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CHARACTERIZATION OF SIMPLE ORBIT GRAPHS

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Abstract. Let G be a (finite) group and let S be a non-empty subset of G. The vertex set of the *orbit graph* $O(G, S)$ is the collection of orbits of left translations induced by s, over all $s \in S$. If u and v are distinct vertices (each representing an orbit of some s and t from S), then for any $g \in G$ appearing in both orbits, there is an edge colored g in $O(G, S)$ joining u and v. Orbit graphs are an important special case of "G-graphs" introduced by Bretto and Faisant in Math. Slovaca **55** (2005). In this paper we characterize *simple* orbit graphs and apply the result to show that a certain class of *simple* orbit graphs is closed under the construction of incidence graphs.

1. Introduction

Let G be a (finite) group and let S be a non-empty subset of G. For each $s \in S$, let λ_s be the left translation $g \mapsto sg$ for all $g \in G$. The *orbit graph* $O(G, S)$ associated with the pair (G, S) is defined as follows [**13**]. For each $s \in S$ and each orbit ω of λ_s, there is a vertex corresponding to the pair (s, ω). For any unordered pair of distinct vertices (s, ω), (s', ω') and for any element $h \in G$ such that $h \in \omega \cap \omega'$, there is an edge with *colour* h that joins (s, ω) and (s', ω').

For any given $s \in S$, the set of all orbits of λ_s forms a partition of G. It follows that the graph $O(G, S)$ is $|S|$-partite and the valency of a vertex representing an orbit of λ_s is equal to $|s|(|S| - 1)$ where $|s|$ is the order of s. It is now easy to see that $O(G, S)$ is a simple graph (that is, a graph with no multiple edges) if and only if $\langle s\rangle \cap \langle t\rangle = \{1\}$ for any two distinct elements s, t in S. Note that in this case for any two distinct elements $s, t \in S$, the set of all orbits of λ_s is disjoint from the set of all orbits of λ_t. We will therefore identify the vertex set of the *simple* orbit graph $O(G, S)$ with the union of orbits of left translations λ_s induced by s, over all $s \in S$. We remark that the orbits of left translation λ_s for $s \in S$ are the right cosets of the cyclic group $\langle s\rangle$ and thus, when dealing with simple orbit graphs, one can freely use the term "orbit" in place of "coset" and vice versa. We prefer using orbits in what follows.

Simple orbit graphs can be regarded as a subclass of *G-graphs* introduced as a potential tool for group isomorphism testing in [**2**]. They also appear useful

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for constructions of various interesting classes of graphs such as symmetric and semisymmetric graphs [**3, 6**], small graphs of given degree and girth [**4**], Hamming graphs and meshes of d-ary trees [**5**]. Further, simple orbit graphs are closely related to Cayley graphs [**13**] and to vertex-transitive non-Cayley graphs [**14**]. Also, simple orbit graphs can be understood as a generalization of double-coset graphs introduced in [**9**] and studied in connection with constructions of semisymmetric graphs [**12, 10**].

The objective of this note is to characterize *simple* orbit graphs in terms of their automorphism groups. With the help of our characterization we will prove that the incidence graph of a simple d-regular bipartite orbit graph ($d \geq 2$) is a simple orbit graph provided that $\mathrm{Aut}\, G$ contains an involutory automorphism swappping the two elements contained in S.

2. Main result

We aim at characterizing *simple* orbit graphs $O(G, S)$. We begin by collecting some simple, but relevant facts about simple orbit graphs. Since any orbit graph $O(G, S)$ such that $|S| = 1$ consists of m copies of a one-vertex graph where $m = [G : \langle S \rangle]$, we may assume that $|S| \geq 2$. Further, for any given element $g \in G$, the graph $O(G, S)$ contains a clique on $|S|$ vertices induced by the set of all the edges coloured g. Finally, the complete graph K_2 cannot be a component of any simple $O(G, S)$ graph. To see this it is sufficient to realize that if K_2 were a component of $O(G, S)$ then the above gives $|S| = 2$, and the valency of any vertex $v \in O(G, S)$ would be equal to $|s| = 1$ for all $s \in S$, a contradiction.

Thus, we may limit our attention to the class of k-partite graphs where $k \geq 2$ such that any graph belonging to the class contains a clique on k vertices and has no component isomorphic to K_2.

Theorem 1. *Let Γ be a k-partite graph, where $k \geq 2$, and let Γ have no component isomorphic to K_2. Then Γ is a simple orbit graph if and only if*

i) $\mathrm{Aut}\, \Gamma$ *contains a subgroup H whose orbits are precisely the partition classes of Γ;*
ii) Γ *contains a clique K on k vertices such that for each vertex $v \in K$ and any given $u \neq v$, $u \in K$, the stabilizer H_v of the vertex v is a cyclic group which acts regularly on the set of all the vertices adjacent to v contained in the same partition class the vertex u belongs to.*

Proof. We first consider the automorphism group $\mathrm{Aut}\, O(G, S)$ of a simple $O(G, S)$ graph with $|S| \geq 2$. Observe that for any given $h \in G$ and each $s \in S$, the action of G by right multiplication on itself induces an action of G on the set of all orbits of λ_s given by $\{g, sg, \ldots, s^{|s|-1}g\} \mapsto \{gh, sgh, \ldots, s^{|s|-1}gh\}$ for $g \in G$, which is transitive, but not regular in general, as was already noted in [**13**]; we denote the action r_h. As the adjacency relation in the graph $O(G, S)$ is induced by the set intersection, $R(G) = \{r_h : h \in G\}$ is the desired subgroup of $\mathrm{Aut}\, O(G, S)$, proving part i).

Combining $|S| \geq 2$ with $\langle s \rangle \cap \langle t \rangle = \{1\}$ for any two distinct elements s, t in S, one can see that the mapping $h \mapsto r_h$ for $h \in G$ is an isomorphism from the group G onto $R(G)$. From this we derive that for each $s \in S$, the stabilizer $R(G)_v$ of the vertex v representing the orbit $\langle s \rangle 1$ is the cyclic group $\langle r_s \rangle \cong \langle s \rangle$. Obviously, the subgraph induced by the set of all orbits containing the unit element of G is a clique on $|S|$ vertices, completing the proof of part ii).

For the reverse direction, let Γ be a k-partite graph where $k \geq 2$, and let Γ have no component isomorphic to K_2. Assume that there is a subgroup $H \subseteq \operatorname{Aut}\Gamma$ and a clique C on k vertices to which i) and ii) apply. We will show that Γ is a simple orbit graph $O(H, S)$ for a suitable $S \subset H$.

Let u, v be two distinct vertices contained in the clique C. Then from ii) applied to the stabilizers of u and v together with our assumption $K_2 \nsubseteq \Gamma$ we have $H_u \cap H_v = \{1\}$ and $H_u \neq H_v$. Letting $S = \{f_v : v \in C\}$ where $\langle f_v \rangle = H_v$, we conclude that $O(H, S)$ is a simple orbit graph with $|S| = k$. We point out that for any two distinct vertices $u, v \in C$, the partition classes corresponding to λ_{f_u} and λ_{f_v} are disjoint.

We will now verify that $\Gamma \cong O(H, S)$. To do so, for any given vertex $v \in C$, we let C_v denote the partition class of Γ which contains v, and by F_v we mean the partition class of $O(H, S)$ corresponding to λ_{f_v}. Thus, $\cup\{C_v : v \in C\}$ and $\cup\{F_v : v \in C\}$ are partitions of the graphs Γ and $O(H, S)$ into independent sets of vertices, respectively.

We define a 1-1 correspondence between the vertex set of the graph $O(H, S)$ and Γ. Take an arbitrary, but fixed vertex $v \in C$. By i) the group H is transitive on the set C_v and so H_u and H_v are conjugate for all $u \in C_v$. This together with the fact that H fixes the class C_v set-wise ensures that $|C_v| = [H : H_v]$, that is, the sets C_v and F_v have the same cardinalities. Obviously, for any two elements g, h appearing in the same orbit of λ_{f_v}, we have $(v)g = (v)h$. It follows that the mapping $\alpha_v : F_v \to C_v$ given by $\langle f_v \rangle h \mapsto (v)h$ for $h \in H$ is bijective. Since any two distinct partition classes of the graph $O(H, S)$ are disjoint, the natural extension $\alpha : \{F_v : v \in C\} \to \{C_v : v \in C\}$, $\alpha|_{F_v} = \alpha_v$ for $v \in C$ is bijective, too.

We wish to show that α preserves adjacencies. Let $u, v \in O(H, S)$ be two adjacent vertices. Then the two orbits u' and v' they represent, say $u' \in F_w$ and $v' \in F_z$, have a (unique) element h in common. Thereby α maps u to $(w)h$ and v to $(z)h$. Note that the two images are adjacent as both w and z belong to the clique C.

Conversely, let $u, v \in \Gamma$ be two adjacent vertices. Then there are two (uniquely determined) vertices $w, z \in C$ such that $u \in C_w$ and $v \in C_z$. Since C_w and C_z are orbits of H, there must be an element $h \in H$ such that $(w)h = u$ and $(z)h \in C_z$. Moreover, H_u and H_w are conjugate and so there is an element $g \in H_u$ such that $(z)hg = v$. Let x, y be two (uniquely determined) orbits of λ_{f_w} and λ_{f_z} respectively, which share the element hg. Then the vertices $x, y \in O(H, S)$ are joined by an edge coloured hg, and their images under the mapping α are $(w)hg = u$ and $(z)hg = v$, respectively. We conclude that the graph Γ is isomorphic to $O(H, S)$. $\square$

For completeness we remark that a certain version of Theorem 1 was announced in [**7**].

Corollary 1. *All components of a simple orbit graph $O(G,S)$ are isomorphic to the simple orbit graph $O(\langle S\rangle, S)$.*

Proof. In [**2**] it was proved that a simple orbit graph $O(G,S)$ is connected if and only if S is a generating set for G. It follows that the restriction of the domain of λ_s onto $\langle S\rangle$ for all $s \in S$, gives rise to the connected graph $O(\langle S\rangle, S)$ which is clearly a component of $O(G,S)$. Since the clique induced by the set of all orbits containing the unit element of G is a subgraph of $O(\langle S\rangle, S)$, the assertion follows from Theorem 1. □

3. Application

The *incidence graph* $I\Gamma$ of a (simple) graph $\Gamma = (V,E)$ is the graph with the vertex set $V \cup E$ in which two vertices $v \in V$ and $e \in E$ are joined by an edge if and only if $v \in e$.

In this section we consider the question whether the incidence graph of a simple orbit graph $O(G,S)$ is also a simple orbit graph. We need the elementary fact that the automorphism group of the incidence graph of a simple graph Γ contains a subgroup isomorphic to $\operatorname{Aut}\Gamma$. To see this it is sufficient to realize that $I\Gamma$ is obtained from Γ by subdividing of each edge and thus, as the action of $\operatorname{Aut}\Gamma$ on the vertex set of Γ induces an action on the set of all its edges, the mapping $\alpha : \operatorname{Aut}\Gamma \to \operatorname{Aut} I\Gamma$ given by $h \mapsto h'$, where $(v)h' = (v)h$ and $(\{u,w\})h' = \{(u)h, (w)h\}$ for $v \in V$ and $\{u,w\} \in E$, is an insertion of the group $\operatorname{Aut}\Gamma$ into $\operatorname{Aut} I\Gamma$. For a subgroup H of the automorphism group of a simple graph Γ we will write H' to denote the image of H under the mapping α.

This observation together with Theorem 1 impose strong restrictions on the automorphism group of a simple orbit graph $O(G,S)$ whose incidence graph is a simple orbit graph. Namely, if $IO(G,S) \cong O(H,T)$, then as the graph $O(H,T)$ is bipartite, by Theorem 1 the orbits of the group $R(H)$ are the bipartition sets of $O(H,T)$ and thus, by the above fact, $O(G,S)$ is a vertex and edge-transitive graph. Moreover, the stabilizer argument gives that $O(G,S)$ is in fact an arc-transitive graph.

The simplest way to produce an $O(G,S)$ graph which meets the properties we have just described (that is a simple orbit graph whose incidence graph is at least potentially a simple orbit graph) is to make sure that S is an orbit of a subgroup of $\operatorname{Aut}(G)$; this guarantees that $O(G,S)$ is a vertex-transitive graph [**13**]. Consequently, to end up with an arc-transitive graph it is sufficient to choose $S = \{s,t\}$ and require that there is an involutory element in $\operatorname{Aut} G$ which swaps the two elements contained in S

Proposition 1. *Let $O(G,S)$ be $d \geq 2$ regular bipartite simple orbit graph. Assume that $\operatorname{Aut} G$ contains an involutory element f which swaps the two elements contained in S. Then $IO(G,S) \cong O(G \rtimes_\phi Z_2, \{(s,0),(1,1)\})$, where $s \in S$ and $(1)\phi = f$.*

Proof. We begin by showing that $IO(G,S)$ is a simple orbit graph. The mapping given by $\langle a\rangle g \mapsto \langle (a)f\rangle (g)f$ for $g \in G$ and $a = s,t$ is an automorphism of $O(G,S)$ [**2**]; we denote it by τ. Obviously, the order of τ is equal to 2 as $f^2 = 1$. Consider the group $H \subseteq \operatorname{Aut} O(G,S)$ generated by $R(G)$ and $\langle\tau\rangle$. By the observation at the beginning of this section, the group H' is a subgroup of $\operatorname{Aut} IO(G,S)$ isomorphic to H. We claim that H' fulfils both conditions i) and ii) stated in Theorem 1. Realizing that $R(G)$ is transitive on each bipartition set and that τ swaps the bipartition sets we obtain that the group H is transitive. The graph $O(G,S)$ is bipartite and in this case Theorem 1 says that $R(G)$ acts regularly on the set of all its edges. Thus H' is the desired group, proving part i). To show ii), let $u,v \in IO(G,S)$ be two vertices such that $u = \langle s\rangle 1$ and $v = \langle t\rangle 1$. Then the subgraph induced by the set $\{u,\{u,v\}\}$ is a clique and obviously, $H'_u = \langle r'_s\rangle$ and $H'_{\{u,v\}} = \langle\tau'\rangle$. We conclude that $IO(G,S)$ is a simple orbit graph $O(H',\{r'_s,\tau'\})$.

We now describe the structure of H'. It is clear that $R(G)\cap\langle\tau\rangle = \{1\}$, and it is easy to check that $\tau r_g \tau = r_{(g)f}$ for $g \in G$. This is equivalent to saying that $H = R(G) \rtimes_\psi \langle\tau\rangle$, where the mapping $\psi : \langle\tau\rangle \longrightarrow \operatorname{Aut} R(G)$ is defined by $\tau \mapsto [r_g \mapsto r_{(g)f}$ for $g \in G]$. In the proof of Theorem 1 we have seen that the mapping $h \mapsto r_h$ for $h \in G$ is an isomorphism from the group G onto $R(G)$. This together with $f \in \operatorname{Aut} G$ imply that $R(G) \rtimes_\psi \langle\tau\rangle \cong G \rtimes_{id} \langle f\rangle$ and this group is obviously isomorphic to $G \rtimes_\phi Z_2$ where $(1)\phi = f$. Briefly, $H = R(G) \rtimes_\psi \langle\tau\rangle \cong G \rtimes_\phi Z_2$. Since the groups H and H' are abstractly isomorphic, we have $H' \cong G \rtimes_\phi Z_2$.

Taking into account that any two simple orbit graphs $O(G,S)$ and $O(G',S')$ are isomorphic whenever there is an isomorphism h from the group G onto G' such that $(S)h = S'$ [**2**], we may conclude that $IO(G,\{s,t\}) \cong O(G\rtimes_\phi Z_2, \{(s,0),(1,1)\}$. □

We use Proposition 1 to give an alternative proof of the following Theorem proved in [**5**]. We introduce a new concept.

For $d \geq 2, n \geq 1$, let $T(d,n)$ be a complete d-ary tree of depth n. Consider a $d^n \times d^n$ matrix of vertices. On each row (resp. column) of the matrix, put $T(d,n)$ such that the vertices of the row (column) are the leaves of the tree. The resulting graph is the mesh $MT(d,n)$ of $T(d,n)$. We remark that the graph $MT(d,n)$ is a generalization of the well-known mesh of trees [**11**], i.e. $MT(2,n)$, and was proposed as a possible interconnection network for parallel computers [**1, 8**] for it combines together the mesh and tree structure.

Let $\{s,t\}$ be the standard basis for the group $Z_d \times Z_d$. Let 0 be the unity of $Z_d \times Z_d$, and f an element in $\operatorname{Aut}(Z_d \times Z_d)$ which swaps s and t. We define the mapping $\phi : Z_2 \to \operatorname{Aut}(Z_d \times Z_d)$ by $(1)\phi = f$.

Theorem A. (Bretto et al., [**5**]) *The mesh of a d-ary tree $MT(d,1)$, where $d \geq 2$, is isomorphic to the simple orbit graph $O((Z_d \times Z_d) \rtimes_\phi Z_2, \{(s,0),(0,1)\})$.*

Proof. First we show that $MT(d,1)$ is, in fact, the incidence graph of a complete bipartite graph $K_{d,d}$. To observe the correspondence, for any given $i,j \in \{1,\ldots,d\}$ let u_i and v_j denote the root of $T(d,1)$ whose leaves are entries of the i-th row and the j-th column of the matrix, respectively. By (u_i,v_j) we denote the (i,j)-th entry of the matrix. Identifying the set $U \cup V$, where $U = \{u_i : i = 1,\ldots,d\}$ and

$V = \{v_i : i = 1, \ldots, d\}$, with a partition of $K_{d,d}$ we can see that the mapping given by $\{u_i, v_j\} \mapsto (u_i, v_j)$ for $i, j \in \{1, \ldots, d\}$ is a bijection from the set of all the edges of the graph $K_{d,d}$ onto the set of all the vertices of $MT(d, 1)$ corresponding to the set of all the entries of the matrix. It is now easy to check that $MT(d, 1) \cong IK_{d,d}$. Finally, for $d \geq 2$ we have $K_{d,d} \cong O((Z_d \times Z_d), \{s, t\})$, as noted in [**6**]. The rest follows from Proposition 1. $\square$

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