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$\mathcal{I}$-CONVERGENCE TO A SET

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Abstract. We will deal with the sequences of points of a metric space. We will introduce $\mathcal{I}$-convergence to a set and give a sufficient condition to a sequence to be $\mathcal{I}$-convergent to a set. A connection between this "limit set" and the set of $\mathcal{I}$-cluster points is investigated.

Introduction

In the paper [**4**] the authors introduced the notion of Γ_2-statistical convergence to a set C for double sequences where some properties of C were required. There arose the question whether it is possible to do an analogous construction for usual sequences of points of a metric space considering $\mathcal{I}$-convergence, i.e. whether it is possible to define $\mathcal{I}$-convergence to a set for sequences of points of arbitrary metric space and whether some results of [**4**] can be obtained for $\mathcal{I}$-convergence to a set.

Notations and definitions

Let (X,ρ) be a metric space. We will use the following notations:

$$\begin{aligned} B(x,\varepsilon) &= \{y \in X : \rho(x,y) < \varepsilon\} && \text{for } x \in X \text{ and } \varepsilon > 0,\\ \rho(x,K) &= \inf\{\rho(x,y) : y \in K\} && \text{for } x \in X \text{ and } K \subset X,\\ B(K,\varepsilon) &= \{x \in X : \rho(x,K) < \varepsilon\} && \text{for } K \subset X \text{ and } \varepsilon > 0. \end{aligned}$$

Definition A. Let $\mathbb{N}$ be the set of positive integers. A non-void family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be a proper ideal in $\mathbb{N}$ if

(i) $A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$,
(ii) if $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$,
(iii) $\mathbb{N} \notin \mathcal{I}$.

(See [**2**].)

Definition B. A proper ideal $\mathcal{I}$ is said to be admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$. (See [**2**].)

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Definition C. Let $\mathcal{I}$ be an admissible ideal. A sequence $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X$ is said to be $\mathcal{I}$-convergent to $\xi \in X$ if for each $\varepsilon > 0$, $A(\varepsilon) = \{n : \rho(x_n, \xi) \geq \varepsilon\} \in \mathcal{I}$. (See [**2**].)

Definition D. A point $\xi \in X$ is said to be an $\mathcal{I}$-cluster point of a sequence $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X$ if for each $\varepsilon > 0$, the set $\{n : \rho(x_n, \xi) < \varepsilon\}$ does not belong to $\mathcal{I}$. The set of all $\mathcal{I}$-cluster points of the sequence $x = \{x_n\}_{n=1}^{\infty}$ is denoted by $\Gamma_x(\mathcal{I})$. (See [**1**].)

Definition E. A sequence $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X$, is said to be $\mathcal{I}$-bounded if there is a compact set $K \subset X$ such that the set $\{n : \rho(x_n, K) > 0\} \in \mathcal{I}$. (Cf. [**2**].)

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Definition 1. Let $\mathcal{I}$ be an admissible ideal and $x = \{x_n\}_{n=1}^{\infty}$ be a sequence, $x_n \in X$. Let $C \subset X$ be a non-void closed set with the following property

$$\{j \in \mathbb{N} : \rho(x_j, C) \geq \varepsilon\} \in \mathcal{I} \qquad \text{for each } \varepsilon > 0. \tag{1}$$

The set C is said to be the minimal closed set fulfilling (1) if for each closed set $C' \subset C$ such that $C \setminus C' \neq \emptyset$, the condition (1) does not hold. (Cf. [**4**].)

Definition 2. A sequence $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X$, is said to be $\mathcal{I}$-convergent to the set C if C is a non-void minimal closed set fulfilling (1). (Cf. [**4**].)

The next assertion is an easy consequence of Definition 2.

Lemma 1. *If a sequence $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X$, $\mathcal{I}$-converges to ξ, then x is $\mathcal{I}$-convergent to the set $C = \{\xi\}$.*

For some sequences there is no minimal closed set fulfilling (1). This shows the following example.

Example. Let $X = \mathbb{R}$ ($\mathbb{R}$ - the real line), $\mathcal{I} = \{A \subset \mathbb{N} : A \text{ is finite}\}$ and sequence $x = \{x_n\}_{n=1}^{\infty}$ be defined as follows: $x_n = n$. Every interval $[a, \infty)$, $a > 0$ fulfills condition (1). Since $\bigcap_{a>0} [a, \infty) = \emptyset$ there is no non-void minimal closed set fulfilling (1).

The next theorem gives a sufficient condition for a sequence to be $\mathcal{I}$-convergent to a set.

Theorem 1. *Let $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X$, be an $\mathcal{I}$-bounded sequence. Then it is $\mathcal{I}$-convergent to the set $\Gamma_x(\mathcal{I})$.*

The next assertion we will be used in the proof of Theorem 1.

Lemma 2. *If $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X$ is $\mathcal{I}$-bounded then $\Gamma_x(\mathcal{I})$ is a non-void compact set.*

Proof. $\mathcal{I}$-boundedness of $x = \{x_n\}_{n=1}^{\infty}$ implies the existence of a compact set K such that $\{j : \rho(x_j, K) > 0\} \in \mathcal{I}$. We show $\Gamma_x(\mathcal{I}) \subset K$. Suppose $\xi \in \Gamma_x(\mathcal{I}) \setminus K$.

Then there is $\varepsilon > 0$ such that $B(\xi,\varepsilon)\cap K = \emptyset$. Then $\{j : x_j \in B(\xi,\varepsilon)\} \subset \{j : x_j \notin K\}$ and $\{j : x_j \notin K\} \in \mathcal{I}$ imply $\{j : x_j \in B(\xi,\varepsilon)\} \in \mathcal{I}$ – a contradiction. It is known that $\Gamma_x(\mathcal{I})$ is a closed set (see [**1, 3**]). Hence $\Gamma_x(\mathcal{I}) \subset K$ is a compact set.

We show $\Gamma_x(\mathcal{I}) \neq \emptyset$ by contradiction. Let none of $\xi \in X$ be an $\mathcal{I}$-cluster point, i. e. for each $\xi \in K$ there exists $\varepsilon(\xi) > 0$ such that $\{j : \rho(x_j,\xi) < \varepsilon(\xi)\} \in \mathcal{I}$. The family $\{B(\xi,\varepsilon(\xi))\}_{\xi\in K}$ is an open cover of K. Since K is a compact there is n such that

$$K \subset \bigcup_{i=1}^{n} B(\xi_i,\varepsilon(\xi_i)) \quad \text{and} \quad \{j : x_j \in K\} \subset \bigcup_{i=1}^{n}\{j : x_j \in B(\xi_i,\varepsilon(\xi_i))\} \in \mathcal{I}.$$

Thus $\{j : x_j \notin K\} \notin \mathcal{I}$. This is a contradiction with $\mathcal{I}$-boundedness of $x=\{x_n\}_{n=1}^{\infty}$.

The proof of Lemma 2 is finished. □

Proof of Theorem 1. To complete the proof we show that:

a) $\Gamma_x(\mathcal{I})$ fulfills condition (1);
b) $\Gamma_x(\mathcal{I})$ is the minimal closed set fulfilling (1).

a) Since $\{x_n\}_{n=1}^{\infty}$ is $\mathcal{I}$-bounded, there is a compact K such that $\{j : x_j \notin K\} \in \mathcal{I}$ and $\Gamma_x(\mathcal{I}) \subset K$ (see the proof of Lemma 2). Let $\varepsilon > 0$. Put $M = K \cap (X \setminus B(\Gamma_x(\mathcal{I}),\varepsilon))$. Obviously M is a compact set and $\{B(\xi,\varepsilon(\xi))\}_{\xi\in M}$ is its open cover. ($\varepsilon(\xi)$ is such that $\{j\colon x_j \in B(\xi,\varepsilon(\xi))\} \in \mathcal{I}$ and $\varepsilon(\xi) < \varepsilon$ for each $\xi \in M$.) Hence there is a finite cover $\mathcal{S}$ of M,

$$\bigcup \mathcal{S} = P = \bigcup_{i=1}^{n} B(\xi_i,\varepsilon(\xi_i)) \supset M \quad \text{and} \quad X\setminus P \subset X \setminus M.$$

Let A' denote the complement of the set A. Then

$$X \setminus (K' \cup P) \subset B(\Gamma_x(\mathcal{I}),\varepsilon) \quad \text{and} \quad B(\Gamma_x(\mathcal{I}),\varepsilon)' \subset K' \cup P.$$

Hence

$$\begin{aligned}\{j : x_j \notin B(\Gamma_x(\mathcal{I}),\varepsilon)\} &\subset \{j : x_j \in K' \cup P\}\\ &\subset \{j : x_j \in K'\} \cup \bigcup_{i=1}^{n}\{j : x_j \in B(\xi_i,\varepsilon(\xi_i))\}.\end{aligned}$$

On the right hand side, there are $n + 1$ summands from $\mathcal{I}$ and consequently $\{j : x_j \notin B(\Gamma_x(\mathcal{I}),\varepsilon)\} \in \mathcal{I}$. Hence $\Gamma_x(\mathcal{I})$ fulfils (1).

b) We show that $\Gamma_x(\mathcal{I})$ is the minimal closed set fulfilling (1). Suppose that there is a closed set C, $C \subset \Gamma_x(\mathcal{I})$, such that $\Gamma_x(\mathcal{I}) \setminus C \neq 0$. Then for some $\xi \in \Gamma_x(\mathcal{I})$, there is $\varepsilon > 0$ such that $B(\xi,\varepsilon)\cap C = \emptyset$. Then

$$B\left(\xi,\frac{\varepsilon}{3}\right) \cap B\left(C,\frac{\varepsilon}{3}\right) = \emptyset \quad \text{and} \quad B\left(\xi,\frac{\varepsilon}{3}\right) \subset X \setminus B\left(C,\frac{\varepsilon}{3}\right).$$

Since $\xi \in \Gamma_x(\mathcal{I})$, we have

$$\left\{j : x_j \in B\left(\xi,\frac{\varepsilon}{3}\right)\right\} \notin \mathcal{I} \quad \text{and} \quad \left\{j : x_j \notin B\left(C,\frac{\varepsilon}{3}\right)\right\} \notin \mathcal{I}.$$

This shows minimality of $\Gamma_x(\mathcal{I})$. □

Theorem 2. *Let a sequence $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X$, be $\mathcal{I}$-convergent to a set C. Then $C = \Gamma_x(\mathcal{I})$.*

Proof. First we show that the inclusion $\Gamma_x(\mathcal{I}) \subset C$ holds. If there is $\xi \in \Gamma_x(\mathcal{I}) \setminus C$, then there exists $\varepsilon > 0$ such that $B(\xi,\varepsilon) \cap B(C,\varepsilon) = \emptyset$ and $B(\xi,\varepsilon) \subset X \setminus B(C,\varepsilon)$. So we have $\{j : x_j \in B(\xi,\varepsilon)\} \subset \{j : x_j \notin B(C,\varepsilon)\}$. Since ξ is an $\mathcal{I}$-cluster point, $\{j : x_j \in B(\xi,\varepsilon)\} \notin \mathcal{I}$ and also $\{j : x_j \notin B(C,\varepsilon)\} \notin \mathcal{I}$, we get a contradiction with the condition (1). Thus $\Gamma_x(\mathcal{I}) \subset C$.

By contradiction we show $C \subset \Gamma_x(\mathcal{I})$. Suppose $\xi \in C \setminus \Gamma_x(\mathcal{I})$. Then there is a $\delta > 0$ such that $\{j : x_j \in B(\xi,\varepsilon)\} \in \mathcal{I}$ holds for every ε, $0 < \varepsilon < \delta$. Let $\eta > 0$. Put $W = B(C \setminus B(\xi,\varepsilon),\eta)$, $Y = B(C,\eta)$ and $Z = B(\xi,\varepsilon)$. Then $X \setminus W \subset (X \setminus Y) \cup Z$. C is the minimal closed set satisfying (1) $\{j : x_j \notin Y\} \in \mathcal{I}$ and $\{j : x_j \in Z\} \in \mathcal{I}$ by our choice of δ. Consequently $\{j : x_j \notin W\} \in \mathcal{I}$. This is a contradiction with the minimality of the closed set C satisfying (1) since $C \setminus B(\xi,\varepsilon) \subsetneq C$.

The proof is finished. $\square$

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