $a$ is left transitive.

In the case of a lattice every ideal is principal if and only if the lattice satisfies the ascending chain condition. A partial extension of this result is proved for trellises in the following lemma.

**Lemma 2.3.** *If $L$ is an acyclic trellis satisfying the ascending p-chain condition, then every ideal $I$ of $L$ is principal and can be written as $I = (a]$ for a unique left transitive element $a$ of $L$.*

*Proof.* Let $I$ be an ideal of $L$. Since $L$ is acyclic and satisfies the ascending p-chain condition, every nonempty subset of $L$ has a maximal element. Hence $I$ has a maximal element say $a$. As $I$ is a subtrellis of $L$, $a$ is the greatest element of $I$ and clearly $I = \langle a \rangle$. Let $x \trianglelefteq y \trianglelefteq a$. As $I$ is an ideal containing $a$, both $y$ and $x$ belong to $I$. Since $a$ is the greatest element of $I$, we have $x \trianglelefteq a$. Therefore $a$ must be left transitive and hence $\langle a \rangle = (a]$. □

**Definition 2.4.** For a trellis with the least element 0, the *height* of an element $a$ is defined to be the supremum of the cardinalities of all p-chains connecting 0 to $a$ and it is denoted by $\mathrm{ht}(a)$. A trellis $L$ is said to be of *finite height* if $L$ is acyclic and there exists an integer $k$ such that $\mathrm{ht}(a) \leq k$, for every element $a$ of $L$.

Lemma 2.3 is useful in dealing with trellises of finite height, since such trellises are acyclic and satisfy the ascending p-chain condition.

In the case of lattices every convex sublattice can be written as the intersection of an ideal and a dual ideal. However, in the case of trellises, even though the intersection of an ideal and a dual ideal is a convex subtrellis if nonempty, the converse holds only if the trellis is a lattice as proved in the following:

**Lemma 2.5.** *If every convex subtrellis of a trellis $L$ is the intersection of an ideal and a dual ideal of $L$, then $L$ is a lattice.*

*Proof.* Let $a, b, c \in L$ with $a \trianglelefteq b \trianglelefteq c$. We have to show that $a \trianglelefteq c$. By the hypothesis, the convex subtrellis $\{a\} = I \cap D$, for an ideal $I$ and a dual ideal $D$ of $L$. Since $a \in I$ and $I$ is an ideal, clearly $a \wedge c \in I$. Also $a \in D$, $a \trianglelefteq b \trianglelefteq c$ and $D$ is a dual ideal so that $c \in D$. Therefore $a \wedge c \in D$. Hence $a \wedge c \in I \cap D = \{a\}$. Thus $a \wedge c = a$ or equivalently $a \trianglelefteq c$, as desired. □

**Definition 2.6.** A proper ideal $I$ of a trellis $L$ is said to be

1. *prime,* if for $a, b \in L$, $a \wedge b \in I$ implies that $a \in I$ or $b \in I$;
2. *maximal,* if $L$ contains no larger proper ideal.

It is known that a lattice $L$ is distributive if and only if every ideal can be written as the intersection of all prime ideals containing it [**3**]. For a trellis of finite height, we have the following result.

**Theorem 2.7.** *Let $L$ be a trellis of finite height. If every ideal of $L$ is the intersection of all prime ideals containing it, then $L$ is a lattice.*

*Proof.* If $L$ is not a lattice, then there exists an element $a$ of least height such that $a \lhd b \lhd c$ and $a \ntriangleleft c$. Then $a \wedge c$ and $a$ must be left transitive elements. Hence by Remark 2.2, $\langle a \wedge c \rangle = (a \wedge c]$ and $\langle a \rangle = (a]$. Since $(a \wedge c] \neq (a]$, $a \notin (a \wedge c]$. By the hypothesis $(a \wedge c] = \cap M$, where $M = \{P \in I(L)$: $P$ is a prime ideal and $a \wedge c \in P\}$. Now $a \wedge c \in P$ for every $P \in M$. This implies $a \in P$ for every $P \in M$. Thus $a \in \cap M = (a \wedge c]$, a contradiction. Hence $L$ must be a lattice. □

**Remark 2.8.** The following results can be proved as in the case of lattices:

1. Let $L$ be a complemented or a relatively complemented trellis. Then every prime ideal of $L$ is maximal.
2. Let $L$ be a trellis and $I$ be a prime ideal of $L$. Then there exists a homomorphism $f$ of $L$ onto $C_2 = \{0, 1\}$ with $0 \lhd 1$ such that $f^{-1}(0) = I$. Hence every prime ideal is the kernel of at least one homomorphism.

## 3. THE KERNEL OF A CONGRUENCE RELATION

**Definition 3.1.** A *congruence relation* $\equiv$ on a trellis $L$ is an equivalence relation such that whenever $a \equiv b$ and $c \equiv d$, then $a \vee c \equiv b \vee d$ and $a \wedge c \equiv b \wedge d$.

**Definition 3.2.** A trellis $L$ is said to be *Hashimoto* if every ideal of $L$ is the kernel of at least one congruence relation.

Hashimoto trellises need not be associative(distributive) in general. One interesting property of Hashimoto trellises is given in the following theorem.

**Theorem 3.3.** *In a Hashimoto trellis, if $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$ hold for some $x$, then $\langle y \rangle = \langle z \rangle$.*

*Proof.* Clearly $x \wedge y \equiv y(\Theta[\langle y \rangle])$. Hence $x \equiv y \vee x = x \vee z(\Theta[\langle y \rangle])$. Also $x \wedge y = x \wedge z \equiv z(\Theta[\langle y \rangle])$. As $x \wedge y \in \langle y \rangle$ and the trellis is Hashimoto, clearly $z \in \langle y \rangle$ or equivalently $\langle z \rangle \subseteq \langle y \rangle$ . Similarly $\langle y \rangle \subseteq \langle z \rangle$. Hence $\langle y \rangle = \langle z \rangle$. □

The next theorem shows that if a trellis is Hashimoto, then its ideal lattice is distributive. First we prove two lemmas.

The following lemma gives a description of the ideal generated by a nonempty subset of a trellis.

**Lemma 3.4.** *For any nonempty subset $H$ of a trellis $L$, $\langle H\rangle$ can be described as follows:*

*Let* $I_0 = H$ *and for* $n \geq 1$,

$$I_n = \{x \in L : x \trianglelefteq p(y_1, y_2, \ldots, y_k), \text{for some } k \geq 1, \text{ where } p \text{ is a polynomial and } y_i \in I_{n-1} \text{ for } 1 \leq i \leq k\}.$$

*Then* $\langle H\rangle = \bigcup\limits_{n=0}^{\infty} I_n$.

*Proof.* First we shall show that for each $n$, $I_{n-1} \subseteq I_n$. For any nonempty subset $X$ of $L$, define $o(X) = \bigcup_{x\in X}(x]$ .

Call $o(X)$ as o-ideal generated by $X$. Clearly $X \subseteq o(X)$. Let $H$ be the given nonempty subset of $L$. Then as in the case of lattices, the subtrellis generated by $H$, denoted by$[H]$ is given by

$$= \{x \in L : x = p(h_1, h_2, \ldots, h_n) \text{ for some } n \geq 1 \text{ and some polynomial } p, \text{where } h_i \in H, 1 \leq i \leq n\}.$$

Hence by definition of $I_n$, $I_n = o([I_{n-1}])$ for $n \geq 1$. This shows that $I_{n-1} \subseteq I_n$ for each $n \geq 1$. Take $K = \bigcup_{n=0}^{\infty} I_n$.
Claim: $K$ is an ideal of $L$.

Let $x, y \in K$. Then $x \in I_n$ and $y \in I_m$ for some $n$ and $m$, say $m \leq n$. Then $I_m \subseteq I_n$ so that $x, y \in I_n$. But then $x \vee y \in I_{n+1} \subseteq K$.

Let $x \in L$, $y \in K$. Since $y \in K$, clearly $y \in I_n$ for some $n$. Now $x \wedge y \trianglelefteq y \in I_n$. This implies $x \wedge y \in I_{n+1} \subseteq K$. Thus $K$ is an ideal.

It remains to show that K $= \langle H\rangle$. Clearly $H = I_0 \subseteq K$ so that $\langle H\rangle \subseteq K$. On the other hand, if $a \in K$, then $a \in I_n$ for some $n$. By induction on $n$, it follows that $I_n \subseteq \langle H\rangle$. In fact, if $n = 0$, $I_0 = H \subseteq \langle H\rangle$. Let $n \geq 0$ and $I_n \subseteq \langle H\rangle$. Since $I_n \subseteq \langle H\rangle$, clearly $[I_n] \subseteq \langle H\rangle$, which in turn implies that $I_{n+1} = \mathrm{o}([I_n]\ ) \subseteq \langle H\rangle$. Thus $\langle H\rangle = \bigcup_{n=0}^{\infty} I_n$ as desired. □

In the following lemma corresponding to each congruence relation on a trellis, a congruence relation on the ideal lattice of the trellis has been constructed.

**Lemma 3.5.** *If $\Theta$ is a congruence relation on a trellis $L$, then the binary relation $\Phi$ on $I(L)$ defined by, " for $J$, $K \in I(L)$, $J \equiv K(\Phi)$ if and only if for each element $a$ in $J$ there is an element $b$ in $K$ such that $a \equiv b(\Theta)$ and for each element $b$ in $K$ there is an element $a$ in $J$ such that $a \equiv b(\Theta)$", is a congruence relation on $I(L)$.*

*Proof.* Clearly $\Phi$ is an equivalence relation on $I(L)$. To show that $\Phi$ satisfies the substitution property, consider $S, J \in I(L)$ with $S \equiv J(\Phi)$ and $K \in I(L)$. We need to prove that

(i) $S \wedge K \equiv J \wedge K(\Phi)$;
(ii) $S \vee K \equiv J \vee K(\Phi)$.

Let $a \in S \wedge K$. Since $a \in S$, there exists an element $b$ in $J$ such that $a \equiv b(\Theta)$. Then $a \equiv a \wedge b(\Theta)$, and $a \wedge b \in J \cap K = J \wedge K$.

Similarly, for each $x \in J \wedge K$ there exists $y \in S \wedge K$ such that $x \equiv y(\Theta)$. Hence $S \wedge K \equiv J \wedge K(\Phi)$, which proves (i).

To prove (ii), we first note that $S \vee K = \langle S \cup K \rangle$. Also by Lemma 3.4, $\langle S \cup K \rangle = \bigcup_{n=0}^{\infty} I_n$, where $I_0 = S \cup K$ and $I_n = o([I_{n-1}])$, for $n \geq 1$. Similarly $J \vee K = \bigcup_{n=0}^{\infty} I'_n$, where $I'_0 = J \cup K$ and $I'_n = o([I'_{n-1}])$, for $n \geq 1$.

We shall show, by induction on $n$, that for each element $a$ in $I_n$, there exists an element $b$ in $I'_n$ such that $a \equiv b(\Theta)$.

Let $a \in I_0 = S \cup K$. Then either $a \in S$ or $a \in K$. If $a \in S$, then as $S \equiv J(\Phi)$, there exists an element $b$ in $J$ such that $a \equiv b(\Theta)$. Now, $b \in J \subseteq J \cup K$. Hence $b \in J \cup K$ such that $a \equiv b(\Theta)$. If $a \in K$, then clearly $a \in K \cup J$ and $a \equiv a(\Theta)$. Thus result is true when $n = 0$.

Let $n \geq 0$ and assume that the result is true for $I_n$. Let $a \in I_{n+1}$. Then $a \trianglelefteq p(y_1, y_2, \ldots, y_k)$, for some $k \geq 1$ , for some polynomial $p$ and $y_i \in I_n$. As $y_i \in I_n$, by induction hypothesis, there is $z_i \in I'_n$ such that $y_i \equiv z_i(\Theta)$, for $1 \leq i \leq k$. Therefore $p(y_1, y_2, \ldots, y_k) \equiv p(z_1, z_2, \ldots, z_k)(\Theta)$. Hence $a = a \wedge p(y_1, y_2, \ldots, y_k) \equiv a \wedge p(z_1, z_2, \ldots, z_k)(\Theta)$ and $a \wedge p(z_1, z_2, \ldots, z_k) \in I'_{n+1}$ .

Hence for each $a$ in $I_{n+1}$ there is an element $b$ in $I'_{n+1}$ such that $a \equiv b(\Theta)$.

Similarly it follows that for each $a \in I'_n$, there is an element $b \in I_n$ such that $a \equiv b(\Theta)$. This proves (ii). Thus $\Phi$ is a congruence relation on $I(L)$. □

Now we are ready to prove the main theorem.

**Theorem 3.6.** *If a trellis $L$ is Hashimoto, then $I(L)$ is distributive.*

*Proof.* To prove that $I(L)$ is distributive, by [**5**], it suffices to show that every principal ideal of $I(L)$ is the kernel of at least one congruence relation on $I(L)$. Let $I$ be the kernel of a congruence relation $\Theta$ on $L$. Let $(I]^*$ denote the principal ideal of $I(L)$ generated by $I$. To show that $(I]^*$ is the kernel of at least one congruence relation on $I(L)$, it is required to show that $(I]^*$ is a congruence class modulo $\Phi$ for some congruence relation $\Phi$ on $I(L)$.

For the congruence relation $\Theta$ on $L$, let $\Phi$ be the congruence relation on $I(L)$ as defined in Lemma 3.5.

To complete the proof it suffices to prove that $(I]^* = [I]\Phi$.

Let $J \in (I]^*$. Then $J \subseteq I$. To show $J \equiv I(\Phi)$, consider any $j \in J$. Then clearly $j \in I$ and $j \equiv j(\Theta)$. On the other hand, let $x \in I$. Then, since $J \subseteq I = [x]\Theta$, clearly $j \equiv x(\Theta)$ holds for any $j \in J$. Thus $J \equiv I(\Phi)$; therefore $J \in [I]\Phi$.

Let $J \in [I]\Phi$ and $j \in J$. Then, since $J \equiv I(\Phi)$, there is an element $x$ in $I$ such that $j \equiv x(\Theta)$. But then $j \in [x]\Theta = I$. Hence $J \in (I]^*$. Thus $(I]^* = [I]\Phi$. □

**Remark 3.7.** The following counterexample shows that for a trellis $L$, distributivity of I(L) need not imply that $L$ is Hashimoto.

**Counterexample 3.8.** In Figure 1, $L$ and $I(L)$ represent a trellis and its ideal lattice, respectively (In the diagram the dashed curve indicates that the transitivity is removed between its end points ). Observe that the ideal $\langle b \rangle$ of $L$ is not the kernel of any congruence relation. If $\langle b \rangle$ is the kernel of a congruence relation $\Theta$ of $L$, then clearly $0 \equiv a(\Theta)$ so that $d = 0 \vee d \equiv a \vee d = 1(\Theta)$. But then $b = d \wedge c \equiv 1 \wedge c = c(\Theta)$ and $c \notin \langle b \rangle$, a contradiction.

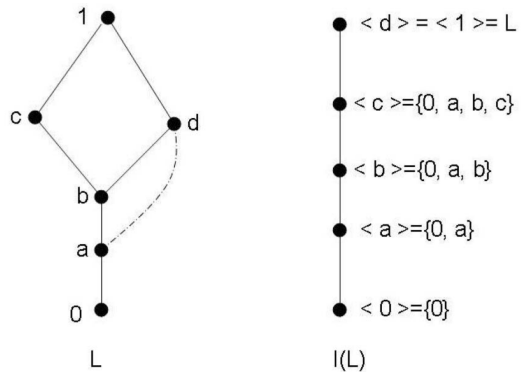


**Figure 1.** $I(L)$ is distributive $\not\Rightarrow$ $L$ is Hashimoto.

A characterization of Hashimoto trellises is given in the following theorem.

**Theorem 3.9.** *The following statements are equivalent for a trellis $L$:*

1. *$L$ is Hashimoto;*
2. *for any ideal $I$ of $L$, $x \equiv y(\Theta[I])$ implies $\langle x\rangle \vee I = \langle y\rangle \vee I$ in $I(L)$.*

*Proof.* Let $L$ be Hashimoto. Let $I$ be an ideal of $L$ and $x \equiv y(\Theta[I])$. Now $x \in [y]\Theta[I \vee \langle y\rangle] = I \vee \langle y\rangle$ since $y \in I \vee \langle y\rangle$ which is the kernel of a congruence relation. Hence $\langle x\rangle \subseteq I \vee \langle y\rangle$ in $I(L)$ so that $\langle x\rangle \vee I \subseteq I \vee \langle y\rangle$ in $I(L)$. Similarly $\langle y\rangle \vee I \subseteq I \vee \langle x\rangle$ in $I(L)$. Hence $\langle x\rangle \vee I = \langle y\rangle \vee I$ in $I(L)$.

Conversely, let 2. hold and let $a$ be an element of an ideal $I$ of $L$. Clearly $I \subseteq [a]\Theta[I]$. On the other hand, if $b \in [a]\Theta[I]$, then $b \equiv a(\Theta[I])$. Hence by the hypothesis, $\langle a\rangle \vee I = \langle b\rangle \vee I$ in $I(L)$. As $a \in I$, clearly $\langle a\rangle \vee I = I$ and therefore $\langle b\rangle \subseteq I$ or equivalently $b \in I$. Thus $I = [a]\Theta[I]$. □

The following theorem generalizes the well-known Hashimoto theorem [**3**] to trellises.

**Theorem 3.10.** *The following statements are equivalent for a trellis $L$:*

1. *every ideal of $L$ is the kernel of exactly one congruence relation of $L$ and every congruence relation of $L$ has a kernel;*
2. *$L$ is bounded below, Hashimoto and if $I = [0](\Theta(a,b))$ for some $a, b \in L$ with $a \trianglelefteq b$, then $\Theta[I] = \Theta(a,b)$.*

*Proof.* 1. implies 2. Since every congruence relation has a kernel, the equality relation $\omega$ also has a kernel and therefore $L$ must be bounded below.

Also $L$ is Hashimoto. Let $a \trianglelefteq b$ and $I = [0]\Theta(a,b)$. Since every ideal of L is the kernel of exactly one congruence relation and $I$ is already the kernel of $\Theta(a,b)$, clearly $\Theta[I] = \Theta(a,b)$.

2. implies 1. Let $I$ be the kernel of a congruence relation $\Psi$. Then clearly $\Theta[I] \subseteq \Psi$. On the other hand, let $x \equiv y(\Psi)$. Then $x \wedge y \equiv y(\Psi)$ and $y \equiv x \vee y(\Psi)$. Let $I_1 = [0]\Theta(x \wedge y, y)$ and $I_2 = [0]\Theta(y, x \vee y)$. As $[0]\Theta(x,y) \subseteq [0]\Psi = I$,

it is clear that $I_1 \subseteq I$ and $I_2 \subseteq I$. Since $I$ is an ideal, $I_1 \vee I_2 \subseteq I$. Now $x \wedge y \equiv y \equiv x \vee y(\Theta[I_1] \vee \Theta[I_2])$. Moreover, $\Theta[I_1] \vee \Theta[I_2] \subseteq \Theta[(I_1 \vee I_2)] \subseteq \Theta[I]$, so that $x \equiv y(\Theta[I])$, which proves that $\Theta[I] = \Psi$.

If $\Theta$ is any congruence relation of $L$, then $[0]\Theta$ is the ideal kernel of $\Theta$ and hence every congruence relation of $L$ has a kernel. □

**Remark 3.11.** A trellis with the least element 0 is said to be *sectionally complemented* if for any two elements $x$, $y$ such that $0 \trianglelefteq x \trianglelefteq y$ there exists an element $x'$ such that $x \wedge x' = 0$ and $x \vee x' = y$. Note that even if a trellis $L$ satisfies the condition (1) of the above theorem, it is not necessary that $L$ or $I(L)$ is sectionally complemented (see Figure 2, where $I(L)$ is the ideal lattice of the trellis $L$ and $\operatorname{Con} L$ is the congruence lattice of $L$. In $\operatorname{Con} L$, $\Theta_1 = \Theta(0, a)$, $\Theta_2 = \Theta(a, b)$ and $\Theta_3 = \Theta(a, c)$).

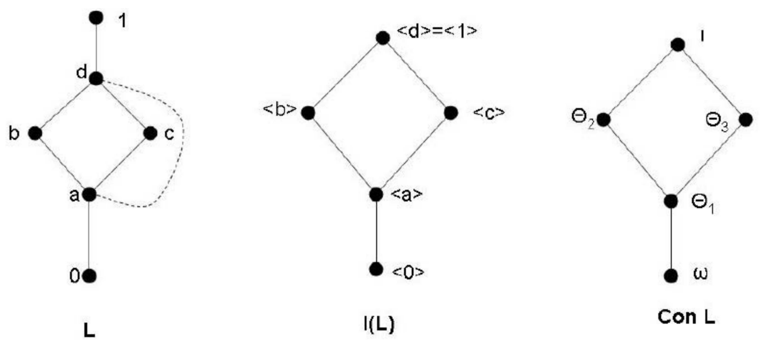


**Figure 2.** .

In the case of a relatively complemented trellis of finite height, we have a nice description for the kernel of a congruence relation. The proof uses some ideas from[**8**, Theorem 45].

**Theorem 3.12.** *Let $I$ be an ideal of a relatively complemented trellis $L$ of finite height. Then the following statements are equivalent:*

1. *$I$ is the kernel of a congruence relation;*
2. *$x \equiv y(\Theta[I])$ if and only if $x \vee i = y \vee i$, for some $i \in I$.*

*Proof.* Let $I$ be an ideal of a trellis $L$ of finite height. Then $I = (a]$ for some left transitive element $a \in I$.

1. implies 2. Suppose $x \equiv y(\Theta\,(a])$. Note that $a \trianglelefteq (x \vee a) \wedge (y \vee a) \trianglelefteq (x \vee a)$. Let $b$ be an $(a, x \vee a)$-complement of $(x \vee a) \wedge (y \vee a)$. Then $0 \equiv a = b \wedge ((x \vee a) \wedge (y \vee a)) \equiv b \wedge (x \vee y) \equiv b \wedge x(\Theta\,(a])$. This shows that

(i) $0 \equiv b \wedge x(\Theta\,(a])$.

Also $x \equiv x \vee a = b \vee ((x \vee a) \wedge (y \vee a)) \equiv b \vee (x \vee y) \equiv b \vee x(\Theta\,(a])$. Therefore

(ii) $x \wedge b \equiv b(\Theta\,(a])$.

Now $0 \equiv b(\Theta\,(a])$ from (i) and (ii). Since $I = (a]$ is the kernel of a congruence relation, clearly $b \in I = (a]$ so that $b \trianglelefteq a$. Moreover, $a \trianglelefteq b$ since $a = b \wedge ((x \vee a) \wedge (y \vee a))$ and hence $a = b$. But then $x \vee a = b \vee ((x \vee a) \wedge (y \vee a)) = a \vee ((x \vee a) \wedge (y \vee a)) = (x \vee a) \wedge (y \vee a)$, whence $x \vee a \trianglelefteq y \vee a$. Similarly $y \vee a \trianglelefteq x \vee a$. Thus $x \vee a = y \vee a$. On the other hand suppose $x \vee i = y \vee i$ for some $i \in I$. Since $0 \equiv i(\Theta\,(a])$, clearly $x \equiv x \vee i = y \vee i \equiv y(\Theta\,(a])$ .

2. implies 1. We are going to show that $I = [0]\Theta[I]$. Evidently $I \subseteq [0]\Theta[I]$. Conversely, let $x \in [0]\Theta[I]$. Then $x \equiv 0(\Theta[I])$, which implies $x \vee i = 0 \vee i$ for some $i \in I$, by 2. Thus $x \trianglelefteq i$, hence $x \in I$. □

The following theorem gives a characterization of trellises in which every ideal is a congruence class under, at most, one congruence relation. The proof would be the same as that of one Theorem 3.10.

**Theorem 3.13.** *The following statements are equivalent for a trellis L:*

1. *every congruence relation has a kernel and every ideal is a congruence class under, at most, one congruence relation;*
2. *L is bounded below and if* $I = [0](\Theta(a, b))$ *for some* $a,\ b \in L$ *with* $a \trianglelefteq b$*, then* $\Theta[I] = \Theta(a, b)$.

**Corollary 3.14.** *In a sectionally complemented trellis every ideal is the kernel of, at most, one congruence relation.*

*Proof.* It is easy to observe that a sectionally complemented trellis satisfies the condition 2. of the above theorem. □

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