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ON SQUARE ITERATIVE ROOTS OF MULTIFUNCTIONS

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Abstract. The most known results on iterative roots are given under the assumption of continuity and monotonicity. In 2009, Li, Jarczyk, Jarczyk and Zhang considered the existence of square iterative roots of multifunctions with exactly one set-valued point. They gave a full description of these roots for strictly monotone, upper semicontinuous multifunctions. In this paper, we investigate the square roots of multifunctions without monotonicity.

1. Introduction

Given a set X and an integer $n > 0$, consider a map $F : X \to X$. An iterative root of order n of F is a map $f : X \to X$ such that

$$f^n = F, \tag{1.1}$$

where f^n denotes the n-th iterate of f, i.e., $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) \equiv x$ for any $x \in X$. Both iterates and iterative roots of one-dimensional maps are important subjects in the theory of functional equations and the theory of dynamical systems. When F is strictly monotone, many results are given, e.g., in [**1, 2, 4, 5, 7, 6, 10**]. The following results are well-known [**7**]:

Lemma 1. (Theorem 11.1.1 in [7]). *Let $f : X \to X$ be a solution of* (1.1). *Then f is surjective (resp. injective, bijective) if and only if F is surjective (resp. injective, bijective).*

We define $E_k[F]$, where k is positive a integer, as the set of those $x \in X$ for which there exists an integer $j \geq 0$ such that

$$F^{j+k}(x) = F^j(x). \tag{1.2}$$

The remainder of the set X, i.e., the set of $x \in X$ which do not fulfill (1.2) for any j and k, will be denoted by $E_0[F]$.

Lemma 2. (Theorem 15.6 in [4]) *Let F be one-to-one map of a set X onto itself and let L_k be the number of orbits under F in $E_k[F]$. In order that* (1.1) *has a solution in X it is necessary and sufficient that, for every k, L_k is either infinite or divisible by d_k, where $d_0 = n$ and, for $k \geq 1$, $d_k = n/m(n, k)$, $m(n, k)$ being the largest divisor of n that is prime to k.*

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For a multifunction $f : X \to 2^X$, its image $f(A)$ of a set $A \subset X$ is defined by $f(A) = \bigcup_{x\in A} f(x)$ and its n-th iterate f^n by the composition of n copies of f

$$f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}.$$

A point $c \in X$ is called a *set-valued point* if the cardinal $\# f(x) > 1$.

In [**3**], the authors discussed a class of multifunctions with lack of iterative roots. And the existence of square iterative roots of monotone multifunctions were considered in [**8**]. In this paper, we continue [**3, 8**] by concerning the purely set-theoretical situation and investigate the square roots of multifunctions without monotonicity.

In what follows, we consider a class of multifunctions $F : X \to 2^X$ of the form

$$F(x) = \begin{cases} F_1(x), & \text{if } x \in X \setminus \{c\}, \\ M, & \text{if } x = c, \end{cases} \tag{1.3}$$

where $c \in X$, $M \subset X$ and F_1 is a bijection satisfying the sufficient condition of Lemma 2. Define $M^* = M \setminus \{c\}$. Clearly, $M^* = M$ when $c \notin M$. Furthermore, by Lemma 2, if F has a square iterative root, it should be in the form of (1.3).

Theorem 1. *Let $F : X \to 2^X$ be the form of* (1.3) *that is nearly bijective with exceptional point c. If $c \notin M$, then F has a square iterative root. Otherwise, F has a square root iff one of the following conditions satisfies*
(1) $f_1(M^*) \subseteq M^*$,
(2) $M^* \subseteq f_1(M^*)$,
(3) $M^* \cap f_1(M^*) = N \subsetneq M^*$ *and* $f_1^{-1}(N) \cup N = M^*$,
where f_1 is a square iterative root of F_1 and $f(c) = M = M^* \cup \{c\}$.

Proof. Suppose that f_1 is a square iterative root of F_1, which implies from Lemma 1 that $f_1 : X \setminus \{c\} \to X \setminus \{c\}$ is a one-to-one map.

In the case $M = M^*$, i.e., $c \notin M$, then F has an iterative square root $f : X \to 2^X$ as followes.

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X \setminus \{c\}, \\ f_1^{-1}(M), & \text{if } x = c. \end{cases}$$

In the other case, i.e., $c \in M$, we note that $c \in f(c)$. Based on this, we have the following three subcases:
(3-1) $M^* \subseteq f_1(M^*)$,
(3-2) $f_1(M^*) \subseteq M^*$,
(3-3) $M^* \cap f_1(M^*) = N$, $N \subset M^*$ is non-empty.

The case of (3-1) implyies that $f_1^{-1}(M^*) \subseteq M^*$. Therefore

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X \setminus \{c\}, \\ f_1^{-1}(M^*) \cup \{c\}, & \text{if } x = c, \end{cases}$$

is a square iterative root of F.

We may also choose a subset $N \subseteq M^*$ such that $f_1(N) = M^*$. Then f can be defined by

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X \setminus \{c\}, \\ N \cup \{c\}, & \text{if } x = c. \end{cases}$$

In the case (3-2), obviously,

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X \setminus \{c\}, \\ M^* \cup \{c\}, & \text{if } x = c, \end{cases}$$

is a square iterative root of F.

In the case of (3-3), since $N \subset f_1(M^*)$, it implies that $f_1^{-1}(N) \subset M^*$. Therefore, $f_1^{-1}(N) \cup N \subseteq M^*$. If $f_1^{-1}(N) \cup N = M^*$, then F has an iterative square root f given by

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X \setminus \{c\}, \\ f_1^{-1}(N) \cup \{c\}, & \text{if } x = c. \end{cases}$$

Otherwise, i.e.,

$$f_1^{-1}(N) \subsetneq N \subset M^*. \tag{1.4}$$

Assume that F has a square iterative root f and $f(c) = M_1^* \cup \{c\}$. Then by iterating, it should satisfy the equality

$$f_1(M_1^*) \cup M_1^* = M^*. \tag{1.5}$$

It follows that $M_1^* \subseteq M^*$ and $f_1(M_1^*) \subseteq M^*$. If $N = \emptyset$, then for any subset $N^* \subseteq M^*$ that $f_1(N^*) \cap M^* \subseteq f_1(M^*) \cap M^* = \emptyset$. Therefore, F has no square iterative roots. On the other hand, i.e., $N \neq \emptyset$, we may choose the biggest subset $N^* \subset M^*$ that $f_1(N^*) \subseteq M^*$. Obviously,

$$N^* = f_1^{-1}(N). \tag{1.6}$$

Otherwise, suppose that $N^* = f_1^{-1}(N) \cup S$ with a nonempty set $S \subset M^*$ and $f_1^{-1}(N) \cap S = \emptyset$. Then $f_1(N^*) \subseteq N \cup f_1(S) \subseteq M^*$, which implies that $f_1(S) \subseteq M^* \cap f_1(M^*) = N$. Hence $S \subseteq f_1^{-1}(N)$ follows that (1.6) is found. Since $N^* \subset M^*$ is the biggest subset that satisfies $f_1(N^*) \subseteq M^*$, it implies that $f(c) \subseteq N^* \cup \{c\}$, which gives

$$F(c) = f(f(c)) \subseteq N \cup N^* \cup \{c\} = N \cup f_1^{-1}(N) \cup \{c\} \subsetneq M^* \cup \{c\}$$

in the view of (1.4). Hence, $F(c) \subsetneq M^* \cup \{c\}$ is a contraction. Thus, in the last subcase, there is no subset in M^* that fulfills (1.5). Therefore, F has no iterative square roots when $f_1^{-1}(N) \cup N \subsetneq M^*$. The proof is completed. □

Corollary 1. *Let F be the form of* (1.3). *Suppose that f_1 is an arbitrary square iterative root of F_1 on $X \setminus \{c\}$. If M^* is the set of all fixed points of F, then F has a square iterative root.*

Proof. Firstly, we define a multifunction $f\colon X \to 2^X$ in the form of (1.3), where f_1 is an arbitrary square root of F_1 on $X \setminus \{c\}$. Then we claim that

$$f_1(M^*) \subseteq M^*.$$

Otherwise, there exists a point $x_0 \in M^*$ that $f_1(x_0) = y_0 \notin M^*$ with $x_0 \neq y_0$. Thus

$$F_1(y_0) = F_1(f_1(x_0)) = f_1(F_1(x_0)) = f_1(x_0) = y_0,$$

which is a contradiction. Therefore

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in I \setminus \{c\}, \\ M^* \cup \{c\}, & \text{if } x = c, \end{cases}$$

is a square iterative root of F. $\square$

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