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TRIGONOMETRIC EXPRESSIONS FOR FIBONACCI AND LUCAS NUMBERS

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Introduction

The amount of literature bears witness to the ubiquity of the Fibonacci numbers and the Lucas numbers. Not only these numbers are popular in expository literature because of their beautiful properties, but also the fact that they 'occur in nature' adds to their fascination. Our purpose is to use a certain polynomial identity to express these numbers in terms of trigonometric functions. It is interesting that these expressions provide natural proofs of old and new divisibility properties for the Fibonacci numbers. One can naturally recover some divisibility properties and discover/observe some others which seem to be new. There are some fascinating open questions about the periodicity of the Fibonacci sequences modulo primes and we shall also prove some partial results on this.

1. Fibonacci and Lucas numbers in trigonometric form

The Fibonacci numbers are recursively defined by $F_{n+1} = F_n + F_{n-1}$ where $F_0 = 0$, $F_1 = 1$. The first few are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

The so-called Cauchy-Binet identity gives an expression in closed form as $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ where $\alpha = (1+\sqrt{5})/2$, the "golden ratio" and $\beta = (1-\sqrt{5})/2 = -1/\alpha$. The Fibonacci numbers have the Lucas numbers as close cousins. The Lucas numbers are defined by the same recursion $L_{n+1} = L_n + L_{n-1}$, but the starting numbers are $L_0 = 2, L_1 = 1$. The first few are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \ldots$$

We recall a polynomial identity (an identity which holds for every complex value of the variable) observed in [**6**]:

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} (xy)^r (x+y)^{n-1-2r} = x^{n-1} + x^{n-2}y + \cdots + y^{n-1}.$$

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Note that it is a simple exercise to prove this polynomial identity by induction on n. The Cauchy-Binet identity can be deduced from the above identity as in [**5**] simply by specializing the values $x = \alpha, y = \beta$. The bridge to this deduction is provided by the summatory expression $F_n = \sum_{r\geq 0} \binom{n-1-r}{r}$ for all $n > 0$ which is also provable by induction on n! See also [**1**] for a combinatorial interpretation of this polynomial identity. We note in passing that Cauchy-Binet type of identity is easily obtained for a general linear recurrence relation of any order m. In that case the n-th term is $a_n = \sum_{i=1}^{m} c_i\lambda_i^n$, where λ_i are the eigenvalues of the characteristic equation and the constants c_i are evaluated by looking at the initial values. We show there is much more scope in exploiting the polynomial identity mentioned above; in particular, we use this and similar polynomial identities to obtain trigonometric and other expressions such as the following.

Theorem 1.

(a) $$F_n = \prod_{r=1}^{[(n-1)/2]} \left(3 + 2\cos\frac{2\pi r}{n}\right)$$

(b) $$L_{2n+1} = \prod_{r=1}^{n} \left(3 - 2\cos\frac{2\pi r}{2n+1}\right)$$

(c) $$L_{2n} = \prod_{r=0}^{n-1} \left(3 - 2\cos\frac{(2r+1)\pi}{2n}\right)$$

(d) $$L_{2n+1} = \sum_{r\geq 0} (-1)^r \binom{2n-r}{r} 5^{n-r}$$

(e) $$L_{2n} = -\,\mathrm{i}(x - x^{-1}) \sum_{r\geq 0} (-1)^r \binom{n-1-r}{r} (x + x^{-1})^{n-1-2r}$$

where $x = \dfrac{3+\sqrt{5}}{2}\,\mathrm{e}^{(\mathrm{i}\,\pi)/(2n)}$.

From these expressions we shall deduce the following divisibility results:

Corollary 1.

(i) F_n *divides* F_{mn},

(ii) L_n *divides* $L_{(2m+1)n}$,

(iii) L_{2n+1} *divides* $F_{2n(2m+1)}$,

(iv) $F_{2n} + F_{2n+2}$ *divides* $F_{(2n+1)m}$,

(v) $F_{n-2k} + F_{n+2k}$ *divides* $F_{mn-2k} + F_{mn+2k}$,

(vi) $F_{n-2k-1} + F_{n+2k+1}$ *divides* $F_{(2m+1)n-2k-1} + F_{(2m+1)n+2k+1}$,

(vii) $F_{n-k} + F_{n+k}$ *divides* $F_{n-k(2l+1)} + F_{n+k(2l+1)}$.

It is worth remarking that the divisibility properties like (i) above can be deduced from the Cauchy-Binet identity equally easily but, there is one subtle difference. Using the Cauchy-Binet identity, one needs to use factorization while the proof deduced from the trigonometric expression "physically shows" all the terms of the denominator "appearing" in the numerator.

The proofs will be given in Section 3 using the polynomial identity. Very interestingly, the Chebychev polynomials are polynomials defined by recursion which generalizes the Fibonacci recursion and in Section 3 we look at them and give another proof of the trigonometric expression. This reveals, in a sense, the mysterious connection between Fibonacci numbers and trigonometric functions.

1.1. A sequence interpolating F_n and L_n

While discussing the Fibonacci numbers, we also run across accidentally the sequence $\{a_n\}$ which is defined by:

$$a_n = \sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} 5^{[(n-1)/2]-r} \qquad \text{for all } n \geq 1.$$

We shall also prove the following lemma

Lemma 1.
(i) $a_n = \prod_{r=1}^{[(n-1)/2]} \left(3 - 2\cos\frac{2\pi r}{n}\right)$.

(ii) The sequence $\{a_n\}$ satisfies the following Cauchy-Binet-type of identity:

$$a_n = \begin{cases} \dfrac{(1+\sqrt{5})^n - (\sqrt{5}-1)^n}{2^n} & \textit{for odd } n \\ \dfrac{(1+\sqrt{5})^n - (\sqrt{5}-1)^n}{2^n\sqrt{5}} & \textit{for even } n \end{cases}$$

(iii) The sequence $\{a_n\}$ satisfies the recursion

$$a_{2n+1} = 5a_{2n} - a_{2n-1}$$
$$a_{2n+2} = a_{2n+1} - a_{2n}$$

(iv) $a_n = F_n$ *or* L_n *according as* n *is even or odd.*

(v) $a_m | a_n$ *if* $m | n$.

Note the first few values of $\{a_n\}$ are $1, 1, 4, 3, 11, 8, 29, 21, 76, 55, 199, 144, 521, 377, \ldots$ As it is not increasing, the divisibility result seems surprising!

2. Proofs using polynomial identity

Proof of Theorem 1(a). Start with the polynomial identity from [**6**]

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} x^r (1+x)^{n-1-2r} = 1 + x + \cdots + x^{n-1}$$

The right hand side equals $(x^n - 1)/(x-1) = \prod_{r=1}^{n-1}(x - \mathrm{e}^{2\,\mathrm{i}\,r\pi/n})$. It is crying out that we combine the terms corresponding to r and $n-r$; if n is even, there is a middle term corresponding to $r = n/2$ which is $x+1$. We obtain

$$\sum_{r=0}^{(n-2)/2} \binom{n-1-r}{r} \left(\frac{-x}{(1+x)^2}\right)^r (1+x)^{n-1} = (x+1) \prod_{r=1}^{(n-2)/2} \left(x^2 - 2x\cos\frac{2\pi r}{n} + 1\right).$$

Let us take for x a solution of the quadratic equation $(x+1)^2 = -x$ (that is, $x^2+3x+1=0$). Thus, one has for even n

$$(1+x)^{n-1} \sum_{r=0}^{(n-2)/2} \binom{n-1-r}{r} = (-x)^{(n-2)/2}(1+x) \prod_{r=1}^{(n-2)/2} \left(3+2\cos\frac{2\pi r}{n}\right).$$

As $(1+x)^2 = -x$, we have for even n that $(1+x)^{n-1} = (1+x)(-x)^{(n-2)/2}$ which, therefore, gives the first formula

$$F_n = \prod_{r=1}^{[(n-1)/2]} \left(3+2\cos\frac{2\pi r}{n}\right) \qquad \text{for all } n \geq 1,$$

where, as usual, the usual convention is that an empty product equals 1. This proves (a). □

Proof of Lemma 1. (i) Let us try to carry over the above proof for the sequence

$$a_n = \sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} 5^{[(n-1)/2]-r}.$$

The polynomial identity

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} x^r (1+x)^{n-1-2r} = \frac{x^n-1}{x-1} = \prod_{r=1}^{n-1} (x - \mathrm{e}^{2\,\mathrm{i}\,r\pi/n})$$

has the right hand side

$$(1+x) \prod_{r=1}^{(n/2)-1} \left(x^2 - 2x\cos\left(\frac{2\pi r}{n}\right) + 1\right) \qquad \text{or} \qquad \prod_{r=1}^{(n/2)-1} \left(x^2 - 2x\cos\left(\frac{2\pi r}{n}\right) + 1\right)$$

according as n is even or odd.

If we now take x to be a solution of $x^2-3x+1=0$ (so $(x+1)^2 = 5x$), we obtain for odd n

$$\sum_{r=0}^{(n-1)/2} (-1)^r \binom{n-1-r}{r} 5^{[(n-1)/2]-r} = \prod_{r=1}^{(n-1)/2} \left(3-2\cos\frac{2\pi r}{n}\right)$$

and for even n

$$\sum_{r=0}^{(n-2)/2} (-1)^r \binom{n-1-r}{r} 5^{[(n-2)/2]-r} = \prod_{r=1}^{(n-2)/2} \left(3-2\cos\frac{2\pi r}{n}\right).$$

Therefore, we obtain the identity for all $n \geq 1$

$$a_n = \prod_{r=1}^{(n-1)/2} \left(3-2\cos\frac{2\pi r}{n}\right).$$

So (i) is proved.

(ii) In the polynomial identity

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} x^r (1+x)^{n-1-2r} = \frac{x^n-1}{x-1}$$

specialize x to a root of $(x+1)^2 = 3x$.

Combining the two expressions, we have

$$a_n = \prod_{r=1}^{[(n-1)/2]} \left(3 - 2\cos\frac{2\pi r}{n}\right) = \begin{cases} \dfrac{(1+\sqrt{5})^n - (\sqrt{5}-1)^n}{2^n} & \text{for odd } n, \\ \dfrac{(1+\sqrt{5})^n - (\sqrt{5}-1)^n}{2^n\sqrt{5}} & \text{for even } n. \end{cases}$$

(iii) As a_n is positive (as it is clear from the right hand side of the Cauchy--Binet-type of identity above) and is an integer (from the definition!) and, since $\left((\sqrt{5}-1)/2\right)^n < 1$, it also follows that

$$a_n = \left[\left(\frac{\sqrt{5}+1}{2}\right)^n\right] \qquad \text{or} \qquad \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^n\right]$$

according as n is odd or even. Now, one may use the Cauchy-Binet-type identity to obtain the recursion which defines a_n's. That is

$$a_{2n+1} = 5a_{2n} - a_{2n-1};$$
$$a_{2n+2} = a_{2n+1} - a_{2n}.$$

(iv) The Cauchy-Binet-type identity or simply the expression

$$a_{2n} = \prod_{r=1}^{n-1} \left(3 - 2\cos\frac{\pi r}{n}\right)$$

makes it clear that $a_{2n} = F_{2n}$ for all n.
As $a_{2n+1} = a_{2n} + a_{2n+2} = F_{2n} + F_{2n+2}$, we have $a_{2n+1} = L_{2n+1}$.

(v) The proof of this divisibility result is the same as for corollary (i) given below. $\square$

Proof of the rest of the Theorem 1. The proofs of (b), (d) are immediate from Lemma 1(i) and (iii).

For (e), we look again at the polynomial identity

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} (xy)^r (x+y)^{n-1-2r} = x^{n-1} + x^{n-2}y + \cdots + y^{n-1}$$

which has for its right hand side the expression $(x^n - y^n)/x - y$ whereas $L_{2n} = \alpha^{2n} + \beta^{2n}$, where $\alpha = (1+\sqrt{5})/2$, $\beta = -1/\alpha$. If we simply take $x = \mathrm{e}^{\mathrm{i}\,\pi/2n}\,\alpha^2$,

$y = x^{-1}$, we have $x^n - y^n = \mathrm{i}(\alpha^{2n} + \beta^{2n}) = \mathrm{i}\, L_{2n}$. Thus, we have

$$L_{2n} = -\,\mathrm{i}(x - x^{-1}) \sum_{r \geq 0} (-1)^r \binom{n-1-r}{r} (x + x^{-1})^{n-1-2r}$$

where $x = \mathrm{e}^{\mathrm{i}\,\pi/2n}\, \alpha^2$. This proves (e).

(c) Now $L_{2n} = \alpha^{2n} + \beta^{2n} = \alpha^{2n} + \alpha^{-2n} = (\alpha^{4n} + 1)/\alpha^{2n} = R_n(\alpha^4)/\alpha^{2n}$ where the polynomial $R_n(x) = x^n + 1$ satisfies

$$R_n(x) = \frac{x^{2n} - 1}{x^n - 1} = \prod_{r=0}^{n-1} \left(x - \mathrm{e}^{2\,\mathrm{i}\,\pi(2r+1)/2n} \right).$$

Thus,

$$\begin{aligned} R_n(x^2) &= \prod_{r=0}^{n-1} \left(x - \mathrm{e}^{2\,\mathrm{i}\,\pi(2r+1)/2n} \right) \left(x - \mathrm{e}^{-2\,\mathrm{i}\,\pi(2r+1)/2n} \right) \\ &= \prod_{r=0}^{n-1} \left(x^2 - 2x \cos \frac{(2r+1)\pi}{2n} + 1 \right). \end{aligned}$$

Finally, if we take $x = \alpha^2$ and note that $\alpha^4 + 1 = 3\alpha^2$ for the golden ratio $\alpha = (1 + \sqrt{5})/2$, we obtain the product expression

$$L_{2n} = \prod_{r=0}^{n-1} \left(3 - 2 \cos \frac{(2r+1)\pi}{2n} \right).$$

This proves (c). □

Proof of Corollary 1. All the parts follow from the product expressions and the identification of the sequence $\{a_n\}$ with the sums of Fibonacci and Lucas numbers. Let us indicate the proof of (i) in detail.
In the expression

$$F_{mn} = \prod_{r=1}^{[(mn-1)/2]} \left(3 + 2 \cos \frac{2\pi r}{mn} \right),$$

there are terms corresponding to $r = n, 2n, \ldots, n[(m-1)/2]$ since $n[(m-1)/2] \leq [(mn-1)/2]$. Each of these terms is also a term for F_m and, in fact, comprises *all* the terms of F_m! Hence F_{mn}/F_m is a product of expressions of the form $3{+}2\cos(2\pi r/mn)$. Each of these is an algebraic integer and thus, the ratio F_{mn}/F_m is simultaneously an algebraic integer and a rational number. Hence the ratio is an integer. Thus (i) is proved.

Similarly (ii) follows when n is odd. Now, observe that L_{2n} divides $L_{2n(2m+1)}$, because in the product

$$L_{2n(2m+1)} = \prod_{r=0}^{n(2m+1)-1} \left(3 - 2 \cos \frac{(2r+1)\pi}{2n(2m+1)} \right)$$

the terms corresponding to $2r + 1 = 2n + 1, 3(2n + 1), \cdots, (2n - 1)(2m + 1)$ are exactly the terms in the product for L_{2n}. Therefore, we have (ii) also for even n.

The rest of the divisibility properties asserted follows from the above divisibility property for L_n's and a_n's by using the expressions

$$F_{n-k} + F_{n+k} = F_k L_n \text{ or } L_k F_n \text{ according as } k \text{ is odd or even.}$$

Note that these well-known expressions themselves follow from the corresponding Cauchy-Binet identities. The corollary is proved. □

Let us finish this theme by writing out a few more such applications of the polynomial identity followed by specializations.

Remark 1. In the polynomial identity, specializations $x = e^{2i\pi/3}$, $x = i$ yield, respectively,

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} = (-1)^{[(n-1)2]} \prod_{r=1}^{[(n-1)/2]} \left(1 + 2\cos\frac{2\pi r}{n}\right)$$
$$= 0, (-1)^{n-1} \textit{ or } (-1)^n \text{ according as } n = 0, 1 \text{ or } 2 \mod 3$$
$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} 2^{[(n-1)/2]-r} = (-2)^{[(n-1)/2]} \prod_{r=1}^{[(n-1)/2]} \cos\frac{2\pi r}{n}$$
$$= 0, (-1)^{(n-1)/4}, (-1)^{(n-2)/4}, \textit{ or } (-1)^{(n-3)/4} \text{ according as } n = 0, 1, 2 \text{ or } 3 \mod 4.$$

Finally, the most general identity obtainable by this method is the following.

Remark 2. For an arbitrary complex number $\mu \neq -2$, we have

$$\left(\frac{\mu+2}{2}\right)^{n-1} \sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} \left(\frac{2\mu}{\mu^2+4}\right)^r$$
$$= \frac{2^n - \mu^n}{2^{n-1}(2-\mu)} = \prod_{r=1}^{[(n-1)/2]} \left(\frac{\mu^2+4}{4} - \mu\cos\frac{2\pi r}{n}\right).$$

When $\mu = -2$, the corresponding identity is

$$\prod_{r=1}^{[(n-1)/2]} 4\cos^2\frac{\pi r}{n} = \frac{n}{2} \quad \text{or} \quad 1$$

according as n is even or odd. The latter identity was referred to by some people (see [**4**]) as 'grandma's identity'.

3. Fibonacci polynomials

Consider the polynomials $F_n(x)$ defined recursively by

$$F_0(x) = 0, \qquad F_1(x) = x, \qquad F_{n+1}(x) = xF_n(x) + F_{n-1}(x).$$

Observe that $F_n(1) = F_n$, the Fibonacci numbers. We remark in passing that the Chebychev polynomials are related to these polynomials. Recalling the standard method of expressing a member of a linear recursion in terms of the characteristic equation (as mentioned in the introduction) one has the following. The recursion is expressed formally by the generating function $\sum_{n\geq 1} F_n(x)y^n = \frac{y}{1-xy-y^2}$. The characteristic polynomial (in y) $1 - xy - y^2$ (for each fixed x) has the 'roots' $(\alpha, \beta) = \frac{-x\pm\sqrt{x^2+4}}{2}$. Note that $\alpha\beta = -1$. Therefore,

$$F_n(x) = \frac{\dfrac{1}{\alpha^{n+1}} - \dfrac{1}{\beta^{n+1}}}{\alpha - \beta} = (-1)^n \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

Now it is easy to find the roots of $F_n(x)$ (they correspond to β/α being a nontrivial $(n+1)$-th root of unity); we get

$$F_n(x) = \prod_{r=1}^{n-1} \left(x - 2\,\mathrm{i}\cos\frac{r\pi}{n} \right).$$

We get

$$\begin{aligned} F_n = F_n(1) &= \prod_{r=1}^{n-1} \left(1 - 2\,\mathrm{i}\cos\frac{r\pi}{n} \right) \\ &= \prod_{r=1}^{[(n-1)/2]} \left(1 + 4\cos^2\frac{r\pi}{n} \right) = \prod_{r=1}^{[(n-1)/2]} \left(3 + 2\cos\frac{2r\pi}{n} \right). \end{aligned}$$

4. Periodicity modulo primes

We recall one open question about the Fibonacci numbers

If p is a fixed prime number, what is the period of the sequence F_n mod p?

Here is a partial answer

Theorem 2.

(a) For any prime $p \neq 5$, we have $F_p \equiv (5/p)$ and $F_{p-(p/5)} \equiv 0 \mod p$.

(b) For every prime p, the sequence $\{F_n\}$ is periodic $\mod p$. The period divides $p-1$ if $(5/p) = 1$; it is a divisor of $2p+2$ but not of $p+1$ when $(5/p) = -1$. In case of the prime 5, the period is 20.

In the above statements, we have used the Legendre symbol (a/p) for a prime p. For instance, a prime p satisfies $(p/5) = 1$ if $p \equiv \pm 1$ mod 5 and satisfies $(p/5) = -1$ if $p \equiv \pm 2$ mod 5.

Proof. (a) We may assume $p \neq 2$ as obviously $F_2 = 1 = (5/2)$ mod 2 and $F_3 = 2$.

We shall use the expression

$$F_n = \frac{\displaystyle\sum_{r=0}^{[(n-1)/2]} \binom{n}{2r+1} 5^r}{2^{n-1}}$$

which is just the binomial expansion of the Cauchy-Binet identity. Then, we have

$$2^{p-1}F_p = \sum_{r=0}^{[(p-1)/2]} \binom{p}{2r+1} 5^r \equiv 5^{(p-1)/2} \mod p$$

since $\binom{p}{s} \equiv 0 \mod p$ for $1 \leq s < p$.

The first statement of (a) follows as $2^{p-1} \equiv 1$ and $5^{(p-1)/2} \equiv (5/p) \mod p$.

Let us now prove the second one.

First, let $(p/5) = -1$, i.e., $p \equiv \pm 2 \mod 5$. Then, $(5/p) = -1$, i.e., $5^{(p-1)/2} \equiv -1 \mod p$. Now,

$$2^p F_{p+1} = \sum_{r=0}^{(p-1)/2} \binom{p+1}{2r+1} 5^r \equiv 1 + 5^{(p-1)/2} \equiv 0 \mod p$$

since $\binom{p+1}{s} \equiv 0 \mod p$ for $0 < s < p$. Thus, p divides $2^p F_{p+1}$ and so, it divides F_{p+1}.

Now, take $(p/5) = 1$, i.e., $p \equiv \pm 1 \mod 5$. Then,

$$2^{p-2}F_{p-1} = \sum_{r=0}^{(p-3)/2} \binom{p-1}{2r+1} 5^r \equiv \sum_{r=0}^{(p-3)/2} -5^r$$

since $\binom{p-1}{2r+1} \equiv -1 \mod p$ for $0 \leq r \leq (p-3)/2$.

Therefore, since $(5/p) = 1$, i.e., $5^{(p-1)/2} \equiv 1 \mod p$, we have

$$4 \cdot 2^{p-2}F_{p-1} \equiv 4 \cdot \sum_{r=0}^{(p-3)/2} -5^r = 1 - 5^{(p-1)/2} \equiv 0.$$

This proves (a).

(b) Once again, we assume that $p \neq 2, 5$ as these two cases are verified individually easily. Recall that $(5/p) = (p/5)$ from the quadratic reciprocity law. Thus, we have mod p,

$$\begin{array}{lll} F_{p-1} \equiv 0, & F_p \equiv 1 & \text{if } (p/5) = 1, \\ F_{p+1} \equiv 0, & F_p \equiv -1 & \text{if } (p/5) = -1. \end{array}$$

The first two equations mean that if $p \equiv \pm 1 \mod 5$, then $F_p \equiv 1$ and $F_{p+1} = F_{p-1} + F_p \equiv 1$, i.e.,

$$F_{p-1+n} \equiv F_n \mod p \qquad \text{for all } n \geq 1.$$

The second pair of equations means that if $p \equiv \pm 2 \mod 5$, then $F_{p+2} = F_p + F_{p+1} \equiv -1$ and $F_{p+3} = F_{p+2} + F_{p+1} \equiv -1 \mod p$.

Thus, $F_{p+1+n} \equiv -F_n$ for all $n \geq 1$. This gives periodicity to a divisor of $2p+2$ but not of $p+1$ when $p \equiv \pm 2 \mod 5$. Our contention is proved. $\square$

Finally, let us end with a simple consequence which was implicit in the above discussion.

Let $p > 5$ be a prime and let q be a prime dividing F_p. Then, $q \equiv \pm 1 \mod p$. Moreover,

$$\begin{array}{llll} q \equiv 1 & \mod p & \text{implies} & q \equiv \pm 1 \mod 5; \\ q \equiv -1 & \mod p & \text{implies} & q \equiv \pm 2 \mod 5. \end{array}$$

Acknowledgment. In 1993, I noticed the connection between Fibonacci numbers and trigonometric functions and did write a little note on this much later in [**4**]. However, I did not know that this connection had been noticed as early as in 1969 (!) in some form (see [**7**]) and more precisely later (see [**2**], [**3**]). I am indebted to the referee for not only pointing this out but also for her/his constructive comments which helped me rewrite this note. Most of all, our acknowledgement is devoted to these beautiful numbers themselves and also towards numerous individuals who brought forth their beautiful properties into focus.

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