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SPACELIKE HYPERSURFACES WITH CONSTANT r-TH MEAN CURVATURE IN ANTI-DE SITTER SPACES

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Abstract. In this paper we investigate spacelike hypersurfaces with constant r-th mean curvature and two distinct principal curvatures in anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$. We give some characterizations of the hyperbolic cylinders in $\mathbb{H}_1^{n+1}(c)$.

1. Introduction

Let $\overline{M}_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorentzian space form with constant sectional curvature c and index 1. When $c > 0$, $\overline{M}_1^{n+1}(c) = \mathbb{S}_1^{n+1}(c)$ is called an $(n+1)$-dimensional de Sitter space; when $c = 0$, $\overline{M}_1^{n+1}(c) = \mathbb{L}^{n+1}$ is called an $(n+1)$-dimensional Minkowski space; when $c < 0$, $\overline{M}_1^{n+1}(c) = \mathbb{H}_1^{n+1}(c)$ is called an $(n+1)$-dimensional anti-de Sitter space. A hypersurface M^n is said to be spacelike if the induced metric on M^n from that of the ambient space is Riemannian metric. Spacelike hypersurfaces with constant mean curvature in Lorentzian space forms are very interesting geometrical objects which have been investigated by many geometers.

S. Montiel [**4, 5**] gave a characterization of hyperbolic cylinders or totally umbilical hypersurfaces in the de Sitter space. T. Ishihara [**3**] studied an n-dimensional $(n \geq 2)$ complete maximal spacelike hypersurface M in the anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$ and proved the norm square of the second fundamental form of M satisfies $S \leq n$. Moreover, $S = n$ if and only if $M^n = \mathbb{H}^m(-\frac{n}{m}) \times \mathbb{H}^{n-m}(-\frac{n}{n-m})$, $(1 \leq m \leq n-1)$.

Cao and Wei [**1**] studied maximal spacelike hypersurfaces with two distinct principal curvatures in the anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$ and gave a characterization of the hyperbolic cylinders in the anti-de Sitter space:

Theorem 1.1. *Let M^n be an n-dimensional $(n \geq 3)$ complete maximal spacelike hypersurface with two distinct principal curvatures λ and μ in an anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$. If $\inf(\lambda-\mu)^2 > 0$, then $M^n = \mathbb{H}^m(-\frac{n}{m}) \times \mathbb{H}^{n-m}(-\frac{n}{n-m})$, $(1 \leq m \leq n-1)$.*

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In [**8**], we extended the above result to complete spacelike hypersurfaces with constant mean curvature and two distinct principal curvatures in an anti-de Sitter space. In fact, we proved the following result.

Theorem 1.2. *Let M^n be an n-dimensional ($n \geq 3$) complete spacelike hypersurface with constant mean curvature immersed in an anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$. Suppose in addition that M^n has two distinct principal curvatures λ and μ with the multiplicities $(n-1)$ and 1, respectively. Satisfying* $\inf(\lambda-\mu)^2 > 0$, *then M^n is a hyperbolic cylinder $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$*

In this paper we will investigate spacelike hypersurfaces with constant r-th mean curvature and with two distinct principal curvatures in the anti-de Sitter spaces and obtain the following result

Theorem 1.3. *Let M^n be a complete spacelike hypersurface of $\mathbb{H}_1^{n+1}(c)$ for $n \geq 3$. Assume that M^n has constant r-th mean curvature and two distinct principal curvatures such that for one of them, the associated space of principal curvature vectors has dimension 1. Then:*

(i) *$H_r = 0$, and therefore, M^n is an $(r-1)$-maximal hypersurface, or*

(ii) *M^n is the locus of a family of moving $(n-1)$-dimensional submanifolds $M_1^{n-1}(s)$. The principal curvature λ of multiplicity $n-1$ is constant along each of the submanifolds $M_1^{n-1}(s)$. The manifolds $M_1^{n-1}(s)$ have constant curvature $((\log|\lambda^r - H_r|^{1/n})')^2 + c - \lambda^2$, which does not change sign. Here the parameter s is the arc length of an orthogonal trajectory of the family $M_1^{n-1}(s)$, and $\lambda = \lambda(s)$ satisfies the ordinary second order differential equation*

$$w'' - w\{H_r(H_r + w^{-n})^{\frac{2-r}{r}} - \frac{n-r}{r}(H_r + w^{-n})^{\frac{2-r}{r}} w^{-n} - c\} = 0, \tag{1.1}$$

or

$$w'' - w\{H_r(H_r - w^{-n})^{\frac{2-r}{r}} + \frac{n-r}{r}(H_r - w^{-n})^{\frac{2-r}{r}} w^{-n} - c\} = 0, \tag{1.2}$$

where $w = |\lambda^r - H_r|^{-\frac{1}{n}}$.

In particular, we also study spacelike hypersurfaces with vanishing r-th mean curvature and with two distinct principal curvatures in an anti-de Sitter space and give some characterizations of hyperbolic cylinders.

2. Preliminaries

Let M^n be a complete spacelike hypersurface in an anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$ of constant sectional curvature $c < 0$. For any $p \in M$, we can choose a local orthonormal frame field $e_1, \dots, e_n, e_{n+1}$ in a neighborhood U of M^n such that $e_1, \dots, e_n$ are tangential to M^n and e_{n+1} is normal to M^n. We use the following convention for the indices

$$1 \leq A, B, C, D \leq n+1, \qquad 1 \leq i, j, k, l \leq n.$$

Denote $\{\omega_A\}$ the corresponding dual coframe and $\{\omega_{AB}\}$ the connection forms of $\mathbb{H}_1^{n+1}(c)$ so that the semi-Riemannian metric and structure equation of $\mathbb{H}_1^{n+1}(c)$ are given, respectively, by

$$d\overline{s}^2 = \sum_i \omega_i^2 - \omega_{n+1}^2, \tag{2.1}$$

$$d\omega_A = \sum_i \omega_{Ai} \wedge \omega_i - \omega_{An+1} \wedge \omega_{n+1},\ \omega_{AB} + \omega_{BA} = 0, \tag{2.2}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} + \overline{\Omega}_{AB}, \overline{\Omega}_{AB} = -\frac{1}{2}\sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D, \tag{2.3}$$

$$K_{ABCD} = c\varepsilon_A \varepsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.4}$$

A well-known argument shows that the forms ω_{in+1} may be expressed as

$$\omega_{in+1} = \sum_j h_{ij}\omega_j,\ h_{ij} = h_{ji}. \tag{2.5}$$

The second fundamental form is given by $A = \sum_j h_{ij}\omega_i \otimes \omega_j$. Furthermore, the mean curvature is given by $H = \frac{1}{n}\sum_i h_{ii}$.

The structure equations of M^n are given by

$$d\omega_i = \sum_i \omega_{ij} \wedge \omega_j,\ \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij},\ \Omega_{ij} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l. \tag{2.7}$$

Using the structure equation, we can obtain the Gauss equation

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.8}$$

The Ricci curvature and the normalized scalar curvature of M are given, respectively, by

$$R_{ij} = c(n-1)\delta_{ij} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{2.9}$$

$$R = \frac{1}{n(n-1)}\sum_i R_{ii}. \tag{2.10}$$

From (2.9) and (2.10) we obtain

$$n(n-1)(R-c) = -n^2H^2 + S, \tag{2.11}$$

where $S = |A|^2 = \sum_{i,j} h_{ij}^2$ is the square of the second fundamental form A.

The Codazzi equation is

$$h_{ijk} = h_{ikj}, \tag{2.12}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{2.13}$$

Associated to the second fundamental form A of M^n one has n invariants S_r, $1 \leq r \leq n$, given by the equality

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k}.$$

If $p \in M$ and e_k is the basis of T_pM formed by eigenvectors of the shape operator A_p, with corresponding eigenvalues λ_k, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[x_1, \ldots, x_n]$ is the r-th elementary symmetric polynomial on the indeterminates $x_1, \ldots, x_n$. The r-th mean curvature of M is given by

$$H_r = \frac{1}{\binom{n}{r}} S_r.$$

In particular, when $r = 1$

$$H_1 = \frac{1}{n} \sum_i \lambda_i = \frac{1}{n} S_1 = H$$

is nothing but the mean curvature of M.

A spacelike hypersurface M^n in Lorentzian space forms $\overline{M}_1^{n+1}(c)$ is called r-maximal if $H_{r+1} \equiv 0$.

3. Spacelike hypersurfaces of constant r-th curvature with two distinct principal curvatures

In this section, we will study hypersurfaces with constant r-th mean curvature and with two distinct principal curvatures in the anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$. Suppose that the multiplicities of the principal curvatures λ and μ are m and $n - m$, respectively. In the following if there is no special statement, we further adopt the notational convention that indices a, b, c range from 1 to m and indices α, β, γ from $m + 1$ to n. We may choose $\{e_A\}_{1 \leq A \leq n+1}$ such that $h_{ab} = \lambda\delta_{ab}$, $h_{\alpha\beta} = \mu\delta_{\alpha\beta}$, $h_{a\alpha} = 0$.

Firstly, we state a theorem that can be proved using the method of Otsuki [**7**].

Theorem 3.1. *Let M^n be a spacelike hypersurface with two distinct principal curvatures in $\mathbb{H}_1^{n+1}(c)$ such that the multiplicities of principal curvatures are all constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1*, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

Proof. We suppose that M^n is a spacelike hypersurface with two distinct principal curvatures in $\mathbb{H}_1^{n+1}(c)$ such that the multiplicities of the principal curvatures λ and μ are m and $n - m$, respectively. We may choose local orthonormal frame field $e_1, \ldots, e_n, e_{n+1}$ in a neighborhood U of M^n such that

$$\omega_{in+1} = \lambda_i \omega_i, \tag{3.1}$$

where λ_i are principal curvatures satisfying $\lambda_a = \lambda$, $\lambda_\alpha = \mu$. Hence we have

$$d\omega_{in+1} = d\lambda_i \wedge \omega_i + \lambda_i d\omega_i = d\lambda_i \wedge \omega_i + \lambda_i \sum_j \omega_{ij} \wedge \omega_j. \tag{3.2}$$

On the other hand, by means of (2.3), (2.4) and (3.1), we get

$$d\omega_{in+1} = \sum_j \lambda_j \omega_{ij} \wedge \omega_j \text{ on } M^n. \tag{3.3}$$

Hence, we get

$$d\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j)\omega_{ij} \wedge \omega_j = 0. \tag{3.4}$$

In particular,

$$d\lambda \wedge \omega_a + (\lambda - \mu) \sum_\alpha \omega_{a\alpha} \wedge \omega_\alpha = 0. \tag{3.5}$$

Putting $d\lambda = \sum_i \lambda_{,i}\omega_i = \sum_a \lambda_{,a}\omega_a + \sum_\alpha \lambda_{,\alpha}\omega_\alpha$, where $\lambda_{,i} = e_i(\lambda)$, (3.5) can be written as

$$\sum_b \lambda_{,b}\omega_b \wedge \omega_a + \sum_\alpha \lambda_{,\alpha}\omega_\alpha \wedge \omega_a + (\lambda - \mu)\sum_\alpha \omega_{a\alpha} \wedge \omega_\alpha = 0. \tag{3.6}$$

It implies that $\lambda_{,b} = 0$ for $b \neq a$ and

$$\sum_\alpha \omega_{a\alpha} \wedge \omega_\alpha = \frac{1}{\lambda - \mu} \sum_\alpha \lambda_{,\alpha}\omega_a \wedge \omega_\alpha. \tag{3.7}$$

Therefore,

$$\begin{aligned} d\omega_a &= \sum_b \omega_{ab} \wedge \omega_b + \sum_\alpha \omega_{a\alpha} \wedge \omega_\alpha \\ &= \sum_b \omega_{ab} \wedge \omega_b + \frac{1}{\lambda - \mu} \sum_\alpha \lambda_{,\alpha}\omega_a \wedge \omega_\alpha. \end{aligned} \tag{3.8}$$

This means that

$$d\omega_a \equiv 0 \quad \mathrm{mod}\ (\omega_1, \ldots, \omega_m), \tag{3.9}$$

for any $1 \leq a \leq m$. Therefore, the system of Pfaff equations $\omega_a = 0$ $(1 \leq a \leq m)$ is completely integrable. In particular, if $m > 1$, then $\lambda_{,a} = 0$ for any $1 \leq a \leq m$. Hence, along the integral submanifold corresponding distribution of the space of principal vectors $\mathrm{Span}\{e_a\}_{1 \leq a \leq m}$, λ is a constant. □

In the following we separate our discussion into two cases.
Case 1: $2 \leq m \leq n - 2$. Using Theorem 3.1, we can obtain the following result.

Theorem 3.2. *Let M^n $(n > 3)$ be a spacelike hypersurface in $\mathbb{H}_1^{n+1}(c)$ with constant r-th mean curvature and with two nonzero distinct principal curvatures. If the multiplicities m and $n - m$ of the principal curvatures λ and μ are greater than 1, then we have*

(i) *Both λ and μ are constants and they satisfy $\lambda\mu = c$. In addition, M^n is locally the hyperbolic cylinder $\mathbb{H}^m(c_1) \times \mathbb{H}^{n-m}(c_2)$;*

(ii) *If M^n is assumed to be complete in $\mathbb{H}_1^{n+1}(c)$, then $M^n = \mathbb{H}^m(c_1)\times\mathbb{H}^{n-m}(c_2)$, where $c_1,\ c_2 < 0$ are constants.*

Proof. Let us denote the integral submanifold through $x \in M^n$, corresponding to λ and μ, by $M_1^m(x)$ and $M_2^{n-m}(x)$, respectively. We write

$$d\lambda = \sum_i \lambda_{,i}\omega_i,\ d\mu = \sum_i \mu_{,i}\omega_i. \tag{3.10}$$

Then Theorem 3.1 implies

$$\lambda_{,a} = 0,\ \mu_{,\alpha} = 0. \tag{3.11}$$

Since

$$S_r = \binom{n}{r}H_r = \sum_{0\le s\le r}\binom{m}{s}\binom{n-m}{r-s}\lambda^s\mu^{r-s}, \tag{3.12}$$

from (3.11) and (3.12), we have

$$\sum_{1\le s\le r} s\binom{m}{s}\binom{n-m}{r-s}\lambda^{s-1}\mu^{r-s}\lambda_{,\alpha} = 0. \tag{3.13}$$

For some α, if $p \in M$ satisfying $\lambda_{,\alpha}(p) \neq 0$ exists, then the open set $\mathcal{U} = \{p \in M \mid \lambda_{,\alpha}(p) \neq 0\}$ is nonempty. From (3.13), it implies

$$\sum_{1\le s\le r} s\binom{m}{s}\binom{n-m}{r-s}\lambda^{s-1}\mu^{r-s} = 0 \text{ on } \mathcal{U}. \tag{3.14}$$

Similarly, from (3.14) it implies inductively $r!\binom{m}{r} = 0$ on $\mathcal{U}$ for $m \ge r$ which is a contradiction, or

$$m!\binom{n-m}{r-m}\mu^{r-m} = 0 \text{ on } \mathcal{U}\ m < r. \tag{3.15}$$

Thus $\mu \equiv 0$ on $\mathcal{U}$ by (3.15) which is a contradiction with $\lambda\mu \neq 0$. Therefore, for any α, we have $\lambda_{,\alpha} \equiv 0$. This means λ is a nonzero constant. It can force that μ is a constant from (3.12). Hence, M^n is an isoparametric hypersurface of $\mathbb{H}^{n+1}(1)$ with two distinct principal curvatures. Therefore, we can obtain our result from [**3**, Theorem 1]. $\square$

<u>Case 2.</u> $m = n-1,\ m \ge 2$. If $H_r = 0$, then M^n is a $(r-1)$-maximal. In the following we assume $H_r \neq 0$ from Theorem 3.1 and $S_r = \binom{n}{r}H_r = \binom{n-1}{r}\lambda^r + \binom{n-1}{r-1}\lambda^{r-1}\mu \neq 0$ is a nonzero constant, we have $\lambda \neq 0$ and

$$\lambda_{,a} = 0, \qquad \mu_{,a} = 0, \tag{3.16}$$

$$\mu = \frac{nH_r - (n-r)\lambda^r}{r\lambda^{r-1}},\ \lambda - \mu = \frac{n(\lambda^r - H_r)}{r\lambda^{r-1}} \neq 0. \tag{3.17}$$

From (2.13) and (3.16), we can obtain

$$\begin{aligned}\sum_k h_{abk}\omega_k &= dh_{ab} + \sum_k h_{kb}\omega_{ka} + \sum_k h_{ak}\omega_{kb}\\ &= dh_{ab} = \delta_{ab}d\lambda = \delta_{ab}\lambda_{,n}\omega_n,\end{aligned}$$

and

$$\sum_k h_{nnk}\omega_k = \quad dh_{nn} = d\mu = \mu_{,n}\omega_n.$$

It follows that

$$h_{abc} = 0,\ h_{abn} = \delta_{ab}\lambda_{,n};\ h_{nna} = 0,\ h_{nnn} = \mu_{,n}. \tag{3.18}$$

Therefore, combining (2.12), (2.13) and (3.18), we have

$$\omega_{an} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_a. \tag{3.19}$$

Thus, using structure equation (2.6), we have $d\omega_n = \sum_a \omega_{na} \wedge \omega_a = 0$. For this reason, we may put $\omega_n = ds$, where s is the arc length of an orthogonal trajectory of the family of the integral submanifolds $M_1^{n-1}(s)$ corresponding to λ. Then for $\lambda = \lambda(s)$, we have $\lambda_{,n} = \lambda'(s)$.

Combining (3.17) and (3.19), we get

$$\omega_{an} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_a = \frac{r\lambda^{r-1}\lambda_{,n}}{n(\lambda^r - H_r)}\omega_a = (\log|\lambda^r - H_r|^{\frac{1}{n}})'\omega_a = -\frac{w'}{w}\omega_a, \tag{3.20}$$

where $w = |\lambda^r - H_r|^{-\frac{1}{n}} > 0$.

Using this formula, on the one hand, from (2.5), (2.6) and (2.7), we have

$$d\omega_a = \sum_b \omega_{ab} \wedge \omega_b + \omega_{an} \wedge \omega_n = \sum_b \omega_{ab} \wedge \omega_b - \frac{w'}{w}\omega_a \wedge \omega_n,$$

and

$$\begin{aligned} d\omega_{an} &= \sum_b \omega_{ab} \wedge \omega_{bn} - R_{anan}\omega_a \wedge \omega_n \\ &= -\frac{w'}{w}\sum_b \omega_{ab} \wedge \omega_b + (\lambda\mu - c)\omega_a \wedge \omega_n. \end{aligned} \tag{3.21}$$

On the other hand,

$$\begin{aligned} d\omega_{an} &= d(-\frac{w'}{w}\omega_a) \\ &= \frac{w''w - w'^2}{w^2} ds \wedge \omega_a - \frac{w'}{w} d\omega_a \\ &= \frac{w''}{w}\omega_a \wedge \omega_n - \frac{w'}{w}\sum_b \omega_{ab} \wedge \omega_b. \end{aligned} \tag{3.22}$$

By comparing (3.21) and (3.22), we obtain

$$w'' - w(\lambda\mu - c) = 0. \tag{3.23}$$

Integrating (3.23), we have

$$w'^2 = w^2[(H_r \pm w^{-n})^{\frac{2}{r}} - c] + C, \tag{3.24}$$

where C is an integration constant.

Similarly, we have

$$
\begin{aligned}
d\omega_{ab} - \sum_c \omega_{ac} \wedge \omega_{cb} &= \omega_{an} \wedge \omega_{nb} - R_{abab}\omega_a \wedge \omega_b \\
&= -\left\{ \left(\frac{w'}{w} \right)^2 - (\lambda^2 - c) \right\} \omega_a \wedge \omega_b.
\end{aligned} \tag{3.25}
$$

Therefore we see that $M_1^{n-1}(s)$ is of constant sectional curvature $\left(\frac{w'}{w}\right)^2 + c - \lambda^2$ which has the same sign for all s.

We can choose the orthonormal frame $\{x; e_1, \dots, e_n, e_{n+1}, e_{n+2}\}$ in $\mathbb{R}_2^{n+2}$ with $e_{n+2} = x$ and put

$$
N = e_1 \wedge \ldots \wedge e_{n-1} \wedge Z, \quad Z = -\frac{w'}{w} e_n - \lambda e_{n+1} + e_{n+2}. \tag{3.26}
$$

Then, using structure equations, by straightforward calculation we can show that

$$
\langle Z, Z \rangle = \left(\frac{w'}{w} \right)^2 + c - \lambda^2, \quad dZ = -\frac{w'}{w} Z\omega_n, \ dN = -\frac{w'}{w} N\omega_n. \tag{3.27}
$$

From (3.27) it shows that the n-vector N in $\mathbb{R}_2^{n+2}$ is constant along $M_1^{n-1}(s)$. Hence, there exists an n-dimensional linear space $E^n(s)$ in $\mathbb{R}_2^{n+2}$ containing $M_1^{n-1}(s)$. Moreover, n-vector N depends only on s and integration gives

$$
N = \frac{w(s_0)}{w} N(s_0). \tag{3.28}
$$

Thus, we can see that $E^n(s)$ is parallel to $E^n(s_0)$.

Hence, we have the following result.

Theorem 3.3. *Let M^n a complete spacelike hypersurface of $\mathbb{H}_1^{n+1}(c)$ for $n \geq 3$. Assume that M^n has constant r-th mean curvature and two distinct principal curvatures such that for one of them, the associated space of principal curvature vectors has dimension* 1. *Then:*

- (i) *$H_r = 0$, and therefore, M^n is an $(r-1)$-maximal hypersurface, or*
- (ii) *M^n is the locus of a family of moving $(n-1)$-dimensional submanifolds $M_1^{n-1}(s)$. The principal curvature λ of multiplicity $n-1$ is constant along each of the submanifolds $M_1^{n-1}(s)$. The manifolds $M_1^{n-1}(s)$ have constant curvature $((\log|\lambda^r - H_r|^{1/n})')^2 + c - \lambda^2$, which does not change sign. Here the parameter s is the arc length of an orthogonal trajectory of the family $M_1^{n-1}(s)$, and $\lambda = \lambda(s)$ satisfies the ordinary second order differential equation* (1.1) *or* (1.2) *where $w = w = |\lambda^r - H_r|^{-\frac{1}{n}}$.*

Corollary 3.4. *In $\mathbb{H}_1^{n+}(c)$, there exist infinitely many spacelike hypersurfaces with constant r-th mean curvature that are not congruent to each other.*

4. $(r-1)$-MAXIMAL SPACELIKE HYPERSURFACES IN THE ANTI-DE SITTER SPACES

In this section, we will investigate spacelike hypersurfaces with vanishing r-th mean curvature in the anti-de Sitter spaces which is called $(r-1)$-Maximal spacelike hypersurfaces. Let M^n be an $(r-1)$-maximal spacelike hypersurface with two distinct principal curvatures in the anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$. In addition suppose the multiplicities of the principal curvatures λ and μ are m and $n-m$, respectively.

Firstly, using Theorem 3.2, it is easy to prove the following result.

Proposition 4.1. *Let M^n $(n>3)$ be an $(r-1)$-maximal spacelike hypersurface in $\mathbb{H}_1^{n+1}(c)$ with two distinct principal curvatures. Suppose the multiplicities m and $n-m$ of the principal curvatures λ and μ are greater than 1, then we have:*

(i) *If $\lambda\mu=0$, then $R\le c$;*
(ii) *If $\lambda\mu\ne 0$, then M^n is locally the hyperbolic cylinder $\mathbb{H}^m(c_1)\times\mathbb{H}^{n-m}(c_2)$.*

Proof. (i) Since $\lambda\mu=0$, then $\lambda=0$ or $\mu=0$. Without loss of generality, we suppose $\mu=0$, so $\lambda\ne0$. From (2.11), we have

$$n(n-1)(R-c)=-n^2H^2+S=(m-m^2)\lambda^2\le 0.$$

It can imply $R\le c$. Moreover, in view of

$$S_r=\binom{n}{r}H_r=\sum_{0\le s\le r}\binom{m}{s}\binom{n-m}{r-s}\lambda^s\mu^{r-s}=0,$$

then $m<r$.

(ii) This is a direct result from Theorem 3.2. □

Corollary 4.2. *Let M^n $(n>3)$ be an $(r-1)$-maximal spacelike hypersurface in $\mathbb{H}_1^{n+1}(c)$ with two distinct principal curvatures. If the multiplicities m and $n-m$ of the principal curvatures λ and μ are greater than $r-1$ $(4\le 2r\le n)$, then we have $\lambda\mu\ne0$ and M^n is locally the hyperbolic cylinder $\mathbb{H}^m(c_1)\times\mathbb{H}^{n-m}(c_2)$.*

Secondly, we proved the following result in [**9**].

Proposition 4.3 ([**9**]). *Let M be an n-dimensional $(n\ge3)$ $(r-1)$-maximal spacelike hypersurface immersed in an anti-de Sitter space $\mathbb{H}_1^{n+1}(c)$. Suppose in addition that M has two distinct principal curvatures λ and μ with the multiplicities $n-1$ and 1, respectively. Then we have:*

(i) *If $\lambda\equiv0$, then $R=c$ and therefore, M^n is 1-maximal, or*
(ii) *If $\inf(\lambda-\mu)^2>0$, then*

$$S\ge\frac{n(r^2-2r+n)}{r(n-r)},\tag{4.1}$$

and $S=\frac{n(r^2-2r+n)}{r(n-r)}$ if and only if M is a hyperbolic cylinder $\mathbb{H}^{n-1}(c_1)\times\mathbb{H}^1(c_2)$.

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