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PRODUCTS OF INTEGRAL-TYPE AND COMPOSITION OPERATORS BETWEEN GENERALLY WEIGHTED BLOCH SPACES

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Abstract. Let φ be a holomorphic self-map of the open unit disk $\mathbb{D}$ on the complex plane and $0 < \alpha, \beta < +\infty$. The boundedness and compactness of products of integral-type and composition operators between generally weighted Bloch spaces are investigated.

1. Introduction and preliminaries

Let $\mathbb{D}$ be the unit disc on the complex plane and φ a holomorphic self-map of $\mathbb{D}$. We denote by $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$, denote by $\mathrm{d}m(z)$ the normalized Lebesgue area measure and define the composition operator C_φ on $H(\mathbb{D})$ by $C_\varphi f = f \circ \varphi$.

The space of analytic functions on $\mathbb{D}$ such that

$$\|f\|_{B_{\log}} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)\log\frac{2}{1-|z|^2} < \infty$$

is called weighted Bloch space $B_{\log}$. $B_{\log}$ and $BMOA_{\log}$ first appeared in the study of boundedness of the Hankel operators on the Bergman space

$$A^1 = \{f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|\,\mathrm{d}m(z) < \infty\}$$

and the Hardy space H^1, respectively. $BMOA_{\log}$ also appeared in the study of a Volterra type operator (see e.g. [**1, 2, 3, 4, 9, 10**]). In [**11**], Yoneda studied the composition operators from $B_{\log}$ to $BMOA_{\log}$. In [**5, 6, 7**], we introduced the space $B^{\alpha}_{\log}, \alpha < 0$, the space of analytic functions on $\mathbb{D}$ such that

$$\|f\|_{B^{\alpha}_{\log}} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^{\alpha}\log\frac{2}{1-|z|^2} < \infty$$

that is called generally weighted Bloch space $B^{\alpha}_{\log}$.

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Let $g \in H(\mathbb{D})$, for $f \in H(\mathbb{D})$ be the integral-type operator I_g and J_g respectively, defined by

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta) \mathrm{d}\zeta,$$

$$J_g f(z) = \int_0^z f(\zeta) g'(\zeta) \mathrm{d}\zeta, \qquad z \in D.$$

The importance of the operators I_g and J_g comes from the fact that

$$I_\phi f(z) + J_\phi f(z) = M_\phi f(z) - f(0)\phi(0), \qquad z \in D,$$

where M_g is the multiplication operator

$$(M_g f)(z) = g(z) f(z), \qquad f \in H(\mathbb{D}), \quad z \in D.$$

The products of composition operators and integral-type operators are defined by

$$C_\varphi J_g f(z) = \int_0^{\varphi(z)} f(\xi) g'(\xi) \mathrm{d}\xi, \qquad J_g C_\varphi f(z) = \int_0^z f(\varphi(\xi)) g'(\xi) \mathrm{d}\xi,$$

$$C_\varphi I_\phi f(z) = \int_0^{\varphi(z)} f'(\xi) \phi(\xi) \mathrm{d}\xi, \qquad I_\phi C_\varphi f(z) = \int_0^z (f \circ \varphi)'(\xi) \phi(\xi) \mathrm{d}\xi.$$

In this article, we consider the characterization of boundedness and compactness of products of integral-type and composition operators between generally weighted Bloch spaces on the unit disk. Throughout the remainder of this paper C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. The boundedness and compactness of $C_\varphi J_g (C_\varphi I_g) : B^\alpha_{\log} \to B^\beta_{\log}$

At the beginning, the following Lemma 2.1 can be seen in [**5**].

Lemma 2.1. *Let $f \in B^\alpha_{\log}$ and $z \in \mathbb{D}$, then*

(a) *For* $0 < \alpha < 1$, $|f(z)| \le \left(1 + \dfrac{1}{(1-\alpha)\log 2}\right) \|f\|_{B^\alpha_{\log}}$;

(b) *For* $\alpha = 1$, $|f(z)| \le \dfrac{\log \frac{4}{1-|z|^2}}{\log 2} \|f\|_{B^\alpha_{\log}}$;

(c) *For* $\alpha > 1$, $|f(z)| \le \left(1 + \dfrac{2^{\alpha-1}}{(\alpha-1)\log 2}\right) \dfrac{1}{(1-|z|^2)^{\alpha-1}} \|f\|_{B^\alpha_{\log}}$.

Lemma 2.2. *Assume that φ is a holomorphic self-map of $\mathbb{D}$ and α, $\beta > 0$. Then $C_\varphi J_g($ or $C_\varphi I_g) : B^\alpha_{\log} \to B^\beta_{\log}$ is compact if and only if for any bounded sequence $(f_j)_{j\in N}$ in $B^\alpha_{\log}$, when $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$, $\|C_\varphi J_g f_j\|_{B^\beta_{\log}} \to 0$ or $\|C_\varphi I_g f_j\|_{B^\beta_{\log}}) \to 0$ as $j \to \infty$.*

The result follows from standard arguments similar to those in [**4**].

It is easy to obtain the following result by a similar method in [**8**] for $0 < \alpha < 1$.

Lemma 2.3. *Assume that φ is a holomorphic self-map of $\mathbb{D}$ and $0 < \alpha < 1$, $\beta > 0$. Then $C_\varphi J_g : B^\alpha_{\log} \to B^\beta_{\log}$ is compact if and only if for any bounded sequence $(f_j)_{j\in N}$ in $B^\alpha_{\log}$, when $f_j \to 0$ uniformly on $\overline{\mathbb{D}}$, $\|C_\varphi J_g f_j\|_{B^\beta_{\log}} \to 0$ as $j \to \infty$.*

Lemma 2.4. *Assume that $h \in H(\mathbb{D})$, $f \in B^\alpha_{\log}$, $\alpha > 0$ for a fixed $z_0 \in \mathbb{D}$. Then there exists a positive constant C independent of f such that*

$$\left|\int_0^{z_0} f(\zeta)h(\zeta)\mathrm{d}\zeta\right| \le C\|f\|_{B^\alpha_{\log}} \max_{|\zeta|\le|z_0|} |h(\zeta)|,$$

$$\left|\int_0^{z_0} f'(\zeta)h(\zeta)\mathrm{d}\zeta\right| \le C\|f\|_{B^\alpha_{\log}} \max_{|\zeta|\le|z_0|} |h(\zeta)|.$$

Proof. For $h \in H(\mathbb{D})$, $f \in B^\alpha_{\log}$, then

$$\begin{aligned}\left|\int_0^{z_0} f(\zeta)h(\zeta)\mathrm{d}\zeta\right| &\le \max_{|\zeta|\le|z_0|} |f(\zeta)| \max_{|\zeta|\le|z_0|} |h(\zeta)| \\ &\le \left(|f(0)| + |z_0| \max_{|\zeta|\le|z_0|} |f'(\zeta)|\right) \max_{|\zeta|\le|z_0|} |h(\zeta)| \\ &\le \max\left\{1, \frac{|z_0|}{(1-|z_0|^2)^\alpha \log\frac{2}{1-|z_0|^2}}\right\} \|f\|_{B^\alpha_{\log}} \max_{|\zeta|\le|z_0|} |h(\zeta)|.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|\int_0^{z_0} f'(\zeta)h(\zeta)\mathrm{d}\zeta\right| &\le |z_0| \max_{|\zeta|\le|z_0|} |f'| \max_{|\zeta|\le|z_0|} |h(\zeta)| \\ &\le \frac{|z_0|}{(1-|z_0|^2)^\alpha \log\frac{2}{1-|z_0|^2}} \|f\|_{B^\alpha_{\log}} \max_{|\zeta|\le|z_0|} |h(\zeta)|.\end{aligned}$$

□

Theorem 2.5. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha \in (0,1)$, $\beta > 0$, then $C_\varphi J_g : B^\alpha_{\log} \to B^\beta_{\log}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\varphi'(z)||g'(\varphi(z))| \log\frac{2}{1-|z|^2} < \infty. \tag{2.1}$$

Proof. Assume that $C_\varphi J_g : B^\alpha_{\log} \to B^\beta_{\log}$ is bounded. Then by the definition of the operator $C_\varphi J_g$,

$$(C_\varphi J_g f)'(z) = f(\varphi(z))g'(\varphi(z))\varphi'(z). \tag{2.2}$$

Let $f_0(z)=1$, then $f_0\in B^{\alpha}_{\log}$. Then by the boundedness of $C_\varphi J_g$

$$(1-|z|^2)^{\beta}|\varphi'(z)|g'(\varphi(z))|\log\frac{2}{1-|z|^2}\leq \|C_\varphi J_g\|\|f_0\|_{B^{\alpha}_{\log}}<\infty. \tag{2.3}$$

Then (2.1) holds by (2.3).

Conversely, assume that (2.1) holds. Then by Lemma 2.1 and (2.2)

$$\begin{aligned}&(1-|z|^2)^{\beta}(C_\varphi J_g f)'(z)\log\frac{2}{1-|z|^2}\\ &\qquad\leq C\|f\|_{B^{\alpha}_{\log}}(1-|z|^2)^{\beta}|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}.\end{aligned}\tag{2.4}$$

Then, by Lemma 2.4, with $h=g'$ and $z_0=\varphi(0)$,

$$|(C_\varphi J_g f_j)(0)|=\left|\int_0^{\varphi(0)} f(\zeta)g'(\zeta)d\zeta\right|\leq C\|f\|_{B^{\alpha}_{\log}}\max_{|\zeta|\leq|\varphi(0)|}|g'(\zeta)|. \tag{2.5}$$

By (2.4), we have

$$\begin{aligned}\|C_\varphi J_g f\|_{B^{\beta}_{\log}}\leq C\big(&\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\\ &+\max_{|\zeta|\leq|\varphi(0)|}|g'(\zeta)|\big)\|f\|_{B^{\alpha}_{\log}}.\end{aligned}$$

By (2.1) and (2.5), the boundedness of $C_\varphi J_g$ is obtained. □

Theorem 2.6. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g\in H(\mathbb{D})$, $\alpha\in(0,1)$, $\beta>0$, then $C_\varphi J_g: B^{\alpha}_{\log}\to B^{\beta}_{\log}$ is compact if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}<\infty.$$

Proof. Assume that $C_\varphi J_g: B^{\alpha}_{\log}\to B^{\beta}_{\log}$ is compact, then it is bounded, hence (2.1) holds by Theorem 2.5.

Conversely, assume that (2.1) holds. Then by Theorem 2.5, $C_\varphi J_g: B^{\alpha}_{\log}\to B^{\beta}_{\log}$ is bounded. By Lemma 2.3 for any bounded sequence $(f_j)_{j\in N}$ in $B^{\alpha}_{\log}$, when $f_j\to 0$ uniformly on $\overline{\mathbb{D}}$, we need only to prove that $\|C_\varphi J_g f_j\|_{B^{\beta}_{\log}}\to 0$ as $j\to\infty$. Then

$$\begin{aligned}&\lim_{j\to\infty}\sup_{z\in\overline{\mathbb{D}}}(1-|z|^2)^{\beta}(C_\varphi J_g f_j)'(z)\log\frac{2}{1-|z|^2}\\ &\leq\sup_{z\in\overline{\mathbb{D}}}(1-|z|^2)^{\beta}|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\lim_{j\to\infty}\|f_j\|_\infty=0.\end{aligned}$$

$$|(C_\varphi J_g f_j)(0)|=\left|\int_0^{\varphi(0)} f_j(\zeta)g'(\zeta)d\zeta\right|\leq C\|f_j\|_\infty\max_{|\zeta|\leq|\varphi(0)|}|g'(\zeta)|\to 0\, as\, j\to\infty.$$

Then the compactness of $C_\varphi J_g$ is completed. □

Theorem 2.7. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g\in H(\mathbb{D})$, $\beta>0$.*

(i) *If*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\log\frac{2}{1-|\varphi(z)|^2}<\infty, \tag{2.6}$$

then $C_\varphi J_g\colon B_{\log}\to B^\beta_{\log}$ *is bounded.*

(ii) *If* $C_\varphi J_g\colon B_{\log}\to B^\beta_{\log}$ *is bounded, then*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\log\log\frac{2}{1-|\varphi(z)|^2}<\infty. \tag{2.7}$$

Proof. (i) For $f\in B_{\log}$, by Lemma 2.1, it holds

$$\begin{aligned}&(1-|z|^2)^\beta(C_\varphi J_g f)'(z)\log\frac{2}{1-|z|^2}\\&\qquad\le C\|f\|_{B_{\log}}(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\log\frac{2}{1-|\varphi(z)|^2}.\end{aligned}$$

By (2.6), we have that $C_\varphi J_g : B_{\log}\to B^\beta_{\log}$ is bounded.

(ii) Assume that $C_\varphi J_g : B_{\log}\to B^\beta_{\log}$ is bounded. For $w\in D$, set

$$f_w(z)=\log\log\frac{2}{1-\overline{w}z}.$$

Then

$$f_w'(z)=\frac{1}{\log\frac{2}{1-\overline{w}z}}\cdot\frac{\overline{w}}{1-\overline{w}z}.$$

Then $|f_w(0)|=\log\log 2$ and

$$\begin{aligned}(1-|z|^2)|f_w'(z)|\log\frac{2}{1-|z|^2}&=\frac{(1-|z|^2)|w|\log\frac{2}{1-|z|^2}}{|1-\overline{w}z|\log\frac{2}{|1-\overline{w}z|}}\\&\le\frac{(1-|z|^2)\log\frac{2}{1-|z|^2}}{|1-z|\log\frac{2}{|1-z|}}<\infty.\end{aligned}$$

Thus $f_w\in B_{\log}$. Hence by the boundedness of $C_\varphi J_g : B_{\log}\to B^\beta_{\log}$, we have

$$\begin{aligned}&(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\log\log\frac{2}{1-|\varphi(z)|^2}\\&\qquad\le C\|C_\varphi J_g f_{\varphi(z)}\|_{B^\beta_{\log}}\le\|C_\varphi J_g\|\cdot\|f_{\varphi(z)}\|_{B_{\log}}<\infty.\end{aligned}$$

□

Theorem 2.8. *Assume that* φ *is a holomorphic self-map of* $\mathbb{D}$*,* $g\in H(\mathbb{D})$*,* $\beta>0$*.*

(i) *If*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}<\infty$$

and

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\log\frac{2}{1-|\varphi(z)|^2}=0, \tag{2.8}$$

then $C_\varphi J_g : B_{\log}\to B^\beta_{\log}$ *is compact.*

(ii) *If* $C_\varphi J_g : B_{\log}\to B^\beta_{\log}$ *is compact, then*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\log\log\frac{2}{1-|\varphi(z)|^2}=0. \tag{2.9}$$

Proof. (i) By (2.8), we have that for any $\varepsilon>0$ there exists an $r_0\in(0,1)$ such that

$$(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\log\frac{2}{1-|\varphi(z)|^2}<\varepsilon, \tag{2.10}$$

for every $|\varphi(z)|>r_0$.

Let $(f_j)_{j\in N}$ be a norm bounded sequence in $B_{\log}$ such that $f_j\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$. By Lemma 2.1, (2.1) and (2.10), we have

$$\begin{aligned}
&(1-|z|^2)^\beta(C_\varphi J_g f_j)'(z)\log\frac{2}{1-|z|^2}\\
&\le \sup_{|\varphi(z)|\le r_0}|f_j(\varphi(z))|\sup_{|\varphi(z)|\le r_0}(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\\
&\quad + C\|f_j\|_{B_{\log}}\sup_{|\varphi(z)|>r_0}(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}\log\frac{2}{1-|\varphi(z)|^2}\\
&\le C\sup_{|\zeta|\le r_0}|f_j(\zeta)|+C\varepsilon\|f_j\|_{B_{\log}}.
\end{aligned}$$

$$\begin{aligned}
|(C_\varphi J_g f_j)(0)| &= \left|\int_0^{\varphi(0)} f(\zeta)g'(\zeta)\mathrm{d}\zeta\right|\\
&\le \max_{|\zeta|\le|\varphi(0)|}|f_j(\zeta)|\max_{|\zeta|\le|\varphi(0)|}|g'(\zeta)|\to 0\quad (j\to\infty).
\end{aligned}$$

Taking the supremum over $z\in\mathbb{D}$ and letting $j\to\infty$, we have $\|C_\varphi J_g f_j\|_{B^\beta_{\log}}\to 0$ as $j\to\infty$. Thus $C_\varphi J_g : B_{\log}\to B^\beta_{\log}$ is compact.

(ii) Assume that $C_\varphi J_g : B_{\log}\to B^\beta_{\log}$ is compact and $(z_n)_{n\in N}$ is a sequence in $\mathbb{D}$ such that $\lim_{n\to\infty}|\varphi(z_n)|=1$. Let

$$f_n(z)=\left(\log\log\frac{2}{1-|\varphi(z_n)|^2}\right)^{-1}\left(\log\log\frac{2}{1-\overline{\varphi(z_n)}z}\right)^2,\qquad n\in N.$$

Then f_n is a uniformly bounded family on $B_{\log}$ that converges to 0 on compact subsets of $\mathbb{D}$. Then $\|C_\varphi J_g f_n\|_{B^\beta_{\log}} \to 0$ as $n \to \infty$.

$$\begin{aligned}\|C_\varphi J_g f_n\|_{B^\beta_{\log}} &\geq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta (C_\varphi J_g f_n)'(z)\log\frac{2}{1-|z|^2}\\ &\geq 1-|z_n|^2)^\beta|\varphi'(z_n)||g'(\varphi(z_n))|\log\frac{2}{1-|z_n|^2}\log\log\frac{2}{1-|\varphi(z_n)|^2}.\end{aligned}$$

Hence

$$\lim_{n\to\infty}(1-|z_n|^2)^\beta|\varphi'(z_n)||g'(\varphi(z_n))|\log\frac{2}{1-|z_n|^2}\log\log\frac{2}{1-|\varphi(z_n)|^2}=0.$$

So (2.9) holds. □

Theorem 2.9. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha > 1$, $\beta > 0$. If*

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha-1}}<\infty, \tag{2.11}$$

then $C_\varphi J_g : B^\alpha_{\log} \to B^\beta_{\log}$ is bounded.

Proof. By Lemma 2.1 and (2.11), for $f \in B^\alpha_{\log}$,

$$\begin{aligned}&(1-|z|^2)^\beta(C_\varphi J_g f)'(z)\log\frac{2}{1-|z|^2}\\ &\qquad\leq C\|f\|_{B^\alpha_{\log}}\frac{(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha-1}}<\infty.\end{aligned}$$

$$\begin{aligned}|(C_\varphi J_g f)(0)| &\leq \max_{|\zeta|\leq|\varphi(0)|}|f(\zeta)|\max_{|\zeta|\leq|\varphi(0)|}|g'(\zeta)|\\ &\leq \max\left\{1,\frac{|\varphi(z_0)|}{(1-|\varphi(z_0)|^2)\log\frac{2}{1-|z_0|^2}}\right\}\|f\|_{B^\alpha_{\log}}\max_{|\zeta|\leq|\varphi(0)|}|g'(\zeta)|.\end{aligned}$$

Then the boundedness of $C_\varphi J_g$ is obtained. □

Theorem 2.10. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha > 1$, $\beta > 0$. If*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}<\infty$$

and

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\beta|\varphi'(z)||g'(\varphi(z))|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha-1}}=0, \tag{2.12}$$

then $C_\varphi J_g \colon B^\alpha_{\log} \to B^\beta_{\log}$ is compact.

Proof. By (2.12), then for any $\varepsilon > 0$, there exists an $r_0 \in (0,1)$ such that

$$\frac{(1-|z|^2)^\beta |\varphi'(z)||g'(\varphi(z))| \log \frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha-1}} < \varepsilon, \qquad \text{for every } \ |\varphi(z)| > r_0.$$

Let $(f_j)_{j\in N}$ be a norm bounded sequence in $B^\alpha_{\log}$ such that $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. By Lemma 2.1, we have

$$\begin{aligned}
&(1-|z|^2)^\beta (C_\varphi J_g f_j)'(z) \log \frac{2}{1-|z|^2} \\
&\le \sup_{|\varphi(z)|\le r_0} |f_j(\varphi(z))| \sup_{|\varphi(z)|\le r_0} (1-|z|^2)^\beta |\varphi'(z)||g'(\varphi(z))| \log \frac{2}{1-|z|^2} \\
&\quad + C\|f_j\|_{B^\alpha_{\log}} \sup_{|\varphi(z)|> r_0} (1-|z|^2)^\beta |\varphi'(z)||g'(\varphi(z))| \log \frac{2}{1-|z|^2} \log \frac{2}{1-|\varphi(z)|^2} \\
&\le C \sup_{|\zeta|\le r_0} |f_j(\zeta)| + C\varepsilon \|f_j\|_{B^\alpha_{\log}}.
\end{aligned}$$

$$|(C_\varphi J_g f_j)(0)| = \left| \int_0^{\varphi(0)} f(\zeta) g'(\zeta) \mathrm{d}\zeta \right| \le C\|f_j\|_{B^\alpha_{\log}} \max_{|\zeta|\le|\varphi(0)|} |g'(\zeta)|.$$

Taking the supremum over $z \in \mathbb{D}$ and letting $j \to \infty$, $\|C_\varphi J_g f_j\|_{B^\beta_{\log}} \to 0$. Thus $C_\varphi J_g \colon B^\alpha_{\log} \to B^\beta_{\log}$ is compact. □

Theorem 2.11. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha \in (0,1)$, $\beta > 0$, then $J_g C_\varphi \colon B^\alpha_{\log} \to B^\beta_{\log}$ is bounded if and only if $J_g C_\varphi \colon B^\alpha_{\log} \to B^\beta_{\log}$ is compact if and only if $g \in B^\beta_{\log}$.*

Theorem 2.12. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\beta > 0$,. If*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g'(z)| \log \frac{2}{1-|z|^2} \log \frac{2}{1-|\varphi(z)|^2} < \infty,$$

then $J_g C_\varphi \colon B_{\log} \to B^\beta_{\log}$ is bounded.

Theorem 2.13. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\beta > 0$, if*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g'(z)| \log \frac{2}{1-|z|^2} < \infty$$

and

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)^\beta |g'(z)| \log \frac{2}{1-|z|^2} \log \frac{2}{1-|\varphi(z)|^2} = 0,$$

then $J_g C_\varphi : B_{\log} \to B^\beta_{\log}$ is compact.

Theorem 2.14. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha > 1$, $\beta > 0$. If*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |g'(z)| \log \frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha-1}} < \infty,$$

then $J_g C_\varphi \colon B^\alpha_{\log} \to B^\beta_{\log}$ is bounded.

Theorem 2.15. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha > 1$, $\beta > 0$. If*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(z)|\log\frac{2}{1-|z|^2} < \infty$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta|g'(z)|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha-1}} = 0,$$

then $J_gC_\varphi\colon B^\alpha_{\log}\to B^\beta_{\log}$ is compact.

Theorem 2.16. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha > 0$, $\beta > 0$,. If*

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{2}{1-|\varphi(z)|^2}} < \infty, \tag{2.13}$$

then $C_\varphi I_g\colon B^\alpha_{\log}\to B^\beta_{\log}$ is bounded.

Proof. By the definition of $C_\varphi I_g$, $(C_\varphi I_g f)'(z) = \varphi'(z)g(\varphi(z))f'(\varphi(z))$. For $f\in B^\alpha_{\log}$, we have

$$\begin{aligned}(1-|z|^2)^\beta(C_\varphi I_g f)'(z)\log\frac{2}{1-|z|^2}&\\ &\le \frac{(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{2}{1-|\varphi(z)|^2}}\|f\|_{B^\alpha_{\log}}.\end{aligned}$$

$$|(C_\varphi I_g f)(0)| = \left|\int_0^{\varphi(0)} f'(\zeta)g(\zeta)\mathrm{d}\zeta\right| \le C\|f\|_{B^\alpha_{\log}}\max_{|\zeta|\le|\varphi(0)|}|g(\zeta)|.$$

By (2.13), we have $C_\varphi I_g : B^\alpha_{\log}\to B^\beta_{\log}$ is bounded. □

Theorem 2.17. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha > 0$, $\beta > 0$,. If*

$$\sup_{z\in D}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))|\log\frac{2}{1-|z|^2} < \infty \tag{2.14}$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{2}{1-|\varphi(z)|^2}} = 0, \tag{2.15}$$

then $C_\varphi I_g\colon B^\alpha_{\log}\to B^\beta_{\log}$ is compact.

Proof. By (2.15), for any $\varepsilon > 0$, there exists an $r\in(0,1)$ such that

$$\frac{(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{2}{1-|\varphi(z)|^2}} < \varepsilon \tag{2.16}$$

for every $r < |\varphi(z)| < 1$.

Let $(f_j)_{j\in N}$ be a norm bounded sequence in $B^\alpha_{\log}$ such that $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Then

$$\begin{aligned}
\|C_\varphi I_g f_j\|_{B^\beta_{\log}} \leq & \sup_{|\varphi(z)|\leq r}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))||f_j'(\varphi(z))|\log\frac{2}{1-|z|^2}\\
&+\sup_{|\varphi(z)|>r}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))||f_j'(\varphi(z))|\log\frac{2}{1-|z|^2}\\
&+\max_{|\zeta|\leq|\varphi(0)|}|f_j'(\zeta)|\max_{|\zeta|\leq|\varphi(0)|}|g(\zeta)|\\
\leq & \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))|\log\frac{2}{1-|z|^2}\sup_{|\zeta|\leq r}|f_j'(\zeta)|\\
&+\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{2}{1-|\varphi(z)|^2}}\|f_j\|_{B^\alpha_{\log}}\\
&+\max_{|\zeta|\leq|\varphi(0)|}|f_j'(\zeta)|\max_{|\zeta|\leq|\varphi(0)|}|g(\zeta)|.
\end{aligned} \tag{2.17}$$

Since $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, by Cauchy's estimate, $f_j' \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Hence by (2.14), (2.16) and (2.17), we have $\|C_\varphi I_g f_j\|_{B^\beta_{\log}} \to 0$ as $j \to \infty$. Hence $C_\varphi I_g \colon B^\alpha_{\log} \to B^\beta_{\log}$ is compact. □

Theorem 2.18. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha > 0$, $\beta > 0$,. If*

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi'(z)||g(z)|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{2}{1-|\varphi(z)|^2}} < \infty,$$

then $I_g C_\varphi \colon B^\alpha_{\log} \to B^\beta_{\log}$ is bounded.

Theorem 2.19. *Assume that φ is a holomorphic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $\alpha > 0$, $\beta > 0$. If*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)||g(z)|\log\frac{2}{1-|z|^2} < \infty$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta|\varphi'(z)||g(z)|\log\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^\alpha\log\frac{2}{1-|\varphi(z)|^2}} = 0,$$

then $I_g C_\varphi \colon B^\alpha_{\log} \to B^\beta_{\log}$ is compact.

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References

1. Attele K. R., M. *Toeplitz and Hankel on Bergman one space*, Hokaido Math J., **21** (1992), 279–293.

2. Benke G., and Chang D. C., *A note on weighted Bergman spaces and the Cesàro operator*, Nagoya Mathematical J., **159** (2000), 25–43.

3. Cima J., Stegenda D., *Hankel operators on H^p*, in: Berkson E. R., Peck N. T.and Ulh J. (Eds.), Analysis at Urbana 1, London Math. Soc. Lecture Note ser., Cambridge Univ. Press, Cambridge, **137** (1989), 133–150.

4. Cowen C. C. and MacCluer B. D., *Composition operators on spaces of analytic functions*, Boca Roton: CRC Press, 1995.

5. Li H., Liu P. and Wang M., *Composition operators between generally weighted Bloch spaces of polydisk*, J. Inequal Pure Appl. Math, **8(3)** (2007), 1–8.

6. Li H. and Liu P., *Composition operators between generally weighted Bloch space and $Q^q_{\log}$ space.* Banach Journal of Mathematical Analysis, **3(1)** (2009), 99–110.

7. Li H. and Yang X., *Products of integral-type and composition operators from generally weighted Bloch space to $F(p,q,s)$ space.* Filomat, **23(3)** (2009), 231–241.

8. Ohno S., Stroethoff K. and Zhao R., *Weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math., **33(1)** (2003), 191–215.

9. Ramey W. and Ulrich D., *Bounded mean oscillation of Bloch pull-backs*, Math. Ann., **291** (1991), 591–606.

10. Siskakis A. and Zhao R. *A Volterra type operator on spaces of analytic functions*, Contemp. Math, **232** (1999), 299–311.

11. Yoneda R., *The composition operators on the weighted Bloch space*, Arch. Math., **78** (2002), 310–317.

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