Acta Math. Univ. Comenianae
Vol. LXXX, 1(2011), pp. 133–142

PRESERVATION OF TENSOR SUM AND TENSOR PRODUCT

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Abstract. This note deals with preservation of tensor sum and tensor product of Hilbert space operators. Basic operations with tensor sum are presented. The main result addresses to the problem of transferring properties from a pair of operators to their tensor sum and to their tensor product. Sufficient conditions are given to ensure that properties preserved by ordinary sum and ordinary product are preserved by tensor sum and tensor product, which are equally relevant for both finite-dimensional and infinite-dimensional spaces.

1. Introduction

Tensor sum and tensor product of Hilbert space operators can be thought of as an extension to infinite-dimensional spaces of the traditional Kronecker sum and Kronecker product of matrices on finite-dimensional spaces. For example, see [**2**, p. 238] and [**3**] where several finite-dimensional applications of both Kronecker sum and Kronecker product can be found. Let A and B be operators on Hilbert spaces. If $A \otimes B$ denotes their tensor product, then their tensor sum is given by

$$(A \otimes I) + (I \otimes B),$$

where I stands for the identity operator. Theoretical aspects of tensor sums have been considered in current literature. For instance, essential spectrum, as well as Weyl and Browder spectra, of tensor sums were investigated in [**10**]. Applications involving tensor sums have also been considered recently in [**5, 6, 7, 8**].

In this paper we are concerned with the problem of preserving properties by tensor sum and product. That is, properties of A and B that can be transferred to the tensor sum $(A \otimes I) + (I \otimes B)$ and to the tensor product $A \otimes B$. After considering some basic operations with tensor sum in Proposition 2 the main result is established in Theorem 1, where preservation by both tensor sum and tensor product is investigated. The compact case is treated in Theorem 2. Applications of Theorem 1 are considered in Corollaries 1 and 2 where, in particular, it is shown how proper contractiveness and strict positivity are both preserved by tensor product and tensor sum, respectively.

Received July 26, 2010; revised September 18, 2010.
2001 *Mathematics Subject Classification.* Primary 47A80; Secondary 47A45.
Key words and phrases. Tensor product; tensor sum; Hilbert space operators.

2. Preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be nonzero complex Hilbert spaces. We shall consider the concept of tensor product space in terms of the single tensor product of vectors as a conjugate bilinear functional on the Cartesian product of $\mathcal{H}$ and $\mathcal{K}$. (See, e.g., [**9**], [**18**] and [**19**] – for an abstract approach see, e.g., [**1**], [**4**] and [**21**].) The single tensor product of $x \in \mathcal{H}$ and $y \in \mathcal{K}$ is a conjugate bilinear functional $x \otimes y \colon \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ defined by $(x \otimes y)\,(u,v) = \langle x\,;u\rangle\;\langle y\,;v\rangle$ for every $(u,v) \in \mathcal{H} \times \mathcal{K}$. The tensor product space is the completion of the inner product space consisting of all (finite) sums of single tensors, which is a Hilbert space with respect to the inner product

$$\Big\langle \sum_i x_i \otimes y_i\,; \sum_j w_j \otimes z_j \Big\rangle = \sum_i \sum_j \langle x_i\,; w_j\rangle\;\langle y_i\,; z_j\rangle$$

for every $\sum_i x_i \otimes y_i$ and $\sum_j w_j \otimes z_j$ in $\mathcal{H} \otimes \mathcal{K}$. (The norm on $\mathcal{H} \otimes \mathcal{K}$ is the one generated by the above inner product.) By an operator on a normed space $\mathcal{X}$ we mean a bounded linear transformation of $\mathcal{X}$ into itself. Let $\mathcal{B}[\mathcal{X}]$ be the normed algebra (equipped with the induced uniform norm) of all operators on $\mathcal{X}$. The *tensor product* of two operators A in $\mathcal{B}[\mathcal{H}]$ and B in $\mathcal{B}[\mathcal{K}]$ is the transformation $A \otimes B \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ defined by

$$(A \otimes B) \sum_i x_i \otimes y_i = \sum_i Ax_i \otimes By_i \quad \text{for every} \quad \sum_i x_i \otimes y_i \in \mathcal{H} \otimes \mathcal{K},$$

which is an operator in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$. Although the tensor product is not a binary operation, it somehow deserves its name since it is distributive with respect to (ordinary) addition. Indeed, the proposition below states some of the basic operations with tensor product of Hilbert space operators (where A^* denotes the adjoint of A and $\|A\|$ the norm of A).

Proposition 1. *For every $\alpha, \beta \in \mathbb{C}$, $A, A_1, A_2 \in \mathcal{B}[\mathcal{H}]$ and $B, B_1, B_2 \in \mathcal{B}[\mathcal{K}]$,*

(a) $\alpha\,\beta\,(A \otimes B) = \alpha A \otimes \beta B$,

(b) $(A_1 + A_2) \otimes (B_1 + B_2) = A_1 \otimes B_1 + A_2 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_2$,

(c) $A_1 A_2 \otimes B_1 B_2 = (A_1 \otimes B_1)\,(A_2 \otimes B_2)$,

(d) $(A \otimes B)^* = A^* \otimes B^*$,

(e) $\|A \otimes B\| = \|A\|\,\|B\|$.

If A and B are invertible, then so is $A \otimes B$ and

(f) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

For an expository paper on tensor product (including a proof of Proposition 1), the reader is referred to [**12**].

3. Tensor Sum

Let A and B be arbitrary operators on $\mathcal{H}$ and on $\mathcal{K}$, respectively. An immediate consequence of Proposition 1(c) reads as follows.

$$A \otimes B = (A \otimes I)\,(I \otimes B) = (I \otimes B)\,(A \otimes I), \tag{1}$$

where the identity on $\mathcal{K}$ makes the tensor product with A and the identity on $\mathcal{H}$ makes the tensor product with B. Recall from Proposition 1(a) that

$$\alpha I \otimes \alpha^{-1} I = I \otimes I,$$

which is the identity operator on $\mathcal{H} \otimes \mathcal{K}$ for every nonzero scalar α, and

$$A \otimes O = O \otimes B = O \otimes O,$$

which is the null operator on $\mathcal{H} \otimes \mathcal{K}$, where the null operator on $\mathcal{K}$ makes the tensor product with A and the null operator on $\mathcal{H}$ makes the tensor product with B. The *tensor sum* of A and B is the transformation $A \boxplus B \colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ defined by

$$A \boxplus B = (A \otimes I) + (I \otimes B), \tag{2}$$

which is an operator in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$. (It is sometimes written $\oplus$ instead of $\boxplus$ but we reserve the symbol $\oplus$ for orthogonal direct sum, as usual.) When the tensor product (in a finite-dimensional setting) is identified with the Kronecker product of matrices, the correspondent expression in (2) is referred to as the Kronecker sum (see e.g., [**2**, p. 238] and [**3**]). This justifies the nomenclature *tensor sum.* However, it is worth noticing that the tensor sum is not commutative. Indeed,

$$A \boxplus O = A \otimes I, \qquad O \boxplus B = I \otimes B,$$

$$A \boxplus I = A \otimes I + I \otimes I, \qquad I \boxplus B = I \otimes I + I \otimes B.$$

In particular, if $\mathcal{H} = \mathcal{K}$ and $A = B$, then

$$A \boxplus O = A \otimes I \neq I \otimes A = O \boxplus A,$$

$$A \boxplus I = A \otimes I + I \otimes I \neq I \otimes I + I \otimes A = I \boxplus A.$$

Basic operations with tensor sum of Hilbert space operators are summarized in the next proposition. Its proof is straightforward, hence omitted.

Proposition 2. *For every $\alpha, \beta \in \mathbb{C}$, $A, A_1, A_2 \in \mathcal{B}[\mathcal{H}]$ and $B, B_1, B_2 \in \mathcal{B}[\mathcal{K}]$,*

(a) $(\alpha + \beta)(A \boxplus B) = \alpha A \boxplus \beta B + \beta A \boxplus \alpha B$,
(b) $(A_1 + A_2) \boxplus (B_1 + B_2) = A_1 \boxplus B_1 + A_2 \boxplus B_2$,
(c) $(A_1 \boxplus B_1)(A_2 \boxplus B_2) = A_1 \otimes B_2 + A_2 \otimes B_1 + A_1 A_2 \boxplus B_1 B_2$,
(d) $(A \boxplus B)^* = A^* \boxplus B^*$,
(e) $\|A \boxplus B\| \leq \|A\| + \|B\|$.

4. Preservation

The next theorem is the central result of this note. It gives sufficient conditions to ensure when a property that is preserved by ordinary product and by ordinary sum is also preserved by tensor product and tensor sum. For simplicity, we assume that the Hilbert spaces throughout this section are separable.

Theorem 1. *Let $\mathcal{C}'$ and $\mathcal{C}$ be classes of operators on Hilbert spaces such that*

(i) $\mathcal{C} \subseteq \mathcal{C}$,

(ii) *every operator unitary equivalent to an operator in $\mathcal{C}'$ or in $\mathcal{C}$ is an operator in $\mathcal{C}'$ or in $\mathcal{C}$, respectively, and*

(iii) *direct sum of countably many copies of an operator in $\mathcal{C}'$ or in $\mathcal{C}$ is an operator in $\mathcal{C}'$ or in $\mathcal{C}$, respectively.*

(a) *If the product of commuting operators acting on the same space, one in $\mathcal{C}'$ and the other in $\mathcal{C}$, is in $\mathcal{C}'$, then the tensor product of two operators, one in $\mathcal{C}'$ and the other in $\mathcal{C}$, is in $\mathcal{C}'$.*

(b) *If the sum of commuting operators acting on the same space, one in $\mathcal{C}'$ and the other in $\mathcal{C}$, is in $\mathcal{C}'$, then the tensor sum of two operators, one in $\mathcal{C}'$ and the other in $\mathcal{C}$, is in $\mathcal{C}'$.*

Proof. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Take A in $\mathcal{B}[\mathcal{H}]$ and B in $\mathcal{B}[\mathcal{K}]$. From (1),

$$A \otimes B = (A \otimes I)\,(I \otimes B) = (I \otimes B)\,(A \otimes I)$$

in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$, where the same notation I is used for the identity on $\mathcal{H}$ and on $\mathcal{K}$. Also recall that tensor product is unitarily equivalent commutative; that is, there exists a unitary transformation $\Pi\colon \mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$ such that

$$\Pi\,(A \otimes B) = (B \otimes A)\,\Pi$$

for every A in $\mathcal{B}[\mathcal{H}]$ and every B in $\mathcal{B}[\mathcal{K}]$, and so $\mathcal{H} \otimes \mathcal{K} \cong \mathcal{K} \otimes \mathcal{H}$ with $\cong$ denoting unitary equivalence. Now, since $\mathcal{H}$ and $\mathcal{K}$ are separable, the tensor products $I \otimes A$ on $\mathcal{K} \otimes \mathcal{H}$ and $I \otimes B$ on $\mathcal{H} \otimes \mathcal{K}$ are unitarily equivalent to the (countable) direct sums $\bigoplus_k A$ on $\bigoplus_k \mathcal{H}$ and $\bigoplus_k B$ on $\bigoplus_k \mathcal{K}$,

$$I \otimes A \cong \bigoplus_k A \quad \text{and} \quad I \otimes B \cong \bigoplus_k B,$$

through unitary transformations $\Phi_{\mathcal{H}}$ and $\Phi_{\mathcal{K}}$ that do not depend on A and B: There are unitary transformations $\Phi_{\mathcal{H}}\colon \bigoplus_k \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ and $\Phi_{\mathcal{K}}\colon \bigoplus_k \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ such that (see e.g., [**12**, Remark 5])

$$\Phi_{\mathcal{H}}(\bigoplus_k A) = (I \otimes A)\,\Phi_{\mathcal{H}} \quad \text{and} \quad \Phi_{\mathcal{K}}(\bigoplus_k B) = (I \otimes B)\,\Phi_{\mathcal{K}}$$

for every A in $\mathcal{B}[\mathcal{H}]$ and every B in $\mathcal{B}[\mathcal{K}]$, and so $\mathcal{K} \otimes \mathcal{H} \cong \bigoplus_k \mathcal{H}$ and $\mathcal{H} \otimes \mathcal{K} \cong \bigoplus_k \mathcal{K}$. (Note that if $\mathcal{H}$ is infinite-dimensional, then $\mathcal{H} \cong \ell^2_+$ and $\bigoplus_k \mathcal{K} = \ell^2_+(\mathcal{K})$; similarly, if $\mathcal{K}$ is infinite-dimensional, then $\mathcal{K} \cong \ell^2_+$ and $\bigoplus_k \mathcal{H} = \ell^2_+(\mathcal{H})$.) Therefore,

$$\begin{aligned} A \otimes B &= \Pi^*(I \otimes A)\,\Pi\,(I \otimes B) = \Pi^*\big[\Phi_{\mathcal{H}}\big(\textstyle\bigoplus_k A\big)\Phi_{\mathcal{H}}^*\big]\,\Pi\,\Phi_{\mathcal{K}}\big(\textstyle\bigoplus_k B\big)\Phi_{\mathcal{K}}^* \\ &= (I \otimes B)\,\Pi^*(I \otimes A)\,\Pi = \Phi_{\mathcal{K}}\big(\textstyle\bigoplus_k B\big)\Phi_{\mathcal{K}}^*\,\Pi^*\big[\Phi_{\mathcal{H}}\big(\textstyle\bigoplus_k A\big)\Phi_{\mathcal{H}}^*\big]\,\Pi \end{aligned}$$

and, by (2),

$$\begin{aligned} A \boxplus B = (A \otimes I) + (I \otimes B) &= \Pi^*(I \otimes A)\,\Pi + (I \otimes B) \\ &= \Pi^*\big[\Phi_{\mathcal{H}}\big(\textstyle\bigoplus_k A\big)\Phi_{\mathcal{H}}^*\big]\,\Pi + \Phi_{\mathcal{K}}\big(\textstyle\bigoplus_k B\big)\Phi_{\mathcal{K}}^*. \end{aligned}$$

Put

$$A' = \Pi^*\big[\Phi_{\mathcal{H}}\big(\textstyle\bigoplus_k A\big)\Phi_{\mathcal{H}}^*\big]\,\Pi \quad \text{and} \quad B' = \Phi_{\mathcal{K}}\big(\textstyle\bigoplus_k B\big)\Phi_{\mathcal{K}}^*$$

in $\mathcal{H} \otimes \mathcal{K}$ so that

$$A \otimes B = A'B' = B'A' \quad \text{and} \quad A \boxplus B = A' + B'.$$

Let $\mathcal{C}'$ and $\mathcal{C}$ be classes of operators satisfying assumptions (i) to (iii). Suppose A and B are in $\mathcal{C}$. If one of them is in $\mathcal{C}'$, then $A' = \Pi^*\big[\Phi_{\mathcal{H}}\big(\bigoplus_k A\big)\Phi_{\mathcal{H}}^*\big]\,\Pi$ and $B' = \Phi_{\mathcal{K}}\big(\bigoplus_k B\big)\Phi_{\mathcal{K}}^*$ are in $\mathcal{C}$, with one of them in $\mathcal{C}'$. Since these operators (that act on the same space $\mathcal{H} \otimes \mathcal{K}$) commute, we may infer the following results.

(a) If both A and B are in $\mathcal{C}$, with one of them in $\mathcal{C}'$, then A' and B' are in $\mathcal{C}$, with one of them in $\mathcal{C}'$. Since A' and B' commute and $A \otimes B = A'B'$, it follows that $A \otimes B$ lies in $\mathcal{C}'$ whenever the classes $\mathcal{C}'$ and $\mathcal{C}$ are such that the product of commuting operators, one in $\mathcal{C}'$ and the other in $\mathcal{C}$, is an operator in $\mathcal{C}'$.

(b) If both A and B are in $\mathcal{C}$, with one of them in $\mathcal{C}'$, then A' and B' are in $\mathcal{C}$, with one of them in $\mathcal{C}'$. Since A' and B' commute and $A \boxplus B = A' + B'$, it follows that $A \boxplus B$ lies in $\mathcal{C}'$ whenever the classes $\mathcal{C}'$ and $\mathcal{C}$ are such that the sum of commuting operators one in $\mathcal{C}'$ and the other in $\mathcal{C}$, is an operator in $\mathcal{C}'$. $\square$

5. Compact Case

The sufficient condition (iii) of Theorem 1 cannot be dismissed. However, If $\mathcal{C}$ stands for collection of all (bounded linear) operators and $\mathcal{C}'$ stands for the collection of all compact operators, both classes comprising operators acting on Hilbert spaces, then it is plain that the conditions (i) and (ii) are satisfied, but not condition (iii) – the identity on infinite-dimensional spaces is not compact, but is a countably infinite direct sum of compacts. Moreover, since the compact operators form a two-sided deal in $\mathcal{B}[\mathcal{H}]$, it follows that the hypothesis in both (a) and (b) of Theorem 1 are also satisfied. In fact, as we shall see below, *tensor product of compact operators is compact, but tensor sum of compact operators on infinite-dimensional spaces is not compact*. Recall that a quasinilpotent operator is one with a null spectral radius (i.e., one whose spectrum is equal to $\{0\}$), and a part of an operator is a restriction of it to an invariant subspace (by a subspace we mean a closed linear manifold).

Theorem 2. *If $A \in \mathcal{B}[\mathcal{H}]$ and $B \in \mathcal{B}[\mathcal{K}]$ are compact, then $A \otimes B \in \mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ is compact. Conversely, if $A \otimes B \in \mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$ is compact and one of $A \in \mathcal{B}[\mathcal{H}]$ or $B \in \mathcal{B}[\mathcal{K}]$ has a nonzero eigenvalue, then the other is compact.*

Proof. If A and B are compact on Hilbert spaces (and so on Banach spaces with Schauder bases), then they are uniform limits of sequences of finite-rank operators $\{A_n\}$ and $\{B_n\}$ (i.e., $A_n \xrightarrow{u} A$ and $B_n \xrightarrow{u} B$), and therefore $A_n \otimes B_n \xrightarrow{u} A \otimes B$ [**17**]. Since each A_n and each B_n is finite-rank, then so is each $A_n \otimes B_n$ (because $\operatorname{range}(A_n \otimes B_n) = \operatorname{range}(A_n) \otimes \operatorname{range}(B_n)$). Hence $A \otimes B$ is the uniform limit of a sequence of finite-rank operators, thus compact. Conversely, suppose $A \otimes B$ is compact and one of A or B, say B, has a nonzero eigenvalue λ. Take an arbitrary eigenvector e in the eigenspace $\ker(\lambda I - B)$, and consider the 1-dimensional subspace $[e]$ of $\mathcal{K}$ spanned by the eigenvector e, which is clearly B-invariant. Thus the regular subspace $\mathcal{H}\otimes[e]$ of $\mathcal{H}\otimes\mathcal{K}$ is $(A\otimes B)$-invariant and so (cf. [**14**] or [**15**]),

$$(A\otimes B)|_{\mathcal{H}\otimes[e]} = A\otimes\lambda \cong A,$$

where the unitary equivalence happens because $\lambda\neq 0$. Therefore A is compact since $(A\otimes B)|_{\mathcal{H}\otimes[e]}$ is compact (restriction of a compact to a subspace is compact). $\square$

However, if A and B are compact operators on infinite-dimensional spaces, then

$$A\boxplus B = (A\otimes I) + (I\otimes B)$$

may not be compact because both $(A\otimes I)$ and $(I\otimes B)$ are not compact if the identities act on infinite-dimensional spaces. For instance, $A=B=D=\operatorname{diag}(\{\frac{1}{j}\}_{j=1}^{\infty})$ on ℓ^2_+ is compact, but $I\otimes D\cong\bigoplus_k D$ is not compact, and so $D\otimes I\cong I\otimes D$ is not compact, and therefore $A\boxplus B=(D\otimes I)+(I\otimes D)$ is not compact.

Remark 1. If $A\otimes B\in\mathcal{B}[\mathcal{H}\otimes\mathcal{K}]$ is compact and one of $A\in\mathcal{B}[\mathcal{H}]$ or $B\in\mathcal{B}[\mathcal{K}]$ has a nonquasinilpotent compact part, then the other is compact.

This in fact is a corollary of the converse of Theorem 2. Indeed, if B has a nonquasinilpotent compact part, then there exists a nonzero subspace $\mathcal{M}$ of $\mathcal{K}$, which is B-invariant, such that $K=B|_{\mathcal{M}}$ in $\mathcal{B}[\mathcal{M}]$ is not quasinilpotent and compact. Since $\mathcal{M}$ is B-invariant, we get (see [**14**] or [**15**])

$$(A\otimes B)|_{\mathcal{H}\otimes\mathcal{M}} = A\otimes K,$$

which is compact because the restriction of a compact operator to a subspace is compact. Since K is compact but not quasinilpotent, it has a nonzero eigenvalue (Fredholm Alternative). Hence, by the converse of Theorem 2, A must be compact.

Also note that Theorem 2 is not a consequence of Theorem 1 since condition (iii) in Theorem 1 is not satisfied by compact operators.

6. Applications

A first application of Theorem 1 deals with tensor product of proper contractions. Recall that an operator T is a contraction if $\|T\|\leq 1$ (i.e., $\|Tx\|\leq\|x\|$ for every x). It is a proper contraction if $\|Tx\|<\|x\|$ for every nonzero x, and a strict contraction if $\|T\|<1$ (i.e., $\sup_{x\neq 0}(\|Tx\|/\|x\|)<1$). It is clear that every strict contraction is a proper contraction, every proper contraction is a contraction, and that these are proper inclusions in a infinite-dimensional space.

According to Proposition 1(e), the tensor product $A\otimes B$ is a contraction (or a strict contraction) if and only if $\|A\|\,\|B\|\leq 1$ (or $\|A\|\,\|B\|<1$). Thus it is trivially verified that if A in $\mathcal{B}[\mathcal{H}]$ and B in $\mathcal{B}[\mathcal{K}]$ are contractions, then so is $A\otimes B$ in $\mathcal{B}[\mathcal{H}\otimes\mathcal{K}]$ and, if in addition one of A or B is a strict contraction, then so is $A\otimes B$. However, a similar result for proper contractions does not follow at once from the norm identity in Proposition 1(e). Indeed, it can be verified that the tensor product of proper contractions is a proper contraction if and only if, for every nonzero finite sum of single tensors $\sum_{i=1}^N x_i \otimes y_i$,

$$\sum_{i=1}^N \sum_{j=1}^N \langle Ax_i\,;Ax_j\rangle\,\langle By_i\,;By_j\rangle < \sum_{i=1}^N \sum_{j=1}^N \langle x_i\,;x_j\rangle\,\langle y_i\,;y_j\rangle$$

whenever $\|Ax\| < \|x\|$ and $\|By\| < \|y\|$ for every nonzero x and y in $\mathcal{H}$ and $\mathcal{K}$ [**13**]. Actually, *the tensor product of proper contractions in fact is a proper contraction*, which can be verified as a corollary of Theorem 1(a).

Corollary 1. *Let A and B be operators acting on separable Hilbert spaces. If one of them is a contraction and the other is a proper contraction, then the tensor product $A \otimes B$ is a proper contraction.*

Proof. Observe that assumptions (i), (ii) and (iii) of Theorem 1 hold for contractions and proper contractions acting on separable Hilbert spaces. That is, it is readily verified that if $\mathcal{C}$ stands for the class of all contractions on separable Hilbert spaces and $\mathcal{C}'$ for the class of all proper contractions on separable Hilbert spaces, then assumptions (i), (ii) and (iii) hold true. Since the product (either left or right) of a contraction with a proper contraction (not necessarily commuting contractions) always is a proper contraction [**16**], it follows by Theorem 1(a) that $A \otimes B$ is a proper contraction whenever one of A or B is a contraction and the other is a proper contraction. □

A second application of Theorem 1 deals with tensor sum of strictly positive operators. Recall that a Hilbert space operator T is nonnegative if $0 \leq \langle Tx\,;x\rangle$ for every vector x (notation: $T \geq O$), positive if $0 < \langle Tx\,;x\rangle$ for every nonzero vector x (notation: $T > O$), and strictly positive if it is an invertible (with a bounded inverse) nonnegative operator (notation: $T \succ O$). Again, it is clear that every strictly positive operator is positive, every positive operator is nonnegative, and that these are proper inclusions in a infinite-dimensional space.

Consider a tensor sum $A \boxplus B$ in $\mathcal{B}[\mathcal{H} \otimes \mathcal{K}]$. Observe from (2) that

$$\begin{aligned}
&\Big\langle (A \boxplus B)\sum_{i=1}^N x_i \otimes y_i; \sum_{i=1}^N x_i \otimes y_i \Big\rangle \\
&\qquad = \Big\langle (A \otimes I)\sum_{i=1}^N x_i \otimes y_i; \sum_{i=1}^N x_i \otimes y_i \Big\rangle + \Big\langle (I \otimes B)\sum_{i=1}^N x_i \otimes y_i; \sum_{i=1}^N x_i \otimes y_i \Big\rangle \\
&\qquad = \sum_{i=1}^N \sum_{j=1}^N \langle Ax_i\,;x_j\rangle\,\langle y_i\,;y_j\rangle + \sum_{i=1}^N \sum_{j=1}^N \langle x_i\,;x_j\rangle\,\langle By_i\,;y_j\rangle
\end{aligned}$$

for every nonzero finite sum of single tensors $\sum_{i=1}^N x_i \otimes y_i$. Thus it can be verified that $A \boxplus B$ is nonnegative, positive or strictly positive if and only if

$$\sum_{i=1}^N \sum_{j=1}^N \langle Ax_i\,;x_j\rangle\,\langle y_i\,;y_j\rangle + \sum_{i=1}^N \sum_{j=1}^N \langle x_i\,;x_j\rangle\,\langle By_i\,;y_j\rangle$$

is nonnegative, positive or positive and bounded away from zero, respectively, for every nonzero finite sum of single tensors $\sum_{i=1}^{N} x_i \otimes y_i$. We apply Theorem 1(b) to show that *a tensor sum is nonnegative, positive or strictly positive if one of the summands is nonnegative and the other is nonnegative, positive, or strictly positive.*

Corollary 2. *Let A and B be operators acting on separable Hilbert spaces. The tensor sum $A \boxplus B$ is nonnegative, positive or strictly positive if one of A or B is nonnegative and the other is nonnegative, positive or strictly positive, respectively.*

Proof. Let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ denote the classes of all nonnegative, positive and strictly positive operators acting on separable Hilbert spaces, respectively. Assumptions (i), (ii) and (iii) of Theorem 1 hold for the pairs $(\mathcal{C}', \mathcal{C})$, $(\mathcal{C}'', \mathcal{C})$ and $(\mathcal{C}'', \mathcal{C}')$ of these three classes, where $\mathcal{C}'' \subset \mathcal{C}' \subset \mathcal{C}$. It is readily verified that the sum of nonnegative operators is again nonnegative, the sum of a nonnegative with a positive is a positive operator, and the sum of a nonnegative with a strictly positive is a strictly positive operator (see, e.g., [**11**, p. 430]). Thus, by Theorem 1(b), $A \boxplus B \geq O$, $A \boxplus B > O$ or $A \boxplus B \succ O$ if one of A or B is nonnegative (say $A \geq O$) and the other is nonnegative, positive or strictly positive (say $B \geq O$, $B > O$, or $B \succ O$) respectively. □

7. Final Remarks

Another application of Theorem 1 involves both tensor product and tensor sum of normal operators. Recall that a Hilbert space operator T is normal if it commutes with its adjoint (i.e., if $T^*T = TT^*$). Theorem 1 ensures that *normality is preserved by both tensor product and tensor sum.* Indeed, If $\mathcal{C} = \mathcal{C}'$ stands for the collection of all normal operators on separable Hilbert spaces, then it is clear that assumptions (i), (ii) and (iii) of Theorem 1 hold true. A corollary of the Fuglede–Putnam Theorem ensures that (ordinary) product and (ordinary) sum of commuting normal operators is again a normal operator (see, e.g., [**11**, p.508]). Therefore, both *$A \otimes B$ and $A \boxplus B$ are normal operators on $\mathcal{H} \otimes \mathcal{K}$ whenever A and B are normal operators on $\mathcal{H}$ and $\mathcal{K}$*, respectively, according to Theorem 1(a,b). However, this can be directly verified (without applying Theorem 1) as follows. By Proposition 1(c,d),

$$(A \otimes B)^*(A \otimes B) = (A^* \otimes B^*)\,(A \otimes B) = A^*A \otimes B^*B,$$

$$(A \otimes B)\,(A \otimes B)^* = (A \otimes B)\,(A^* \otimes B^*) = A\,A^* \otimes B\,B^*.$$

Moreover, by Proposition 2(c,d),

$$(A \boxplus B)^*(A \boxplus B) = (A^* \boxplus B^*)\,(A \boxplus B) = (A^* \otimes B) + (A \otimes B^*) + (A^*A \boxplus B^*B),$$

$$(A \boxplus B)\,(A \boxplus B)^* = (A \boxplus B)\,(A^* \boxplus B^*) = (A \otimes B^*) + (A^* \otimes B) + (A\,A^* \boxplus B\,B^*).$$

Therefore, if $A^*A = A\,A^*$ and $B^*B = B\,B^*$, then

$$(A \otimes B)^*(A \otimes B) = (A \otimes B)\,(A \otimes B)^* \text{ and } (A \boxplus B)^*(A \boxplus B) = (A \boxplus B)\,(A \boxplus B)^*.$$

More results on tensor products along this line can be found in [**20**] and [**12**].

It is also worth noticing on a possible attempt to generalize the results presented in this paper towards multiple tensor products and multiple tensor sums in the following sense. The tensor product of a pair of Hilbert spaces and of a pair of operators can be naturally extended to a finite collection of complex Hilbert spaces and to a finite collection of operators as follows. For any integer $m \geq 2$, let $\{\mathcal{H}_i\}_{i=1}^m$ be a finite collection of Hilbert spaces. The single tensor product of an m-tuple of vectors $(x_1, \ldots, x_m)$ with each x_i in $\mathcal{H}_i$ is the conjugate multilinear functional $\bigotimes_{i=1}^m x_i \colon \prod_{i=1}^m \mathcal{H}_i \to \mathbb{C}$ defined by $\big(\bigotimes_{i=1}^m x_i\big)(u_1, \ldots, u_m) = \prod_{i=1}^m \langle x_i\,; u_i\rangle$ for every $(u_1, \ldots, u_m) \in \prod_{i=1}^m \mathcal{H}_i$. The tensor product space $\bigotimes_{i=1}^m \mathcal{H}_i$ is the completion of the inner product space of all (finite) sums of single tensor products $\bigotimes_{i=1}^m x_{i,k}$ with $x_{i,k} \in \mathcal{H}_i$, which is again a Hilbert space with respect to the inner product

$$\left\langle \sum_k \bigotimes_{i=1}^m x_{i,k}\,; \sum_\ell \bigotimes_{i=1}^m w_{i,\ell} \right\rangle = \sum_k \sum_\ell \prod_{i=1}^m \langle x_{i,k}\,; w_{i,\ell}\rangle$$

for every $\sum_k \bigotimes_{i=1}^m x_{i,k}$ and $\sum_\ell \bigotimes_{i=1}^m w_{i,\ell}$ in $\bigotimes_{i=1}^m \mathcal{H}_i$. The tensor product $\bigotimes_{i=1}^m \mathcal{M}_i$ of subspaces $\mathcal{M}_i$ of $\mathcal{H}_i$ is a subspace of the tensor product space $\bigotimes_{i=1}^m \mathcal{H}_i$. This comes from the fact that if $\{h_{i,\gamma_i}\}_{\gamma_i \in \Gamma_i}$ is an orthonormal basis for each $\mathcal{H}_i$, then $\big\{\bigotimes_{i=1}^m h_{i,\gamma_i}\big\}_{(\gamma_1, \ldots, \gamma_m) \in \Pi_{i=1}^m \Gamma_i}$ is an orthonormal basis for $\bigotimes_{i=1}^m \mathcal{H}_i$ (see, e.g., [**21**, Theorem 3.12(b)]). The tensor product of a finite collection $\{A_i\}_{i=1}^m$ of operators, each A_i acting on $\mathcal{H}_i$, is given by

$$\left(\bigotimes_{i=1}^m A_i\right) \sum_k \bigotimes_{i=1}^m x_{i,k} = \sum_k \bigotimes_{i=1}^m A_i x_{i,k} \quad \text{for every} \quad \sum_k \bigotimes_{i=1}^m x_{i,k} \in \bigotimes_{i=1}^m \mathcal{H}_i.$$

This defines an operator in $\mathcal{B}[\bigotimes_{i=1}^m \mathcal{H}_i]$ with the following properties.

$$\left(\bigotimes_{i=1}^m A_i\right)^* = \bigotimes_{i=1}^m A_i^*, \quad \text{and} \quad \left(\bigotimes_{i=1}^m A_i\right)^{-1} = \bigotimes_{i=1}^m A_i^{-1}$$

if each A_i is invertible. Also

$$\left\|\bigotimes_{i=1}^m A_i\right\| = \prod_{i=1}^m \|A_i\|, \quad \text{and} \quad \left(\bigotimes_{i=1}^m A_i\right)\left(\bigotimes_{i=1}^m B_i\right) = \left(\bigotimes_{i=1}^m A_i B_i\right)$$

if $\{B_i\}_{i=1}^m$ is a collection of m operators with each B_i acting on each $\mathcal{H}_i$. Moreover, the multiple tensor product $\bigotimes_{i=1}^m A_i$ is promptly endowed with associativity, which means that

$$\bigotimes\nolimits_{i=1}^m A_i = \bigotimes\nolimits_{i=1}^{j-1} A_i \otimes A_j \otimes \bigotimes\nolimits_{i=j+1}^m A_i$$

for every integer $j \in [2, m-1]$ if $m > 2$. Thus the tensor product results in Theorems 1 and 2 may undergo a natural extension to a finite number of operators along the lines developed in [**14**]. Similarly, an extension for the tensor sum of a finite collection $\{A_i\}_{i=1}^m$ of operators will also enjoy associativity. Indeed, if each A_i acts on the same $\mathcal{H}$ for all $i = 1, 2, 3$, and if I denotes the identity on $\mathcal{H}$, then

$$(A_1 \boxplus A_2) \boxplus A_3 = A_1 \otimes I \otimes I + I \otimes A_2 \otimes I + I \otimes I \otimes A_3 = A_1 \boxplus (A_2 \boxplus A_3).$$

Thus a trivial induction leads to the following generalization of (2)

$$\boxplus_{i=1}^{m} A_i = \sum_{i=1}^{m} \left(\bigotimes_{j=1}^{i-1} I_j \right) \otimes A_i \otimes \left(\bigotimes_{j=i+1}^{m} I_j \right),$$

where I_j is the identity on each $\mathcal{H}_j$ and the empty tensor sum (i.e., $\bigotimes_{j=k}^{\ell} I_j$ for $\ell < k$) is always missing. So the multiple tensor product $\boxplus_{i=1}^{m} A_i$ is also entitled to be treated in light of Theorem 1. This and its possible outgrowths might be a promising suggestion for a future research.

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