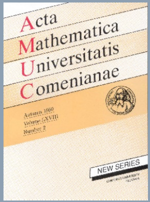

ON SOLUTIONS OF A SYSTEM OF RATIONAL DIFFERENCE EQUATIONS

YU YANG, LI CHEN AND YONG-GUO SHI

Abstract. In this paper we investigate the system of rational difference equations

$$x_n = \frac{a}{y_{n-p}}, \qquad y_n = \frac{b y_{n-p}}{x_{n-q} y_{n-q}}, \qquad n = 1, 2, \ldots,$$

where q is a positive integer with $p < q$, $p \nmid q$, p is an odd number and $p \geq 3$, both a and b are nonzero real constants and the initial values $x_{-q+1}, x_{-q+2}, \ldots, x_0, y_{-q+1}, y_{-q+2}, \ldots, y_0$ are nonzero real numbers. We show all real solutions of the system are eventually periodic with period $2pq$ (resp. $4pq$) when $(a/b)^q = 1$ (resp. $(a/b)^q = -1$) and characterize the asymptotic behavior of the solutions when $a \neq b$, which generalizes Özban's results [Appl. Math. Comput. **188** (2007), 833–837].

1. Introduction

Consider the system of rational difference equations

$$x_n = \frac{a}{y_{n-p}}, \qquad y_n = \frac{b y_{n-p}}{x_{n-q} y_{n-q}}, \qquad n = 1, 2, \ldots, \tag{1.1}$$

Received February 1, 2010; revised September 29, 2010.

2010 *Mathematics Subject Classification.* Primary 39A11, 37B20.

Key words and phrases. System of difference equations; homogeneous equations of degree one; eventually periodic solutions.

This research was supported by the undergraduate scientific research project of Neijiang Normal University.

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where q is a positive integer with $p < q$, p is a positive integer, both a and b are nonzero real constants and the initial values $x_{-q+1}, x_{-q+2}, \dots, x_0, y_{-q+1}, y_{-q+2}, \dots, y_0$ are nonzero real numbers.

The system of equations (1.1) is equivalent to the single rational equation of order $p + q$

$$x_n = \frac{c x_{n-p} x_{n-p-q}}{x_{n-q}}, \qquad c = \frac{a}{b}. \tag{1.2}$$

This is obtained by eliminating the variable $y_n = a/x_{n+p}$ as follows:

$$\frac{a}{x_{n+p}} = \frac{ab/x_n}{x_{n-q}(a/x_{x_{n+p-q}})} = \frac{b x_{n+p-q}}{x_n x_{n-q}}.$$

Taking the reciprocal and shifting all indices back p units gives (1.2). Equations (1.1) belong to a class of "homogeneous equations of degree one" (cf. [**9**, **10**] and references therein). By the substitution $t_n = x_n/x_{n-p}$, system (1.1) can be written as a "triangular vector map or system" where one equation is independent of the other:

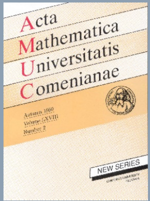
$$t_n = \frac{c}{t_{n-q}}, \qquad s_n = t_n s_{n-p}.$$

Dynamics of triangular maps have been studied by several other people (see a nice survey [**12**] and a beautiful result [**1**]).

In particular, Çinar in [**3**] proved that all positive solutions of the system of rational difference equations

$$x_n = \frac{1}{y_{n-1}}, \qquad y_n = \frac{y_{n-1}}{x_{n-2} y_{n-2}}, \qquad n = 1, 2, \dots$$

with the period four. That such a nonlinear rational system has a period so simple as 4 is surprising. Later, Yang et al in [**15**] generalized his result and obtained all positive solutions of system (1.1) with $p|q$ and $a = b$ have period $2q$. For the case $p|q$ and $a \neq b$, they also investigated the behavior of positive solutions. Similar nonlinear systems of rational difference equations were investigated,



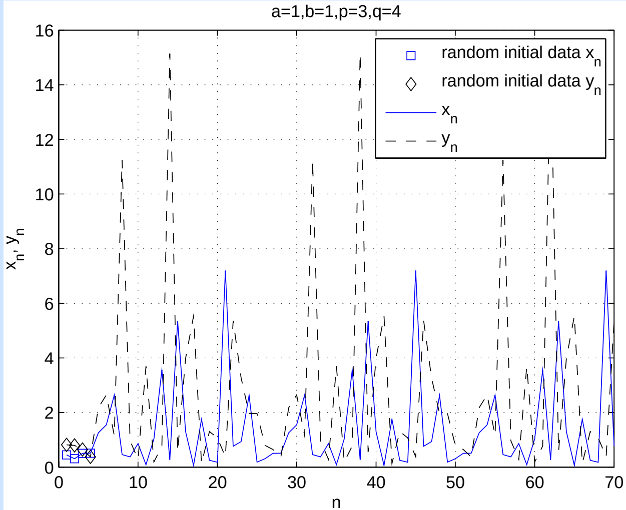
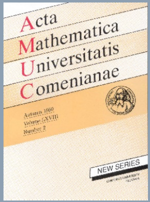


Figure 1. A positive solution of (1.1) is eventually periodic with period 24 where $a = b = 1$, $p = 3, q = 4$. This result is given in [7] .

for instance, by Clark and Kulenovic [4], Özban [6], Papaschinopoulos and Schinas [8], Camouzis and Papaschinopoulos [2], Iričanin and Stević [5], Shojaei et al [11], and Yang [13, 14]. Recently, Özban [7] investigated the behavior of the positive solutions of system (1.1) where $p = 3$, $p \nmid q$. For the case $b = a \in \mathbb{R}^+$, $p = 3$, $q > 3$, $p \nmid q$, the author obtained all positive solutions of the system of difference equations (1.1) that are eventually periodic (see the definition below and Figure 1)



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Close
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with period $6q$. For the case $b \neq a \in \mathbb{R}^+$, $p = 3$, $q > 3$, $p \nmid q$, he also characterized the asymptotic behavior of the positive solutions of system (1.1).

In this paper we study the behavior of the real solutions of system (1.1) where p is odd with $p < q$, $p \nmid q$, and so we generalize Özban's results of [7]. Before stating our main results, we set the following definition used in this paper.

Definition 1 ([16]). A solution $\{(x_n, y_n)\}_{n=-(q-1)}^{\infty}$ of (1.1) is eventually periodic if there exist an integer $n_0 \geq -q+1$ and a positive integer w such that

$$(x_{n+n_0+w}, y_{n+n_0+w}) = (x_{n+n_0}, y_{n+n_0}), \qquad n = 1, 2, \dots,$$

and w is called a period.

An eventually periodic sequence such as $\{1, 1, 2, 3, 2, 3, 2, 3, 2, 3, \dots\}$ that is periodic from some point onwards can serve as an example.

2. Main results

Lemma 1. *Let* $\{(x_n, y_n)\}_{n=-(q-1)}^{\infty}$ *be an arbitrary solution of* (1.1). *Then*

$$x_n y_n = x_{n+2q} y_{n+2q}, \qquad n = -q+1, -q+2, \dots$$

Proof. From (1.1) we have

$$x_{n+2q} y_{n+2q} = \frac{a}{y_{n+2q-p}} \frac{b y_{n+2q-p}}{x_{n+q} y_{n+q}} = \frac{ab}{x_{n+q} y_{n+q}} \tag{2.1}$$

and

$$x_{n+q} y_{n+q} = \frac{a}{y_{n+q-p}} \frac{b y_{n+q-p}}{x_n y_n} = \frac{ab}{x_n y_n}. \tag{2.2}$$

Then substituting (2.2) into (2.1), we get

$$x_{n+2q}y_{n+2q} = x_n y_n, \qquad n = -q+1, -q+2, \dots$$

□

Theorem 1. *Let p be odd, $c := a/b$ and $\{(x_n, y_n)\}_{n=-(q-1)}^{\infty}$ be an arbitrary solution of (1.1).*

(i) *If $|c| < 1$, then for each integer l with $1 \le l \le 2pq$, the subsequence $\{x_{2pqj+l-p}\}_{j=0}^{\infty}$ converges to zero exponentially and the subsequence $\{y_{2pqj+l-p}\}_{j=0}^{\infty}$ tends to infinity exponentially.*

(ii) *If $c^q = 1$, then all solutions of the system of difference equations (1.1) are eventually periodic with period $2pq$; If $c^q = -1$, then all solutions of the system of difference equations (1.1) are eventually periodic with period $4pq$.*

(iii) *If $|c| > 1$, then for each integer l with $1 \le l \le 2pq$, the subsequence $\{x_{2pqj+l-p}\}_{j=0}^{\infty}$ tends to infinity exponentially and the subsequence$\{y_{2pqj+l-p}\}_{j=0}^{\infty}$ converges to zero exponentially.*

Proof. For each $n \ge 1$, substituting $x_n = a/y_{n-p}$ into $y_{n+q} = by_{n+q-p}/(x_n y_n)$, we get

$$y_n y_{n+q} = \frac{1}{c} y_{n-p} y_{n+q-p}. \tag{2.3}$$

Repeated application of (2.3) yields

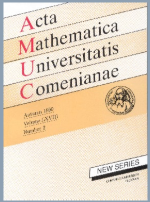
$$y_{n-p}y_{n+q-p} = c^2 y_{n+p}y_{n+q+p} = c^3 y_{n+2p}y_{n+q+2p} = \dots$$

or

$$y_{n-p}y_{n+q-p} = c^{t+1} y_{n+pt}y_{n+q+pt}, \qquad t = 0, 1, \dots, \quad n = 1, 2, \dots \tag{2.4}$$

Since $q > p$ and $p \nmid q$, it follows that $q = pk + m$ for some positive integer k where $m < p$. Hence the last equation turns into

$$y_{n-p}y_{n+(pk+m)-p} = c^{t+1} y_{n+pt}y_{n+(pk+m)+pt}, \quad t = 0, 1, \dots, \quad n = 1, 2, \dots \tag{2.5}$$



For $t = k-1$, we have

$$y_{n-p}y_{n+(pk+m)-p} = c^k y_{n+pk-p}y_{n+(2pk+m)-p}, \quad k = 1,2,\dots, \quad n = 1,2,\dots \tag{2.6}$$

Multiplying both sides of Eq. (2.6) by $\prod_{i=2}^{p} y_{n+i(pk+m)-p}$, we obtain

$$y_{n-p}\prod_{i=1}^{p} y_{n+i(pk+m)-p} = c^k y_{n+pk-p}y_{n+(2pk+m)-p}\prod_{i=2}^{p} y_{n+i(pk+m)-p}. \tag{2.7}$$

Then, by taking $n = n+pk$ and $t = (p-1)k+m-1$ in (2.5), we get

$$y_{n+pk-p}y_{n+(2pk+m)-p} = c^{(p-1)k+m}\prod_{i=p}^{p+1} y_{n+i(pk+m)-p} \tag{2.8}$$

which combined with (2.7), leads to

$$y_{n-p}\prod_{i=1}^{p-1} y_{n+i(pk+m)-p} = c^{pk+m}\prod_{i=2}^{p+1} y_{n+i(pk+m)-p}. \tag{2.9}$$

Moreover, taking $n = n+j(pk+m)$, $j = 1,2,\dots,m-1$ and $t = pk+m-1$ in (2.5), we get

$$\prod_{i=j}^{1+j} y_{n+i(pk+m)-p} = c^{pk+m}\prod_{i=p+j}^{p+j+1} y_{n+i(pk+m)-p}. \tag{2.10}$$

When p is odd, it follows that

$$\prod_{i=1}^{p-1} y_{n+i(pk+m)-p} = c^{\frac{(pk+m)(p-1)}{2}} \prod_{i=p+1}^{2p-1} y_{n+i(pk+m)-p},$$

$$\prod_{i=2}^{p+1} y_{n+i(pk+m)-p} = c^{\frac{(pk+m)(p-1)}{2}} \left(\prod_{i=p+2}^{2p} y_{n+i(pk+m)-p} \right) y_{n+(p+1)(pk+m)-p}.$$

These together with (2.9) imply that

$$y_{n-p} = c^{pk+m} y_{n+2p(pk+m)-p},$$

or

$$y_{n-p} = c^q y_{n+2pq-p}, \qquad n = 1, 2, \dots \tag{2.11}$$

since $q = pk + m$. It is clear that repeated application of (2.11) yields

$$y_{n+2pqj-p} = c^{qj} y_{n-p}, \qquad j = 1, 2, \dots, \quad n = 1, 2, \dots \tag{2.12}$$

Moreover from $x_n = a/y_{n-p}$ and $y_{n-p} = c^q y_{n+2pq-p}$, it follows that

$$x_n = c^q a/y_{n+2pq-p} \qquad \text{or} \qquad x_n = c^q x_{n+2pq},$$

or

$$x_{n+2pq-p} = c^q x_{n-p}, \qquad n = 1, 2, \dots \tag{2.13}$$

Again repeated application of (2.13) leads to

$$x_{n+2pqj-p} = c^{qj} x_{n-p}, \quad j = 1, 2, \dots, \quad n = 1, 2, \dots \tag{2.14}$$

Consequently: (i) follows from Eqs.(2.12) and (2.14) and the fact that $|c| < 1$. (iii) follows from equations Eqs.(2.12) and (2.14), and the fact that $|c| > 1$.

It remains to show (ii). If $c^q = 1$ (resp. $c^q = -1$), it follows from (2.13) and (2.11) that

$$x_n = x_{n+2pq}, \qquad y_n = y_{n+2pq}, \quad n = 1, 2, \dots \tag{2.15}$$

$$\text{(resp.}\quad x_n = x_{n+4pq}, \qquad y_n = y_{n+4pq}, \quad n = 1, 2, \dots). \tag{2.16}$$

A short computation reveals that

$$x_{2pqj-p} = x_{-p}y_{-p}\frac{x_0}{a} \neq x_{-p},$$

$j = 1, 2, \dots$ for arbitrary initial values. In fact, from (2.15) (resp. (2.16)), it suffices to show that $x_{2pq-p} = x_{-p}y_{-p}x_0/b$ (resp. $x_{4pq-p} = x_{-p}y_{-p}x_0/b$). From Lemma 1, we have $x_n y_n = x_{n+2q}y_{n+2q} = \dots = x_{n+2pq}y_{n+2pq}$. Thus by taking $n = -p$, we have

$$x_{-p}y_{-p} = x_{2pq-p}y_{2pq-p}, \qquad \text{(resp.}\quad x_{-p}y_{-p} = x_{4pq-p}y_{4pq-p}). \tag{2.17}$$

From (2.3), we have

$$\frac{y_{n-p}}{y_n} = \frac{y_{n+q}}{y_{n+q-p}} = \dots = \frac{y_{n+(2p-1)q}}{y_{n+(2p-1)q-p}}. \tag{2.18}$$

By taking $n = q$ in (2.18), we get

$$\frac{y_{q-p}}{y_q} = \frac{y_{2pq}}{y_{2pq-p}}, \qquad \text{(resp.}\quad \frac{y_{q-p}}{y_q} = \frac{y_{4pq}}{y_{4pq-p}}). \tag{2.19}$$

Folloing from (2.17), (2.19) and $y_{2pq} = y_0$, we obtain

$$x_{2pq-p} = \frac{x_{-p}y_{-p}}{y_{2pq-p}} = x_{-p}y_{-p}\frac{y_{q-p}}{y_q y_{2pq}} = x_{-p}y_{-p}\frac{y_{q-p}}{y_q y_0}, \tag{2.20}$$

$$\text{(resp.}\quad x_{4pq-p} = x_{-p}y_{-p}\frac{y_{q-p}}{y_q y_0}).$$

By taking $n = q$ in the second equation of system (1.1), we have

$$\frac{y_{q-p}}{y_q y_0} = \frac{x_0}{b}.$$

This together with (2.20) imply that

$$x_{2pq-p} = \frac{x_{-p}y_{-p}x_0}{b}, \qquad \left(\text{resp. } x_{4pq-p} = \frac{x_{-p}y_{-p}x_0}{b}\right).$$

□

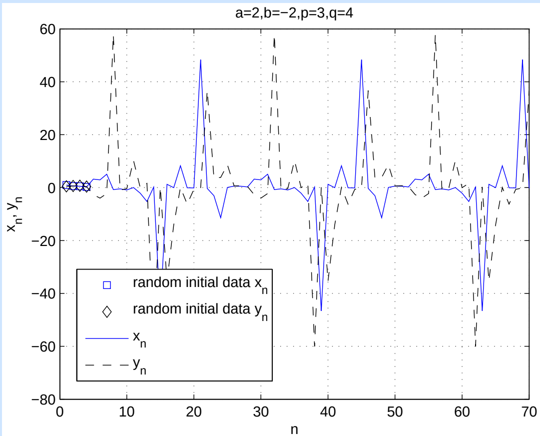


Figure 2. $c^q = 1$, $w = 24$.

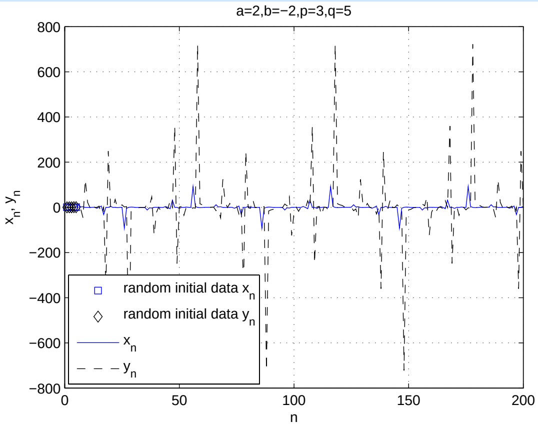
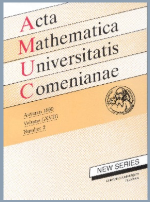


Figure 3. $c^q = -1$, $w = 60$.



Remark 1. Some numerical experiments are carried out by MATLAB software. Choosing $a = -b = 2$, $p = 3$, $q = 4$, and random initial data, we see that $c^q = 1$ and the solutions of (1.1) are eventually periodic with period 24 in Fig. 2. Choosing $a = -b = 2$, $p = 3$, $q = 5$ and random initial data, we see that $c^q = -1$ and the solutions of (1.1) are eventually periodic with period 60 in Fig. 3.

A natural question is what the solutions look like if p is even. We plot Figs. 4–7 with different c and different q. None of them can tell that the corresponding solution of (1.1) is eventually periodic even if $c = 1$.

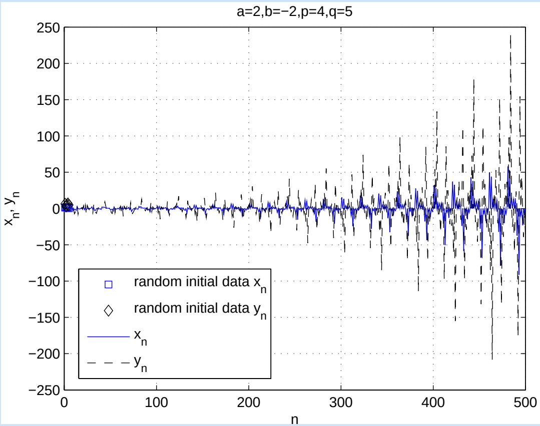


Figure 4. p is even, $c = -1$.

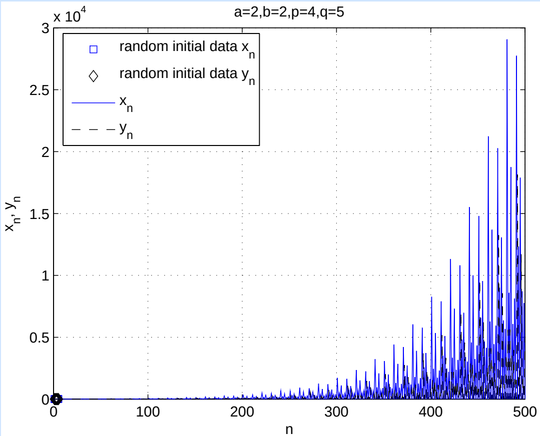


Figure 5. p is even, $c = 1$.

Go back
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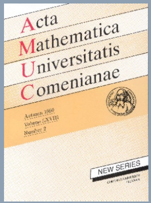

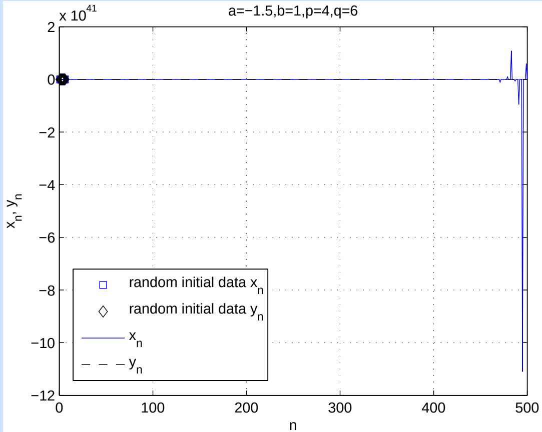

Figure 6. p, q are even, $c = -1.5$.

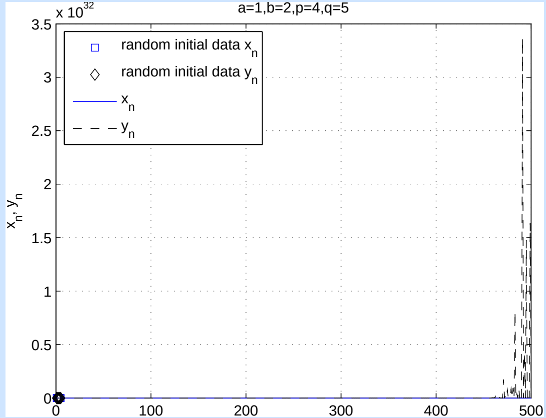

Figure 7. p is even, q is odd, $c = 0.5$.

Acknowledgment. The authors are very grateful to the referees for many helpful comments and suggestions.

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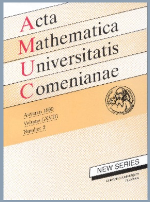

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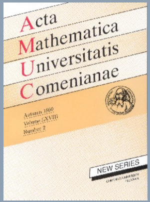

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