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ON SOLUTIONS OF A SYSTEM OF RATIONAL DIFFERENCE EQUATIONS

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Abstract. In this paper we investigate the system of rational difference equations

$$x_n = \frac{a}{y_{n-p}}, \qquad y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}, \qquad n = 1, 2, \ldots,$$

where q is a positive integer with $p < q$, $p \nmid q$, p is an odd number and $p \geq 3$, both a and b are nonzero real constants and the initial values $x_{-q+1}, x_{-q+2}, \ldots, x_0, y_{-q+1}, y_{-q+2}, \ldots, y_0$ are nonzero real numbers. We show all real solutions of the system are eventually periodic with period $2pq$ (resp. $4pq$) when $(a/b)^q = 1$ (resp. $(a/b)^q = -1$) and characterize the asymptotic behavior of the solutions when $a \neq b$, which generalizes Özban's results [Appl. Math. Comput. **188** (2007), 833–837].

1. Introduction

Consider the system of rational difference equations

$$x_n = \frac{a}{y_{n-p}}, \qquad y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}, \qquad n = 1, 2, \ldots, \tag{1}$$

where q is a positive integer with $p < q$, p is a positive integer, both a and b are nonzero real constants and the initial values $x_{-q+1}, x_{-q+2}, \ldots, x_0, y_{-q+1}, y_{-q+2}, \ldots, y_0$ are nonzero real numbers.

The system of equations (1) is equivalent to the single rational equation of order $p + q$

$$x_n = \frac{cx_{n-p}x_{n-p-q}}{x_{n-q}}, \qquad c = \frac{a}{b}. \tag{2}$$

This is obtained by eliminating the variable $y_n = a/x_{n+p}$ as follows:

$$\frac{a}{x_{n+p}} = \frac{ab/x_n}{x_{n-q}(a/x_{x_{n+p-q}})} = \frac{bx_{n+p-q}}{x_n x_{n-q}}.$$

Taking the reciprocal and shifting all indices back p units gives (2). Equations (1) belong to a class of "homogeneous equations of degree one" (cf. [**9, 10**] and

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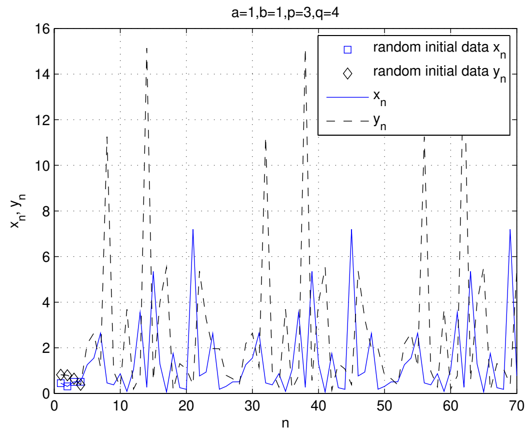

Figure 1. A positive solution of (1) is eventually periodic with period 24 where $a = b = 1$, $p = 3, q = 4$. This result is given in [**7**] .

references therein). By the substitution $t_n = x_n/x_{n-p}$, system (1) can be written as a "triangular vector map or system" where one equation is independent of the other:

$$t_n = \frac{c}{t_{n-q}}, \qquad s_n = t_n s_{n-p}.$$

Dynamics of triangular maps have been studied by several other people (see a nice survey [**12**] and a beautiful result [**1**]).

In particular, Çinar in [**3**] proved that all positive solutions of the system of rational difference equations

$$x_n = \frac{1}{y_{n-1}}, \qquad y_n = \frac{y_{n-1}}{x_{n-2}y_{n-2}}, \qquad n = 1, 2, \ldots$$

with the period four. That such a nonlinear rational system has a period so simple as 4 is surprising. Later, Yang et al in [**15**] generalized his result and obtained all positive solutions of system (1) with $p|q$ and $a = b$ have period $2q$. For the case $p|q$ and $a \neq b$, they also investigated the behavior of positive solutions. Similar nonlinear systems of rational difference equations were investigated, for instance, by Clark and Kulenovic [**4**], Özban [**6**], Papaschinopoulos and Schinas [**8**], Camouzis and Papaschinopoulos [**2**], Iričanin and Stević [**5**], Shojaei et al [**11**], and Yang [**13, 14**]. Recently, Özban [**7**] investigated the behavior of the positive solutions of system (1) where $p = 3$, $p \nmid q$. For the case $b = a \in \mathbb{R}^+$, $p = 3$, $q > 3$, $p \nmid q$, the author obtained all positive solutions of the system of difference equations (1) that are eventually periodic (see the definition below and Figure 1) with period $6q$. For the case $b \neq a \in \mathbb{R}^+$, $p = 3$, $q > 3$, $p \nmid q$, he also characterized the asymptotic behavior of the positive solutions of system (1).

In this paper we study the behavior of the real solutions of system (1) where p is odd with $p < q$, $p \nmid q$, and so we generalize Özban's results of [**7**]. Before stating our main results, we set the following definition used in this paper.

Definition 1 ([**16**])**.** A solution $\{(x_n, y_n)\}_{n=-(q-1)}^{\infty}$ of (1) is eventually periodic if there exist an integer $n_0 \geq -q+1$ and a positive integer w such that

$$(x_{n+n_0+w}, y_{n+n_0+w}) = (x_{n+n_0}, y_{n+n_0}), \qquad n = 1, 2, \dots,$$

and w is called a period.

An eventually periodic sequence such as $\{1, 1, 2, 3, 2, 3, 2, 3, 2, 3, \dots\}$ that is periodic from some point onwards can serve as an example.

2. Main results

Lemma 1. *Let $\{(x_n, y_n)\}_{n=-(q-1)}^{\infty}$ be an arbitrary solution of* (1)*. Then*

$$x_n y_n = x_{n+2q} y_{n+2q}, \qquad n = -q+1, -q+2, \dots$$

Proof. From (1) we have

$$x_{n+2q} y_{n+2q} = \frac{a}{y_{n+2q-p}} \frac{b y_{n+2q-p}}{x_{n+q} y_{n+q}} = \frac{ab}{x_{n+q} y_{n+q}} \tag{3}$$

and

$$x_{n+q} y_{n+q} = \frac{a}{y_{n+q-p}} \frac{b y_{n+q-p}}{x_n y_n} = \frac{ab}{x_n y_n}. \tag{4}$$

Then substituting (4) into (3), we get

$$x_{n+2q} y_{n+2q} = x_n y_n, \qquad n = -q+1, -q+2, \dots$$

□

Theorem 1. *Let p be odd, $c := a/b$ and $\{(x_n, y_n)\}_{n=-(q-1)}^{\infty}$ be an arbitrary solution of* (1)*.*

(i) *If $|c| < 1$, then for each integer l with $1 \leq l \leq 2pq$, the subsequence $\{x_{2pqj+l-p}\}_{j=0}^{\infty}$ converges to zero exponentially and the subsequence $\{y_{2pqj+l-p}\}_{j=0}^{\infty}$ tends to infinity exponentially.*

(ii) *If $c^q = 1$, then all solutions of the system of difference equations* (1) *are eventually periodic with period $2pq$; If $c^q = -1$, then all solutions of the system of difference equations* (1) *are eventually periodic with period $4pq$.*

(iii) *If $|c| > 1$, then for each integer l with $1 \leq l \leq 2pq$, the subsequence $\{x_{2pqj+l-p}\}_{j=0}^{\infty}$ tends to infinity exponentially and the subsequence $\{y_{2pqj+l-p}\}_{j=0}^{\infty}$ converges to zero exponentially.*

Proof. For each $n \geq 1$, substituting $x_n = a/y_{n-p}$ into $y_{n+q} = b y_{n+q-p}/(x_n y_n)$, we get

$$y_n y_{n+q} = \frac{1}{c} y_{n-p} y_{n+q-p}. \tag{5}$$

Repeated application of (5) yields

$$y_{n-p} y_{n+q-p} = c^2 y_{n+p} y_{n+q+p} = c^3 y_{n+2p} y_{n+q+2p} = \dots$$

or

$$y_{n-p}y_{n+q-p} = c^{t+1}y_{n+pt}y_{n+q+pt}, \qquad t = 0, 1, \ldots, \quad n = 1, 2, \ldots \tag{6}$$

Since $q > p$ and $p \nmid q$, it follows that $q = pk + m$ for some positive integer k where $m < p$. Hence the last equation turns into

$$y_{n-p}y_{n+(pk+m)-p} = c^{t+1}y_{n+pt}y_{n+(pk+m)+pt}, \quad t = 0, 1, \ldots, \quad n = 1, 2, \ldots \tag{7}$$

For $t = k - 1$, we have

$$y_{n-p}y_{n+(pk+m)-p} = c^{k}y_{n+pk-p}y_{n+(2pk+m)-p}, \quad k = 1, 2, \ldots, \quad n = 1, 2, \ldots \tag{8}$$

Multiplying both sides of Eq. (8) by $\prod_{i=2}^{p} y_{n+i(pk+m)-p}$, we obtain

$$y_{n-p}\prod_{i=1}^{p} y_{n+i(pk+m)-p} = c^{k}y_{n+pk-p}y_{n+(2pk+m)-p}\prod_{i=2}^{p} y_{n+i(pk+m)-p}. \tag{9}$$

Then, by taking $n = n + pk$ and $t = (p-1)k + m - 1$ in (7), we get

$$y_{n+pk-p}y_{n+(2pk+m)-p} = c^{(p-1)k+m}\prod_{i=p}^{p+1} y_{n+i(pk+m)-p} \tag{10}$$

which combined with (9), leads to

$$y_{n-p}\prod_{i=1}^{p-1} y_{n+i(pk+m)-p} = c^{pk+m}\prod_{i=2}^{p+1} y_{n+i(pk+m)-p}. \tag{11}$$

Moreover, taking $n = n + j(pk+m)$, $j = 1, 2, \ldots, m-1$ and $t = pk + m - 1$ in (7), we get

$$\prod_{i=j}^{1+j} y_{n+i(pk+m)-p} = c^{pk+m}\prod_{i=p+j}^{p+j+1} y_{n+i(pk+m)-p}. \tag{12}$$

When p is odd, it follows that

$$\prod_{i=1}^{p-1} y_{n+i(pk+m)-p} = c^{\frac{(pk+m)(p-1)}{2}}\prod_{i=p+1}^{2p-1} y_{n+i(pk+m)-p},$$

$$\prod_{i=2}^{p+1} y_{n+i(pk+m)-p} = c^{\frac{(pk+m)(p-1)}{2}}\left(\prod_{i=p+2}^{2p} y_{n+i(pk+m)-p}\right)y_{n+(p+1)(pk+m)-p}.$$

These together with (11) imply that

$$y_{n-p} = c^{pk+m}y_{n+2p(pk+m)-p},$$

or

$$y_{n-p} = c^{q}y_{n+2pq-p}, \qquad n = 1, 2, \ldots \tag{13}$$

since $q = pk + m$. It is clear that repeated application of (13) yields

$$y_{n+2pqj-p} = c^{qj}y_{n-p}, \qquad j = 1, 2, \ldots, \quad n = 1, 2, \ldots \tag{14}$$

Moreover from $x_n = a/y_{n-p}$ and $y_{n-p} = c^q y_{n+2pq-p}$, it follows that

$$x_n = c^q a/y_{n+2pq-p} \qquad \text{or} \qquad x_n = c^q x_{n+2pq},$$

or

$$x_{n+2pq-p} = c^q x_{n-p}, \qquad n = 1, 2, \ldots \tag{15}$$

Again repeated application of (15) leads to

$$x_{n+2pqj-p} = c^{qj} x_{n-p}, \quad j = 1, 2, \ldots, \quad n = 1, 2, \ldots \tag{16}$$

Consequently: (i) follows from Eqs.(14) and (16) and the fact that $|c| < 1$. (iii) follows from equations Eqs.(14) and (16), and the fact that $|c| > 1$.

It remains to show (ii). If $c^q = 1$ (resp. $c^q = -1$), it follows from (15) and (13) that

$$x_n = x_{n+2pq}, \qquad y_n = y_{n+2pq}, \quad n = 1, 2, \ldots \tag{17}$$

$$(\text{resp.} \quad x_n = x_{n+4pq}, \qquad y_n = y_{n+4pq}, \quad n = 1, 2, \ldots). \tag{18}$$

A short computation reveals that

$$x_{2pqj-p} = x_{-p} y_{-p} \frac{x_0}{a} \neq x_{-p},$$

$j = 1, 2, \ldots$ for arbitrary initial values. In fact, from (17) (resp. (18)), it suffices to show that $x_{2pq-p} = x_{-p} y_{-p} x_0/b$ (resp. $x_{4pq-p} = x_{-p} y_{-p} x_0/b$). From Lemma 1, we have $x_n y_n = x_{n+2q} y_{n+2q} = \cdots = x_{n+2pq} y_{n+2pq}$. Thus by taking $n = -p$, we have

$$x_{-p} y_{-p} = x_{2pq-p} y_{2pq-p}, \qquad (\text{resp.} \quad x_{-p} y_{-p} = x_{4pq-p} y_{4pq-p}). \tag{19}$$

From (5), we have

$$\frac{y_{n-p}}{y_n} = \frac{y_{n+q}}{y_{n+q-p}} = \cdots = \frac{y_{n+(2p-1)q}}{y_{n+(2p-1)q-p}}. \tag{20}$$

By taking $n = q$ in (20), we get

$$\frac{y_{q-p}}{y_q} = \frac{y_{2pq}}{y_{2pq-p}}, \qquad (\text{resp.} \qquad \frac{y_{q-p}}{y_q} = \frac{y_{4pq}}{y_{4pq-p}}). \tag{21}$$

Folloing from (19), (21) and $y_{2pq} = y_0$, we obtain

$$\begin{aligned} x_{2pq-p} &= \frac{x_{-p} y_{-p}}{y_{2pq-p}} = x_{-p} y_{-p} \frac{y_{q-p}}{y_q y_{2pq}} = x_{-p} y_{-p} \frac{y_{q-p}}{y_q y_0}, \\ (\text{resp.} \quad x_{4pq-p} &= x_{-p} y_{-p} \frac{y_{q-p}}{y_q y_0}). \end{aligned} \tag{22}$$

By taking $n = q$ in the second equation of system (1), we have

$$\frac{y_{q-p}}{y_q y_0} = \frac{x_0}{b}.$$

This together with (22) imply that

$$x_{2pq-p} = \frac{x_{-p} y_{-p} x_0}{b}, \qquad (\text{resp.} \quad x_{4pq-p} = \frac{x_{-p} y_{-p} x_0}{b}).$$

□

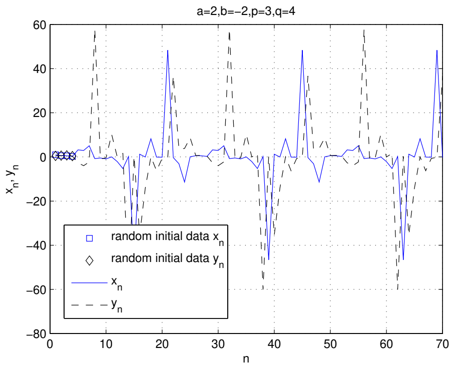

Figure 2. $c^q = 1$, $w = 24$.

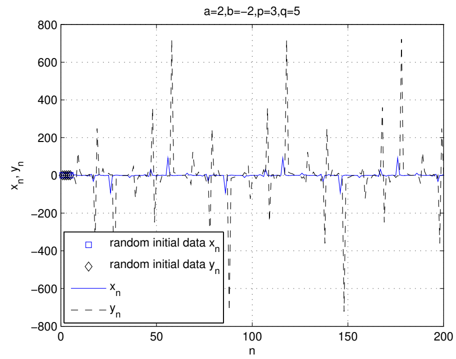

Figure 3. $c^q = -1$, $w = 60$.

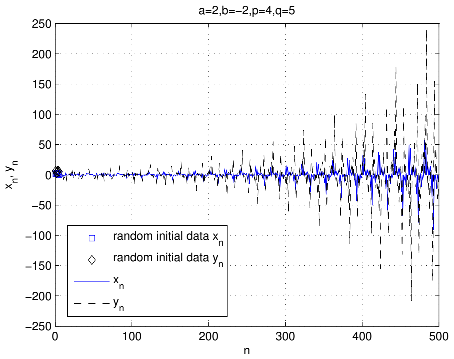

Figure 4. p is even, $c = -1$.

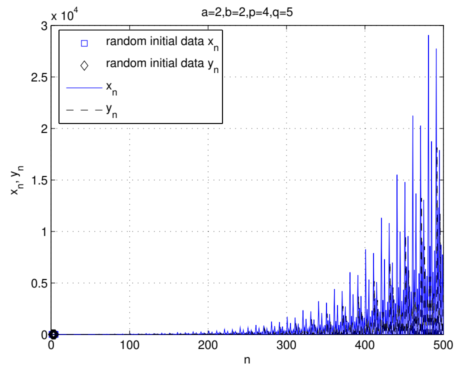

Figure 5. p is even, $c = 1$.

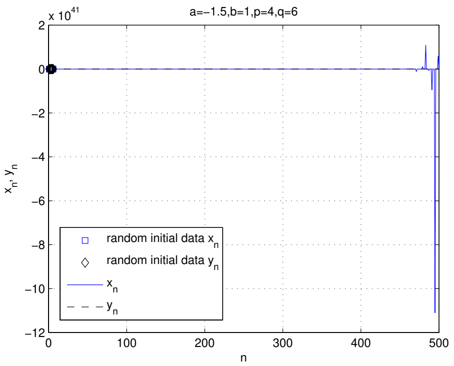

Figure 6. p, q are even, $c = -1.5$.

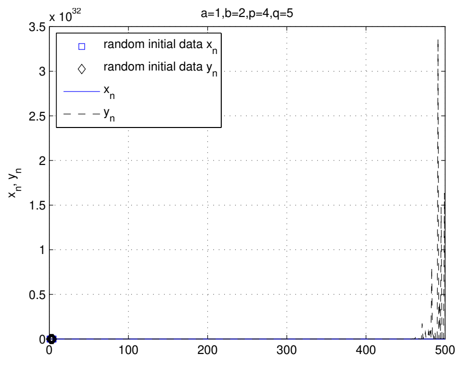

Figure 7. p is even, q is odd, $c = 0.5$.

Remark 1. Some numerical experiments are carried out by MATLAB software. Choosing $a = -b = 2$, $p = 3$, $q = 4$, and random initial data, we see that $c^q = 1$ and the solutions of (1) are eventually periodic with period 24 in Fig. 2. Choosing $a = -b = 2$, $p = 3$, $q = 5$ and random initial data, we see that $c^q = -1$ and the solutions of (1) are eventually periodic with period 60 in Fig. 3.

A natural question is what the solutions look like if p is even. We plot Figs. 4–7 with different c and different q. None of them can tell that the corresponding solution of (1) is eventually periodic even if $c = 1$.

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