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SPECIAL CONGRUENCE TRIPLES FOR A REGULAR SEMIGROUP

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Abstract. With the usual notation for congruences on a regular semigroup S, in a previous communication we studied the lattice Λ generated by $\Gamma = \{\sigma, \tau, \mu, \beta\}$ relative to properties such as distributivity and similar conditions. For K and T the kernel and trace relations on the congruence lattice of S, we form an abstraction of the triple $(\Lambda; K|_\Lambda, T_\Lambda)$ called a c-triple. In this study a number of relations on the free lattice generated by Γ appears. Here we study implications and independence of these relations, both on c-triples as well as on congruence lattices of regular semigroups. We consider the behavior of the members of Γ under forming finite direct products, construct examples and supplement some results in the paper referred to above.

1. Introduction and summary

In [**7**] we considered an abstraction of the following situation. Let S be a regular semigroup, $\mathscr{C}(S)$ be its congruence lattice and

$$\Gamma = \{\sigma, \tau, \mu, \beta\}$$

where σ is the least group, τ is the greatest idempotent pure, μ is the greatest idempotent separating, and β is the least band congruence on S, respectively. Let Λ be the sublattice of $\mathscr{C}(S)$ generated by Γ. Let K and T be the kernel and trace relations on $\mathscr{C}(S)$, respectively. The abstraction of the triple $(\Lambda; K|_\Lambda, T|_\Lambda)$ is called a *c-triple*, that is $(\Lambda; K, T)$ where Λ is a lattice generated by a 4-element set Γ, and K, T are relations on Λ, all of these satisfying certain conditions. The subject of that paper comprizes the following cases: any three elements of Γ generate a distributive lattice, Λ is distributive, K is a congruence, and a further special case. In paper [**7**] we explained the background of this problem within the theory of congruences on regular semigroups.

The subject of the present paper is a study of the relations occurring in consideration of c-triples in the abstract setting as well as in concrete cases which arise in regular semigroups. This pertains mainly to their independence and implications.

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A minimum of terminology and notation can be found in Section 2 for not which we relegate most of it to the paper discussed above. Section 3 contains a study of the behavior of the congruences σ, τ, μ and β relative to forming finite direct products of semigroups. This is followed in Section 4 by several lemmas needed later. Section 5 consists of a number of examples. All this serves as a preparation for results in Section 6 which concern certain morphisms in the preceding paper and represent the main part of the paper. Section 7 contains three diagrams which exhibit the independence of certain basic relations. The paper is concluded by Section 8 with a discussion of some problems naturally arising in this context.

2. Terminology and notation

For concepts and symbolism we generally follow the book [**2**]. We now list some special terminology and notation.

Let X be a set. The equality and the universal relations on X are denoted by ε_X and ω_X, respectively, with or without subscript, its cardinality by $|X|$.

Let S be a semigroup. Then $E(S)$ denotes its set of idempotents and $\mathscr{C}(S)$ its congruence lattice. The identity element of a monoid is usually denoted by e. If S has a zero 0 and A is a subset of S, then $A^* = A \setminus \{0\}$. Further, S^1 denotes the semigroup S with an identity element adjoined if S is not a monoid, otherwise $S^1 = S$.

For a regular semigroup S, the congruence defined in Section 1 is written with a subscript, that is

$$\Gamma_S = \{\sigma_S, \tau_S, \mu_S, \beta_S\}$$

and the lattice they generate by Λ_S, only if needed for clarity. In this context, K and T denote the K- and T-relations on $\mathscr{C}(S)$, respectively, without subscript. This is the concrete aspect of this symbolism. In fact, we will be interested in the restrictions $K|_{\Lambda_S}$ and $T|_{\Lambda_S}$ only. The abstract meaning of the symbols σ, τ, μ, β is that they are letters standing for generators of a lattice Λ, in this case we use the notation $\Gamma_\Lambda = \{\sigma, \tau, \mu, \beta\}$.

We call $(\Lambda; K, T)$ a *c-triple* [**7**, Definition 2.1] if Λ is a lattice generated by Γ_Λ with the least element ε and greatest element ω, K is a $\wedge$-congruence and T is a congruence on Λ satisfying: $K \cap T = \varepsilon_\Lambda$, $[\varepsilon, \tau]$ and $[\beta, \omega]$ are K-classes, $[\varepsilon, \mu]$ and $[\sigma, \omega]$ are T-classes, and

$$\beta \wedge (\sigma \vee \mu) = \mu \vee (\sigma \wedge \beta) \Longrightarrow (\sigma \vee \mu) \wedge (\tau \vee \beta) = \tau \vee (\sigma \wedge \beta) \vee \mu.$$

It is easy to verify that for a regular semigroup and the lattice Λ generated by $\{\sigma_S, \tau_S, \mu_S, \beta_S\}$, $K|_\Lambda$ and $T|_\Lambda$, these conditions are satisfied.

For various purposes, the elements of Γ_Λ will be subject to some of the following conditions.

(A) $\tau \leq \sigma$.
(B) $\mu \leq \beta$.
(C) $\sigma \wedge (\tau \vee \beta) = \tau \vee (\sigma \wedge \beta)$.
(D) $\beta \wedge (\sigma \vee \mu) = \mu \vee (\sigma \wedge \beta)$.
(E) $\sigma \wedge (\tau \vee \mu) = \tau \vee (\sigma \wedge \mu)$.

(F) $\beta \wedge (\tau \vee \mu) = \mu \vee (\tau \wedge \beta)$.
(G) $\sigma \wedge (\tau \vee \mu) \wedge \beta = (\sigma \wedge \mu) \vee (\tau \wedge \beta)$.
(H) $(\sigma \vee \mu) \wedge (\tau \vee \beta) = \tau \vee (\sigma \wedge \beta) \vee \mu$.
(I) $\sigma \vee \mu \, K \, \mu$.
(J) $\sigma \wedge \mu \, K \, \sigma$.
(K) $\tau \vee \mu \, K \, \mu$.
(L) $\tau \vee \beta \, T \, \tau$.

The negation of a condition (X) is denoted by $\sim$(X). The above conditions do not appear until the end of Section 4. After that they play a central role. We emphasize that they occur in two ways: abstractly as conditions for c-triples and concretely for various regular semigroups.

The results of [**7**] do not appear until Section 6 and the paper is self-contained until that point. After that they are used in an essential way. Since the notation and statements needed from that paper are quite extensive, we do not repeat them here but refer to them by exact reference.

3. Finite direct products

This section serves as the first of three sections which are needed in the main body of the paper.

For $i = 1, 2, \ldots, n$, let S_i be a semigroup and $\rho_i \in \mathscr{C}(S_i)$. On the direct product $S = \prod_{i=1}^{n} S_i$ define a relation by

$$(a_i) \prod_{i=1}^{n} \rho_i \, (b_i) \Longleftrightarrow a_i \, \rho_i \, b_i \text{ for } i = 1, 2, \ldots, n.$$

For $n = 2$, we write $S_1 \times S_2$ and $\rho_1 \times \rho_2$.

The first lemma is valid for general semigroups.

Lemma 3.1. *For $i = 1, 2, \ldots, n$, let S_i be a semigroup, $\lambda_i, \rho_i \in \mathscr{C}(S_i)$ and $\oplus \in \{\wedge, \vee\}$. Then*

$$\prod_{i=1}^{n} \lambda_i \oplus \prod_{i=1}^{n} \rho_i = \prod_{i=1}^{n} (\lambda_i \oplus \rho_i).$$

Proof. The assertion is trivial for $n = 1$. We now consider the case $n = 2$. In order to simplify the notation, let S and S' be semigroups, $\lambda, \rho \in \mathscr{C}(S)$ and $\lambda', \rho' \in \mathscr{C}(S')$. Then

$$\begin{aligned}
(a,p) \ (\lambda \wedge \rho) \times (\lambda', \rho') \ (b,q) &\Longleftrightarrow a \ \lambda \wedge \rho \ b, \quad p \ \lambda' \wedge \rho' \ q \\
&\Longleftrightarrow a \, \lambda \, b, \quad a \, \rho \, b, \quad p \, \lambda' \, q, \quad p \, \rho' \, q \\
&\Longleftrightarrow (a,p) \ \lambda \times \lambda' \ (b,p), \quad (a,p) \ \rho \times \rho' \ (b,p) \\
&\Longleftrightarrow (a,p) \ (\lambda \times \lambda') \wedge (\rho \times \rho') \ (b,p)
\end{aligned}$$

which takes care of meet. Further

$$(a,p) \ (\lambda \vee \rho) \times (\lambda' \vee \rho') \ (b,q) \quad \Longleftrightarrow \quad a \ \lambda \vee \rho \ b, \quad p \ \lambda' \vee \rho' \ q$$

$\Longleftrightarrow$ there exist sequences in S and S'

$$a\,\lambda\,x_1\,\rho\,x_2\,\lambda\,\ldots\,x_m\,\rho\,b, \quad p\,\lambda'\,y_1\,\rho'\,y_2\,\lambda'\,\ldots\,y_n\,\rho'\,q.$$

If $m \neq n$, we can repeat some of x_i or y_i in order to achieve sequences of this type with $m = n$. Hence we may assume that $m = n$. It follows that

$$(a,p)\ \lambda\times\lambda'\ (x_1,y_1)\ \rho\times\rho'\ (x_2,y_2)\ \lambda\times\lambda'\ \cdots\ (x_n,y_n)\ \rho\times\rho'\ (b,q) \tag{3.1}$$

and thus $(a,p)\ (\lambda\times\lambda')\vee(\rho\times\rho')\ (b,q)$. Conversely, if we assume the last relation, we obtain a sequence of the form (3.1) and by reversing our steps (without the complication of comparing m and n), we deduce that the assertion also holds for the join.

This proves the case $n = 2$. The general case follows from it by straightforward induction. $\square$

The next theorem, valid for regular monoids and of its own interest, pertains to the congruences in $\Gamma = \{\sigma,\tau,\mu,\beta\}$ where for a semigroup S, we write again $\Gamma_S = \{\sigma_S,\tau_S,\mu_S,\beta_S\}$. Note the obvious fact that for any semigroups S_i where $\prod$ indicates the Cartesian product of sets.

Theorem 3.2. *For $i = 1,2,\ldots,n$, let S_i be a regular monoid, $S = \prod_{i=1}^n S_i$ and $\theta \in \Gamma$. Then $\theta_S = \prod_{i=1}^n \theta_{S_i}$.*

Proof. The assertion is trivial for $n = 1$. We now treat the case $n = 2$. To simplify the notation we consider regular monoids S and V and their direct product.

$\theta = \sigma$. Note that σ is the least idempotent identifying congruence on a regular semigroup. Hence

$$(a,p)\,\sigma_{S\times V}\,(b,q) \Longleftrightarrow \text{there exists a sequence in } S\times V,$$

$$\left.\begin{aligned} (a,p) &= (s_1,x_1)(c_1,w_1)(t_1,y_1)\\ (s_1,x_1)(d_1,z_1)(t_1,y_1) &= (s_2,x_2)(c_2,w_2)(t_2,y_2)\\ &\cdots\\ (s_n,x_n)(d_n,z_n)(t_n,y_n) &= (b,q)\end{aligned}\right\} \tag{3.2}$$

where $(s_i,x_i),(t_i,y_i) \in S\times V$ and $(c_i,w_i),(d_i,z_i) \in E(S\times V)$ for $i = 1,2,\ldots,n$. It follows that

$$a = s_1c_1t_1,\ s_1d_1t_1 = s_2c_2t_2,\ \ldots\ s_nd_nt_n = b, \tag{3.3}$$

$$p = x_1w_1y_1,\ x_1z_1y_1 = x_2w_2t_2,\ \ldots\ x_nz_ny_n = q, \tag{3.4}$$

where $s_i,t_i \in S$, $c_i,d_i \in E(S)$, $x_i,y_i \in V$, $w_i,z_i \in E(V)$ for $i = 1,2,\ldots,n$, and thus $a\,\sigma_S\,b$, $p\,\sigma_V\,q$ whence $(a,p)\ \sigma_S\times\sigma_V\ (b,q)$. Conversely, if the last relation holds, then we have sequences of the forms (3.3) and (3.4) of possibly different length. Repeating some of the elements of these sequences, we see that we may suppose that they are of the same length. In this way we arrive at a sequence of the form (3.2) which yields that $(a,p)\,\sigma_{S\times V}\,(b,q)$. Therefore $\sigma_{S\times V} = \sigma_S\times\sigma_V$.

$\theta = \beta$. Observe that β is the least congruence which identifies each element of a regular semigroup with its square. We may thus follow the same steps as above except that instead of $(c_i,w_i),(d_i,z_i) \in E(S\times V)$ we now have for some $(g_i,h_i) \in S\times V$ for $i = 1,2,\ldots,n$.

$\theta = \mu$. Recall that μ is the greatest idempotent separating congruence on a regular semigroup. It follows at once that

$$(a,p)\,\mathcal{H}_{S\times V}\,(b,q) \Longleftrightarrow a\,\mathcal{H}_S\,b,\ p\,\mathcal{H}_V\,q$$

and thus

$$\begin{aligned}
& (a,p)\,\mu_{S\times V}\,(b,q) \\
\Longleftrightarrow\quad & (s,x)(a,p)(t,y)\,\mathcal{H}_{S\times V}\,(s,x)(b,q)(t,y) \text{ for all } (s,x),(t,y)\in S\times V \\
\Longleftrightarrow\quad & (sat,xpy)\,\mathcal{H}_{S\times V}\,(sbt,xqy) \text{ for all } (s,x),(t,y)\in S\times V \\
\Longleftrightarrow\quad & sat\,\mathcal{H}_S\,sbt \text{ for all } s,t\in S,\ xpy\,\mathcal{H}_V\,xqy \text{ for all } x,y\in V \\
\Longleftrightarrow\quad & a\,\mu_S\,b,\ p\,\mu_V\,q \Longleftrightarrow (a,p)\ \mu_S\times\mu_V\ (b,q),
\end{aligned}$$

as required.

$\theta = \tau$. Recall that τ is the greatest idempotent pure congruence on a regular semigroup and hence the principal congruence on the set of its idempotents. On the one hand,

$$\begin{aligned}
& (a,p)\,\tau_{S\times V}\,(b,q) \\
\Longleftrightarrow\ & ((s,x)(a,p)(t,y)\in E(S\times V) \Leftrightarrow (s,x)(b,q)(t,y)\in E(S\times V) \\
& \quad \text{for all } (s,x),(t,y)\in E(S\times V)) \\
\Longleftrightarrow\ & ((sat,xpy)\in E(S\times V) \Leftrightarrow (sbt,xqy)\in E(S\times V) \\
& \quad \text{for all } (s,x),(t,y)\in E(S\times V)) \\
\Longleftrightarrow\ & (sat\in E(S), xpy\in E(V) \Leftrightarrow sbt\in E(S), xqy\in E(V) \\
& \quad \text{for all } s,t\in S, x,y\in V)
\end{aligned}$$

and on the other hand

$$\begin{aligned}
& (a,p)\ \tau_S\times\tau_V\ (b,q) \\
\Longleftrightarrow\quad & \begin{cases} sat\in E(S)\Leftrightarrow sbt\in E(S) \text{ for all } s,t\in S, \\ xpy\in E(V)\Leftrightarrow xqy\in E(V) \text{ for all } x,y\in V. \end{cases}
\end{aligned}$$

It then follows that $\tau_S\times\tau_V\subseteq\tau_{S\times V}$. Conversely, let (3.5) hold and assume that $sat\in E(S)$. For x the identity of V and y an inverse of p, we have $xpy\in E(V)$ which by (3.5) yields $sbt\in E(S)$. By symmetry, we conclude that $a\,\tau_S\,b$. It follows similarly that $p\,\tau_V\,q$ which implies that $(a,p)\ \tau_S\times\tau_V\ (b,q)$. Therefore $\tau_{S\times V}\subseteq\tau_S\times\tau_V$ and equality prevails.

This establishes the case $n=2$. The general case follows by simple induction using the case $n=2$. □

The above proof can be easily modified if both S and V are semigroups but neither is a monoid. Then the identity adjoined to $S\times V$ can be written as $(1,1)$ where the first one can be considered as an adjoined identity of S and the second one as an adjoined identity of V. The problem arises when one of S and V is a monoid and the other one is not.

We can now use Lemma 3.1 and Theorem 3.2 to prove the following statement.

Corollary 3.3. *Let w be an element of the free lattice generated by Γ and for any regular semigroup V set $w_V = w(\sigma_V, \tau_V, \mu_V, \beta_V)$. For $i = 1, 2, \dots, n$, let S_i be a regular monoid and set $S = \prod_{i=1}^n S_i$. Then $w_S = \prod_{i=1}^n w_{S_i}$.*

Proof. For the special case when $w \in \Gamma$, the present assertion reduces to that of Theorem 3.2. Using this, Lemma 3.1 implies that meets and joins of congruences σ_S, τ_S, μ_S, β_S can be performed componentwise. Using the same lemma, we can repeat performing meets and joins of the resulting congruences again by components. This can be repeated as many times as necessary until the desired conclusion is reached. □

For kernels, we have the following simple result.

Lemma 3.4. *For $i = 1, 2, \dots, n$, let S_i be a regular semigroup, $\rho_i \in \mathscr{C}(S_i)$ and set $S = \prod_{i=1}^n S_i$. Then*

Proof. Indeed,

$$
\begin{aligned}
& (a_i) \in \ker \textstyle\prod_{i=1}^n \rho_i \iff (a_i) \ \prod_{i=1}^n \rho_i \ (e_i) \text{ for some } (e_i) \in E(S) \\
\iff & a_i \, \rho_i \, e_i \text{ for some } e_i \in E(S_i), \ i = 1, 2, \dots, n \\
\iff & a_i \in \ker \rho_i \text{ for } i = 1, 2, \dots, n \iff (a_i) \in \textstyle\prod_{i=1}^n \ker \rho_i .
\end{aligned}
$$

□

We are finally ready for the desired result.

Theorem 3.5. *Let u and v be elements of the free lattice generated by Γ. For $i = 1, 2, \dots, n$, let S_i be a regular monoid and set $S = \prod_{i=1}^n S_i$. Let $P \in \{K, T, =\}$. Then S satisfies $u\,P\,v$ if and only if S_i satisfies $u\,P\,v$ for $i = 1, 2, \dots, n$.*

Proof. By Corollary 3.3 and Lemma 3.4, we obtain

$$
\begin{aligned}
S \text{ satisfies } u\,K\,v \iff & u_S\,K\,v_S \iff \textstyle\prod_{i=1}^n u_{S_i} \ K \ \prod_{i=1}^n v_{S_i} \\
\iff & \ker\left(\textstyle\prod_{i=1}^n u_{S_i}\right) = \ker\left(\prod_{i=1}^n v_{S_i}\right) \\
\iff & \textstyle\prod_{i=1}^n \ker u_{S_i} = \prod_{i=1}^n \ker v_{S_i} \\
\iff & \ker u_{S_i} = \ker v_{S_i} \text{ for } i = 1, 2, \dots, n \\
\iff & u_{S_i}\,K\,v_{S_i} \text{ for } i = 1, 2, \dots, n.
\end{aligned}
$$

Now using Corollary 3.3, we get

$$
\begin{aligned}
S \text{ satisfies } u\,T\,v \iff & \operatorname{tr} u_S = \operatorname{tr} v_S \iff \operatorname{tr}\left(\textstyle\prod_{i=1}^n u_{S_i}\right) = \operatorname{tr}\left(\prod_{i=1}^n v_{S_i}\right) \\
\iff & \operatorname{tr} u_{S_i} = \operatorname{tr} v_{S_i} \text{ for } i = 1, 2, \dots, n \\
\iff & S_i \text{ satisfies } u\,T\,v \text{ for } i = 1, 2, \dots, n.
\end{aligned}
$$

This proves the assertion for K and T; the claim for equality now follows from $K \cap T = \varepsilon$ or can be proved directly (essentially the same way as for T). □

4. Lemmas

This section serves as preparation for the next one and concerns Brandt semigroups, their ideal extensions, and Reilly semigroups.

Lemma 4.1. *Let $S = B(G, I)$ be a Brandt semigroup. Then $\sigma = \omega$; $\tau = \omega$ if $|G| = |I| = 1$ and $\tau = \varepsilon$ otherwise; $\mu = \mathcal{H}$; $\beta = \mathcal{H}$ if $|I| = 1$ and $\beta = \omega$ otherwise.*

Proof. Straightforward. □

We shall need the following notation. Let S be an ideal extension of S_0 by S_1 determined by a partial homomorphism $\varphi : S_1^* \to S_0$. For $i = 0, 1$, let $\rho_i \in \mathscr{C}(S_i)$ be such that $\{0\}$ is a ρ_1-class and for $a, b \in S_1^*$, $a\,\rho_1\,b$ implies $a\varphi\,\rho_0\,b\varphi$. Define $[\rho_0, \rho_1] = \rho_0 \cup \big(\rho_1|_{S_1^*}\big)$. Let $\psi = \varphi \cup \iota_{S_0}$. For $\rho_0 \in \mathscr{C}(S_0)$, define $[\rho_0]$ by

$$a\,[\rho_0]\,b \Longleftrightarrow a\psi\,\rho_0\,b\psi \quad (a, b \in S).$$

Then $[\rho_0, \rho_1], [\rho_0] \in \mathscr{C}(S)$; for an extensive discussion, see [**4**]. For $i = 0, 1$, we let ε_i, ω_i, η_i and μ_i denote the equality, the universal relation, the least semilattice congruence and the greatest idempotent separating congruence on S_i, respectively.

Lemma 4.2. *For $i = 0, 1$, let $S_i = B(G_i, I_i)$ be Brandt semigroups where $|I_0| > 1$ and S be an ideal extension of S_0 by S_1 determined by a partial homomorphism*

$$\varphi : (i, g, j) \longrightarrow (i\xi, u_i^{-1}(g\omega)u_j, j\xi).$$

Then $\sigma = \omega$; $\tau = [\varepsilon_0]$ if the condition $(\mathscr{X})$ below holds and $\tau = \varepsilon$ otherwise; $\mu = \mathcal{H}$; $\beta = [\omega_0, \eta_1]$ if $|I_1| = 1$ and $\beta = \omega$ otherwise; where

$(\mathscr{X})$ *$i\xi = j\xi$, $g\omega = u_i u_j^{-1} \Longrightarrow i = j$, $g = e_1$ the identity of G_1.*

Proof. We follow [**2**, Lemma XIV.4.4] for notation. Recall that $\xi : I_1 \to I_0$ and $u : I_1 \to G_0$ with $u : i \mapsto u_i$ are functions, $\omega : G_1 \to G_0$ is a homomorphism and let e_i be the identity of G_i for $i = 0, 1$.

First $\sigma = \omega$ since S has a zero and $\mu = \mathcal{H}$ since $\mathcal{H}$ is a congruence on S. If $|I_1| = 1$, then S_1 is a group with zero and β has the indicated form. If $|I_1| > 1$, then clearly $\beta = \omega$.

By [**5**, Theorem 5.2(ii)], τ does not saturate S_0 if and only if

$$a \in S_1^*,\ a\varphi \in E(S_0) \Longrightarrow a \in E(S_1).$$

Equivalently

$$(i\xi, u_i^{-1}(g\omega)u_j, j\xi) \in E(S_0) \Longrightarrow (i, g, j) \in E(S_1),$$

that is

$$i\xi = j\xi,\ u_i^{-1}(g\omega)u_j = e_0 \Longrightarrow i = j,\ g = e_1$$

which is evidently equivalent to condition $(\mathscr{X})$. By Lemma 4.1, we have $\tau_0 = \varepsilon_0$ and $\tau_1 = \varepsilon_1$. The expression for τ now follows directly from [**5**, Theorem 5.2(ii)]. □

Lemma 4.3. *Let $S = B(G, \alpha)$ be a Reilly semigroup. Then $\tau = \sigma$ if α is injective and $\tau = \varepsilon$ otherwise.*

Proof. According to [**3**, Corollary 2.2], $\sigma = \sigma_{(M;e,0)}$ and thus

$$\ker \sigma = \{(m, g, m) \in S \mid g \in M\}$$

where $M = \bigcup_{n \geq 0} \ker \alpha^n$.

Assume first that α is injective. Then so is α^n for every $n \geq 0$ and $M = \{e\}$. It follows that $\ker \sigma = E(S)$ which implies that $\sigma \subseteq \tau$. But $\tau \subseteq \sigma$ always holds and we get $\sigma = \tau$.

Suppose next that α is not injective. Then there exists $g \in G$ such that $g \neq e$ and $g\alpha = e$. Let $0 \leq n < m$. Then

$$\begin{array}{rcl} (n, e, n)(n, g, n) & = & (n, g, n) \notin E(S), \\ (m, e, m)(n, g, n) & = & (m, g\alpha^{m-n}, m) \; = \; (m, e, m) \in E(S) \end{array}$$

which shows that $((n, e, n), (m, e, m)) \notin \tau$. It follows that $\operatorname{tr} \tau = \varepsilon$. Since always $\ker \tau = E(S)$, we conclude that $\tau = \varepsilon$. $\square$

The final lemma lists sufficient conditions for the validity of some of our conditions and will come in quite handy.

Lemma 4.4. *Each of the conditions on the left (e.g. $\sigma = \omega$) implies the condition on the right on the same line.*

$\sigma = \omega,\ \mu = \varepsilon,\ \mu = \beta,\ \beta = \omega \implies$ (D).
$\sigma = \omega,\ \sigma = \tau,\ \tau = \varepsilon,\ \mu = \varepsilon \implies$ (E).
$\tau = \varepsilon,\ \mu = \varepsilon,\ \mu = \beta,\ \beta = \omega \implies$ (F).
$\sigma = \tau,\ \tau = \varepsilon,\ \mu = \varepsilon,\ \mu = \beta,\ (\sigma = \omega,$ (F)$),\ (\beta = \omega,$ (E)$) \implies$ (G).
$\sigma = \tau \implies$ (H).

Proof. All of this follows by direct inspection. $\square$

5. Examples

Here we construct several examples which will be used in the succeeding sections in crucial ways: for proving independence of certain conditions and constructing further examples by means of finite direct products in the main results of the paper. Recall that $\sim$(X) stands for the negation of the statement (X).

Example 5.1. Conditions (D)–(H), (J), (L) hold and (I), (K) fail.

Let $\mathbb{Z}_2 = \mathbb{Z}/(2)$, $R_2 = \{\lambda_1, \lambda_2\}$ be a right zero semigroup, B_2 be a 5-element combinatorial Brandt semigroup and S be the ideal extension of $S_0 = \mathbb{Z}_2 \times R_2$ by B_2 determined by the partial homomorphism

$$\varphi : (i, j) \longmapsto (u_i + u_j, \lambda_1) \quad ((i, j) \in B_2^*)$$

where we write (i, j) for $(i, 1, j)$ in B_2 and $u_1 = \overline{0}$, $u_2 = \overline{1}$.

We first list the classes of some of the congruences:

$$\begin{array}{rl} \sigma : & \{(\overline{0}, \lambda_1), (\overline{0}, \lambda_2), (1, 1), (2, 2)\},\ \{(\overline{1}, \lambda_1), (\overline{1}, \lambda_2), (1, 2), (2, 1)\}, \\ \mu : & \{(\overline{0}, \lambda_1), (\overline{1}, \lambda_1)\}, \{(\overline{0}, \lambda_2), (\overline{1}, \lambda_2)\}, \{(1, 1)\}, \{(1, 2)\},\ \{(2, 1)\}, \{(2, 2)\}, \\ \beta : & \{(\overline{0}, \lambda_1), (\overline{1}, \lambda_1), (1, 1), (1, 2), (2, 2)\},\ \{(\overline{0}, \lambda_2), (\overline{1}, \lambda_2)\}, \\ \sigma \wedge \beta : & \{(\overline{0}, \lambda_1), (1, 1), (2, 2)\},\ \{(\overline{0}, \lambda_2)\},\ \{(\overline{1}, \lambda_1), (1, 2), (2, 1)\},\ \{(\overline{1}, \lambda_2)\}, \\ \mu \vee (\sigma \wedge \beta) : & \{(\overline{0}, \lambda_1), (\overline{1}, \lambda_1), (1, 1), (1, 2), (2, 1), (2, 2)\},\ \{(\overline{0}, \lambda_2), (\overline{1}, \lambda_2)\}. \end{array}$$

Hence $\sigma \vee \mu = \omega$ and thus $\beta \wedge (\sigma \vee \mu) = \mu = \mu \vee (\sigma \wedge \beta)$ so that (D) holds.

We always have $\tau \subseteq \sigma$. Since $\ker \sigma = E(S)$, see above, we also have $\sigma \subseteq \tau$ and equality prevails. But then Lemma 4.4 yields that (E), (G) and (H) hold. Also $\tau \vee \mu = \sigma \vee \mu = \omega$, but $\ker \mu \neq S$ so that $(\omega, \mu) \notin K$, and both (K) and (I) fail. Since $\sigma = \tau$, we get $\ker \sigma = \ker (\sigma \wedge \mu)$ and hence (J) holds. We have seen that $\tau = \sigma$. Since $\operatorname{tr} \sigma = \omega$, we have $\sigma \vee \beta \, T \, \sigma$ and thus (L) holds as well.

Example 5.2. Conditions (D)–(H) hold and (I)–(L) fail.

Let $S_0 = B(\mathbb{Z}_4, \{1,2\})$ and $S_1 = B(\mathbb{Z}_2, \{1,2\})$ be Brandt semigroups, S be the ideal extension of S_0 by S_1 determined by the partial homomorphism

$$\varphi : (i, g, j) \longmapsto (i\xi, u_i^{-1}(g\omega)u_j, j\xi) \quad ((i,g,j) \in S_1^*)$$

where

$$\omega = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{2} \end{pmatrix}, \quad u_1 = \overline{0}, \quad u_2 = \overline{1}, \quad \xi = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

That φ is a partial homomorphism that follows from [**2**, Lemma XIV.4.4]. Since S has a zero, we have $\sigma = \omega$; one sees easily that also $\beta = \omega$. Hence Lemma 4.4 implies that conditions (D)–(G) hold. From $\sigma = \beta = \omega$ it follows that also (H) holds.

We now verify that condition $(\mathscr{X})$ in Lemma 4.2 holds. Hence assume that $i\xi = j\xi$ and in the additive notation $-u_i + g\omega + u_j = \overline{0}$. If $i = j$, we have $g\omega = \overline{0}$ so $g = \overline{0}$. Assume that $i \neq j$. We may suppose that $i = 1$ and $j = 2$. Then $-u_1 + g\omega + u_2 = \overline{0}$ becomes $g\omega = \overline{3}$. But $g\omega \in \{\overline{0}, \overline{2}\}$, so this case is impossible. Therefore $(\mathscr{X})$ holds and Lemma 4.2 implies that $\tau = [\varepsilon_0]$.

Clearly $\mu = \mathcal{H} = [\mu_0, \mu_1]$. By [**4**, Lemma 6.1(ii)], we obtain

$$\ker \mu = \ker \mu_0 \cup (\ker \mu_1)^*,$$

which in conjunction with $\sigma = \omega$ implies

$$\ker (\sigma \wedge \mu) = \ker \mu \neq S = \ker \sigma = \ker (\sigma \vee \mu)$$

and thus both (I) and (J) fail. By [**4**, Lemma 4.5(ii)], we get

$$\tau \vee \mu = [\varepsilon_0] \vee [\mu_0, \mu_1] = [\mu_0]$$

and thus by [**4**, Lemma 7.2],

$$\ker (\tau \vee \mu) = \ker \mu_0 \cup \{a \in S_1^* \mid a\varphi \in \ker \mu_0\}.$$

Since

$$(1, \overline{0}, 2)\varphi = (1, \overline{1}, 1) \in \ker \mu_0, \quad (1, \overline{0}, 2) \notin (\ker \mu_1)^*,$$

we have $(\tau \vee \mu, \mu) \notin K$ so condition (K) fails. Finally, $\tau \vee \beta = \omega$ and $\tau = [\varepsilon_0]$ so that $(\tau \vee \beta, \tau) \notin T$ and (L) fails.

Example 5.3. Conditions (D)–(K) hold and (L) fails.

Let $S = B(G, \alpha)$ be a Reilly semigroup, where α is not injective. Since S is a bisimple inverse semigroup, we get that $\beta = \omega$. By Lemma 4.4, we deduce that (D) and (F) hold. Lemma 4.3 yields that $\tau = \varepsilon$ which again by Lemma 4.4 gives that (E) and (G) hold. From $\beta = \omega$ and $\tau = \varepsilon$; follows that (H) holds.

We adopt the notation of [**3**]. By [**3**, Corollary 2.2 and Theorem 4.2] with $M = \bigcup_{n\geq 0} \ker \alpha^n$, we have

$$\sigma = \sigma_{(M,e,0)}, \quad \mu = \rho_G, \quad \sigma \vee \mu = \sigma_{(G,e,0)},$$

so that

$$\ker(\sigma \vee \mu) = \{(m, g, m) \in S \mid m \geq 0, g \in G\} = \ker \mu.$$

It follows that (I) holds. Validity of (J) follows directly from [**3**, Proposition 5.3]. Since $\tau = \varepsilon$, also (K) holds. Finally $\tau = \varepsilon$ and $\beta = \varepsilon$ yield that (L) fails.

We are now able to make certain conclusions. Both Examples 5.1 and 5.2 show that conditions in [**7**, Theorem 5.2] do not imply those in [**7**, Theorem 6.3]. Both Examples 5.1 and 5.3 show that conditions in [**7**, Theorem 6.3] do not imply conditions in [**7**, Theorem 7.5]; together they show that conditions (I) and (L) are independent and hence neither can be omitted in [**7**, Theorem 7.5(ii)].

6. Realization of c-triples

One aim here is to construct examples which fit exactly the requirements of [**7**, Theorems 7.5, 6.3 and 5.2]. For a regular semigroup S, this involves an isomorphism of Λ_i and Λ_S which extends the mapping $\alpha \mapsto \alpha_S$ ($\alpha \in \Gamma$) as well as possible requirements concerning K- and T-relations. To this end, we construct several examples from those in the preceding section by forming finite direct products and applying results of Section 3.

Lemma 6.1. *Let $(\Lambda; K, T)$ be a c-triple satisfying condition (I). Then*

$$[\varepsilon, \tau], \quad [\sigma \wedge \mu, \sigma], \quad [\mu, \sigma \vee \mu], \quad [\beta, \mu]$$

are K-classes.

Proof. By the definition of a c-triple in Section 2 we have that $[\varepsilon, \tau]$ and $[\beta, \omega]$ are K-classes. Since $\operatorname{tr} \mu_K \subseteq \operatorname{tr} \mu = \varepsilon$, we have $\mu_K = \mu$, and since (I) holds, also $\mu^K = \sigma \vee \mu$. Hence the interval $[\mu, \sigma \vee \mu]$ is a K-class. By (I) we get $\sigma \wedge (\sigma \vee \mu)\, K\, \sigma \wedge \mu$ and thus $\sigma\, K\, \sigma \wedge \mu$. In addition, $\operatorname{tr}(\sigma \wedge \mu) \subseteq \operatorname{tr} \mu = \varepsilon$ so that $\sigma_K = \sigma \wedge \mu$. Since $\operatorname{tr} \sigma = \omega$ for any $\theta\, K\, \sigma$, we have $\theta \wedge \sigma\, K\, \theta$ and $\theta \wedge \sigma\, T\, \theta$ so that $\theta \subseteq \sigma$ which implies that $\sigma^K = \sigma$. It follows that $[\sigma \wedge \mu, \sigma]$ is a K-class. $\square$

We start with [**7**, Theorem 7.5] in the first theorem and continue with [**7**, Theorem 6.3] in the second and end with [**7**, Theorem 5.2] in the third; for the remaining [**7**, Theorem 4.2] we have no suitable example. For the first theorem, we need the following notation.

$\mathbb{Z}_2$	=	$\mathbb{Z}/(2)$ — additive integers modulo 2,
C	=	$(\mathcal{M}^0(\{1,2\}, \mathbb{Z}_2, \{1,2\}; P))^1$ with $P = \left[\begin{smallmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{smallmatrix}\right]$,
B	—	the bicyclic semigroup. It is well-known that $(m,n)\,\sigma\,(p,q) \Longleftrightarrow m - n = p - q$.
L_2^1	—	a 2-element left zero semigroup with an identity adjoined.

	$\mathbb{Z}_2$	C	B	L_2^1	$\mathscr{T}_2$
σ	ε	ω	σ	ω	ω
τ	ε	ε	σ	ω	ρ
μ	ω	$\mathcal{H}$	ε	ε	ε
β	ω	$\mathcal{H}$	ω	ε	$\mathcal{D}$
$\sigma \vee \mu$			σ		
$\tau \vee \mu$					ρ
$\tau \vee \beta$		$\mathcal{H}$			
$\sigma \wedge \mu$	ε	$\mathcal{H}$			ε
$\sigma \wedge \beta$					$\mathcal{D}$
$\tau \wedge \beta$			σ	ε	ρ
$\ker \mu$	$\mathbb{Z}_2$	C		L_2^1	
$\ker(\sigma \vee \mu)$			$E(B)$		
$\operatorname{tr} \tau$			ω		
$\operatorname{tr}(\tau \vee \beta)$	ε	ε		ω	
(D)					$\sigma = \omega$
(K)					$\mu = \tau \vee \mu$
separation	$\sigma \wedge \mu \neq \mu$	$\omega \neq \tau \vee \beta$	$\sigma \vee \mu \neq \omega$	$\tau \neq \tau \wedge \beta$	$\sigma \wedge (\tau \vee \mu) \wedge \beta$
		$\varepsilon \neq \sigma \wedge \mu$	$\tau \wedge \beta \neq \varepsilon$		$\neq \sigma \wedge \beta$

Table 1

In Table 1, for the quantities indicated in the first column, each succeeding column gives the values for the semigroups listed in the first row. Conditions (I) and (L) hold for the first four semigroups by the information in the fourth block of rows in view of Lemma 4.4. For the fifth semigroup, the validity of (D) follows from Lemma 4.4 and the validity of (K) since $\tau = \tau \vee \mu$. The last block of rows indicates separation, namely which pairs of congruences are distinct. The verification of the assertions contained in the table is lengthy but straightforward, and is omitted.

Theorem 6.2. *Let* $S_3 = \mathbb{Z}_2 \times C \times B \times L_2^1$*, the direct product. Then the morphism* δ_3 *in* [**7**, Theorem 7.5(iv)] *is an isomorphism of* $(\Lambda_3; K_3, T_3)$ *onto* $(\Lambda_{S_3}; K_{S_3}, T_{S_3})$.

Proof. From Table 1, we can see that the semigroups $\mathbb{Z}_2$, C, B and L_2^1 satisfy conditions (I) and (L). Hence by Theorem 3.5, also S_3 satisfies these conditions. By [**7**, Theorem 7.5(iv)], the mapping

$$\delta_3 : (\Lambda_3; K_3, T_3) \longrightarrow (\Lambda_{S_3}; K_{S_3}, T_{S_3})$$

is a morphism. From the last two rows of Table 1, we conclude that the collection of semigroups $\mathbb{Z}_2$, C, B and L_2^1 separates certain pairs of vertices of [**7**, Diagram 3]. In view of Theorem 3.5, these vertices are also separated in the semigroup S_3. Simple inspection of [**7**, Diagram 3] shows that the separation of these pairs of vertices implies that the morphism δ_3 identifies no two distinct vertices of the diagram. Therefore δ_3 is injective; it is always surjective.

From [**7**, Diagram 3], we see that the complete collection of K-classes of Λ_3 is

$$[\varepsilon,\tau],\ [\sigma\wedge\mu,\sigma],\ [\mu,\sigma\vee\mu],\ [\beta,\mu].$$

The morphism δ_3 extends the mapping $\gamma_{\Lambda_3,\Lambda_{S_3}}$ and hence maps each of these intervals onto the corresponding interval in Λ_{S_3}, that is each letter gets a subscript S_3. By Lemma 6.1, the latter form the complete collection of K-classes of Λ_{S_3}. Thus δ_3^{-1} is K-preserving so that

$$\delta_3^{-1} : (\Lambda_{S_3}; K_{S_3}, T_{S_3}) \longrightarrow (\Lambda_3; K_3, T_3)$$

is a morphism. Consequently, δ_3 is an isomorphism. □

For the second theorem, we also need

$\mathscr{T}_2$ — the semigroup of all transformations on a 2-element set written on the right and composed as such. Denote by ρ the Rees congruence on $\mathscr{T}_2$ relative to its kernel.

Theorem 6.3. *Let $S_2' = S_3 \times \mathscr{T}_2$, the direct product. Then the morphism δ_2' in* [**7**, Theorem 6.3(iv)] *is an isomorphism of $(\Lambda_2; K_2', T_2)$ onto $(\Lambda_{S_2'}; K_{S_2'}, T_{S_2'})$.*

Proof. By Theorem 6.2 and [**7**, Theorem 7.5], S_3 satisfies condition (I), and thus, by [**7**, Lemma 3.2], also (D) and (K). Since in $\mathscr{T}_2$ we have $\sigma = \omega$, Lemma 4.4 implies that $\mathscr{T}_2$ satisfies (D). From Table 1 we get that $\mu = \tau \vee \mu$ and thus $\mathscr{T}_2$ satisfies (K) as well. Now Theorem 3.5 implies that S_2' satisfies both (D) and (K). By [**7**, Theorem 7.5(iv)],

$$\delta_2' : (\Lambda_2; K_2', T_2) \longrightarrow (\Lambda_{S_2'}; K_{S_2'}, T_{S_2'})$$

is a morphism. We also see from Table 1 that $\mathscr{T}_2$ separates $\sigma\wedge(\tau\vee\mu)\wedge\beta$ from $\sigma\wedge\beta$ which implies that δ_2' causes no collapsing. Therefore δ_2' is injective and is always surjective.

We show next that δ_2' carries K-classes of Λ_2 onto K-classes of $\Lambda_{S_2'}$. Since δ_2' is a morphism, and thus is K-preserving, we know that it carries K-classes into K-classes. Recall that K-classes of Λ_2 in this case are given in [**7**, Corollary 6.1, Case 3], that is

$$[\beta,\omega],\ [\mu\vee(\sigma\wedge\beta),\sigma\vee\mu],\ [\sigma\wedge\beta,\sigma],\ [\varepsilon,\tau],\ [\mu,\tau\vee\mu],\ [\sigma\wedge\mu,\tau\vee(\sigma\wedge\mu)]. \tag{6.1}$$

We also know that in $\Lambda_{S_2'}$, $[\beta,\omega]$ and $[\varepsilon,\tau]$ are K-classes since $(\Lambda_{S_2'}; K_{S_2'}, T_{S_2'})$ is a c-triple. It remains to show that there is no collapsing of the other K-classes in (6.1). From Table 1, we get that the pairs

$$(\sigma,\sigma\vee\mu),\ (\sigma,\mu),\ (\mu,\sigma\wedge\mu),\ (\sigma\vee\mu,\sigma\wedge\mu)$$

are not K-related and $(\mu,\sigma\vee\mu) \notin K$ for $\mathscr{T}_2$. In the light of Theorem 3.5, we conclude that this holds for S_2' as well. Thus there is no collapsing and δ_2' indeed maps K-classes onto K-classes. It follows that

$$(\delta_2')^{-1} : (\Lambda_{S_2'}; K_{S_2'}, T_{S_2'}) \longrightarrow (\Lambda_2; K_2', T_2)$$

is a morphism. Consequently, δ_2' is an isomorphism. □

For the third theorem, we need the notation:
B_2^1 – combinatorial 5-element Brandt semigroup with an identity adjoined,
E_2 – the semigroup in Example 5.2,
B^1 – the bicyclic semigroup with an identity adjoined.

Table 2 is similar to Table 1. The second block of its rows provides the reason for the validity of conditions (D), (F) and (G). Again the verification of the results contained in the table is lengthy, but straightforward, and is omitted.

	$\mathbb{Z}_2$	B_2^1	E_2	B^1
σ	ε	ω	ω	$[\sigma_0]$
τ	ε	ε	$[\varepsilon_0]$	$[\sigma_0]$
μ	ω	ε	$[\mu_0, \mu_1]$	ε
β	ω	$[\omega_0, \varepsilon_1]$	ω	$[\omega_0, \varepsilon_1]$
(D)	$\beta = \omega$	$\sigma = \omega$	$\beta = \omega$	$\mu = \varepsilon$
(F)	$\beta = \omega$	$\mu = \varepsilon$	$\beta = \omega$	$\mu = \varepsilon$
(G)	$\sigma = \tau$	(F), $\sigma = \omega$	(F), $\sigma = \omega$	$\sigma = \tau$
$\tau \wedge \beta$			$[\varepsilon_0]$	$[\sigma_0, \varepsilon_1]$
$\sigma \wedge \mu$	ε		$[\mu_0, \mu_1]$	
$\sigma \wedge \beta$			ω	
$\tau \vee \mu$			$[\mu_0]$	
$\sigma \vee \mu$				$[\sigma_0]$
$\tau \vee \beta$		$[\omega_0, \varepsilon_1]$		
$\sigma \wedge (\tau \vee \mu) \wedge \beta$			$[\mu_0]$	
separation	$\sigma \wedge \mu \neq \mu$	$\tau \vee \beta \neq \omega$	$\tau \wedge \beta \neq \varepsilon$ $\sigma \wedge \mu \neq \varepsilon$ $\sigma \wedge (\tau \vee \mu) \wedge \beta$ $\neq \sigma \wedge \beta$ $(\sigma \wedge (\tau \vee \mu) \wedge \beta,$ $\sigma \wedge \mu) \notin K$	$\sigma \vee \mu \neq \omega$ $\tau \wedge \beta \neq \tau$ $\tau \wedge \beta \neq \varepsilon$

Table 2

Theorem 6.4. *Let $S_2 = \mathbb{Z}_2 \times B_2^1 \times E_2 \times B^1$, the direct product. Then the morphism δ_2 in* [**7**, Theorem 6.3(iv)] *is an isomorphism of $(\Lambda_2; K_2, T_2)$ onto $(\Lambda_{S_2}; K_{S_2}, T_{S_2})$.*

Proof. For each of the direct factors of S_2, Table 2 provides the reason why conditions (D), (F) and (G) hold in view of Lemma 4.4. This table also gives the separation properties of each of the factors needed to distinguish certain strategic pairs in [**7**, Diagram 2]. This implies separation of any two distinct vertices of the lattice Λ_2. An obvious inductive argument extends [**6**, Proposition 9.9] to the direct product of any finite number of inverse semigroups. Applying this to the above direct product implies that S_2 satisfies conditions (D), (F) and (G). By [**7**, Theorem 6.3(iv)],

$$\delta_2 : (\Lambda_2; K_2, T_2) \longrightarrow (\Lambda_{S_2}; K_{S_2}, T_{S_2})$$

is a morphism. The fourth block in Table 2 implies that δ_2 is injective in view of Theorem 3.5; it is always surjective.

Next consider the semigroup E_2. From Table 2, we get

$$\sigma \wedge (\tau \vee \mu) \wedge \beta = [\mu_0], \quad \sigma \wedge \mu = [\mu_0, \mu_1].$$

In view of [**3**, Lemma 7.2], we have

$$\ker[\mu_0] = \ker \mu_0 \cup \{a \in S_1^* \mid a\varphi \in \ker \mu_0\},$$

and clearly $\ker[\mu_0, \mu_1] = \ker \mu_0 \cup (\ker \mu_1)^*$. Since

$$(1, \overline{0}, 2)\varphi = (1, \overline{0}, 1) \in \ker \mu_0, \quad (1, \overline{0}, 2) \notin \ker \mu,$$

it follows that $\ker[\mu_0] \neq \ker[\mu_0, \mu_1]$. Therefore

$$(\sigma \wedge (\tau \vee \mu) \wedge \beta, \sigma \wedge \mu) \notin K \tag{6.2}$$

which gives Case 1 of [**7**, Lemma 6.1] since in that case $\{\sigma \wedge \mu\}$ is a K-class and the three classes in that lemma are exclusive. In the light of the extended version of [**6**, Proposition 9.9] indicated above, we conclude that (6.2) is also valid for S_2. It follows that the K-relation on Λ_2 is minimal. From this we derive that δ_2^{-1} is K-preserving. Therefore

$$\delta_2^{-1} : (\Lambda_{S_2}; K_{S_2}, T_{S_2}) \longrightarrow (\Lambda_2; K_2, T_2)$$

is a morphism. Consequently, δ_2 is an isomorphism. □

Note that isomorphism between two triples in our category means that essentially they have the same lattices and the same K-relations since we require the morphisms to be K-preserving. It may have been more natural to require them also to be T-preserving. However, this would have created very different conditions.

Using [**7**, Lemma 3.2 and Corollary 6.4] in the proofs of Theorem 6.2 and 6.3, we could have shortened the argument to show that the inverses of our morphisms are K-preserving. In addition, from these references, one may conclude that the resulting isomorphisms are T-preserving. This implies that, roughly speaking, they have the same lattice and the same K- and T-relations.

7. Independence of conditions for c-triples

In the absence of examples of regular semigroups which would exhibit independence of conditions (D)–(G), we provide examples of c-triples below. In Diagrams 1–4, full lines stand for K-relation, dashed lines for T-relation and dotted lines for inclusion. These diagrams pertain to c-triples. It does not guarantee that there exists a regular semigroup whose relevant congruences have this form. The long verification showing that these diagrams indeed represent c-triples is omitted.

According to Example 5.1 or 5.2, (D) $\not\Rightarrow$ (K) and by Diagram 1, also (K) $\not\Rightarrow$ (D). Hence conditions (D) and (K) are independent for c-triples (but may still not be for regular semigroups). Diagram 1 also shows that for c-triples, neither (E), (F) nor (F), (G) imply (D). Note that in Examples 5.1–5.3, all of (D), (E), (F), (G) are valid.

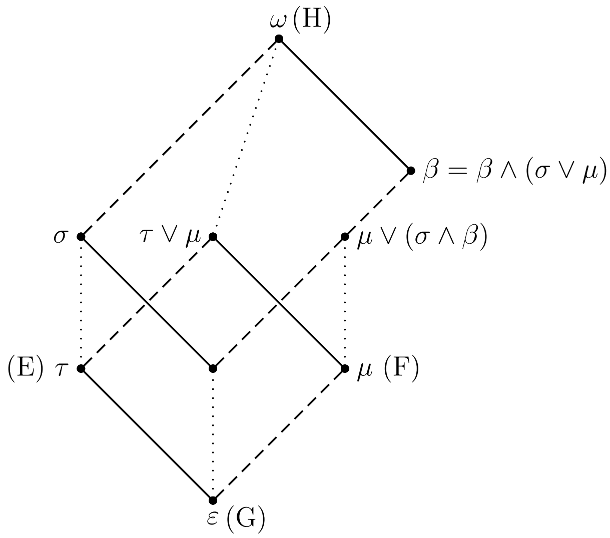

Diagram 1. $\sim$(D), (E)–(H), $\sim$(I), $\sim$(J), (K), $\sim$(L).

It follows from Diagrams 1–4 that for c-triples, the conditions (D), (E), (F) and also the conditions (D), (F), (G) are independent.

8. Problems

We propose:
Problem 1: Does every c-triple arise from a regular semigroup?
Problem 2: Do conditions (D), (E), (F) imply (G) for regular semigroups?
Problem 3: Are conditions (D), (E), (F) independent for regular semigroups?
Problem 4: Are conditions (D), (F), (G) independent for regular semigroups?

Problem 1 is a seminal question. In other words, by the definition of a c-triple, have we captured the essence of the triple $(\Lambda_S; K|_{\Lambda_S}, T|_{\Lambda_S})$?

If Problem 1 has an affirmative solution, to prove it one would have to start with a c-triple and construct a regular semigroup for it, a daunting proposition for it amount to a kind of coordinatization. If it has a negative solution, it begs for a counterexample. In that case, one should reinforce the definition of a c-triple and hope that the new definition would be strong enough when tested for the realization by a regular semigroup. If not, one may repeat this procedure (which may not end after a finite number of steps).

For special classes of semigroups, further conditions may be valid. As an example, we have the following result.

Proposition 8.1. *In a completely regular semigroup S, the equality*

$$\sigma \wedge [\mu \vee (\tau \wedge \beta)] = (\sigma \wedge \mu) \vee (\tau \wedge \beta) \tag{8.1}$$

holds and (F) implies (G).

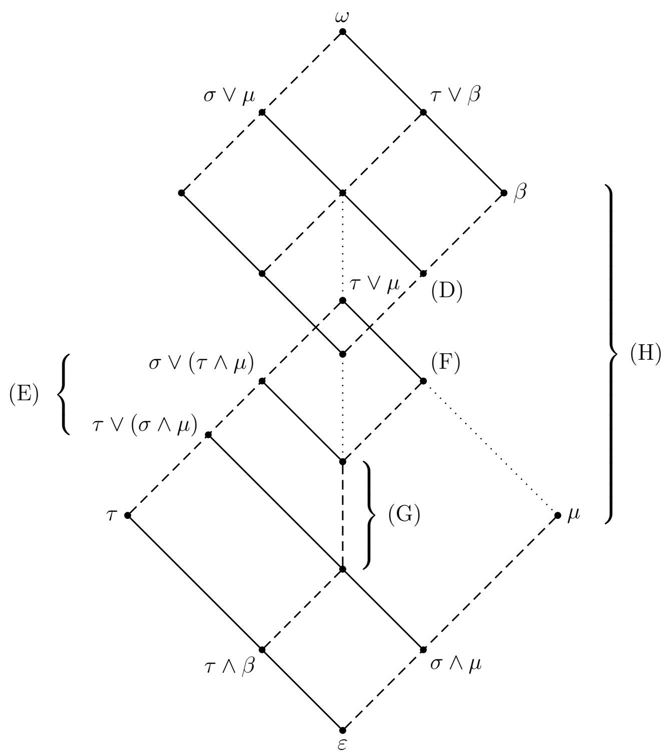

Diagram 2. (D), $\sim$(E), (F), $\sim$(G).

Proof. Since $\beta \leq \mathcal{D}$ in S, we have $\tau \wedge \beta \leq \tau \wedge \mathcal{D}$ which by [**1**, Lemma 3.1] implies that $\ker \mu = \ker [\mu \vee (\tau \wedge \beta)]$. Hence

$$\begin{aligned}\ker \{\sigma \wedge [\mu \vee (\tau \wedge \beta)]\} &= \ker \sigma \cap \ker [\mu \vee (\tau \wedge \beta)] = \ker \sigma \cap \ker \mu \\ &\leq \ker [(\sigma \wedge \beta) \vee (\tau \wedge \beta)];\end{aligned}$$

also $(\sigma \wedge \mu) \vee (\tau \wedge \beta) \leq \sigma \wedge [\mu \vee (\tau \wedge \beta)]$ since $\tau \leq \beta$ and

$$\operatorname{tr} \{\sigma \wedge [\mu \vee (\tau \wedge \beta)]\} = \operatorname{tr} (\tau \wedge \beta) = \operatorname{tr} [(\sigma \wedge \mu) \vee (\tau \wedge \beta)].$$

Formula (8.1) follows.

Now assume that (F) holds. Then using formula (8.1), we obtain

$$\sigma \wedge (\tau \vee \mu) \wedge \beta = \sigma \wedge [\mu \vee (\tau \wedge \beta)] = (\sigma \wedge \mu) \vee (\tau \wedge \beta)$$

and (G) holds. $\square$

The first consequence of Proposition 8.1 is that Problem 2 has an affirmative answer for completely regular semigroups.

Since we are interested in condition (G), the following two conditions are quite interesting for us.

(N) $\sigma \wedge \mu = \varepsilon$. (O) $\tau \wedge \beta = \varepsilon$.

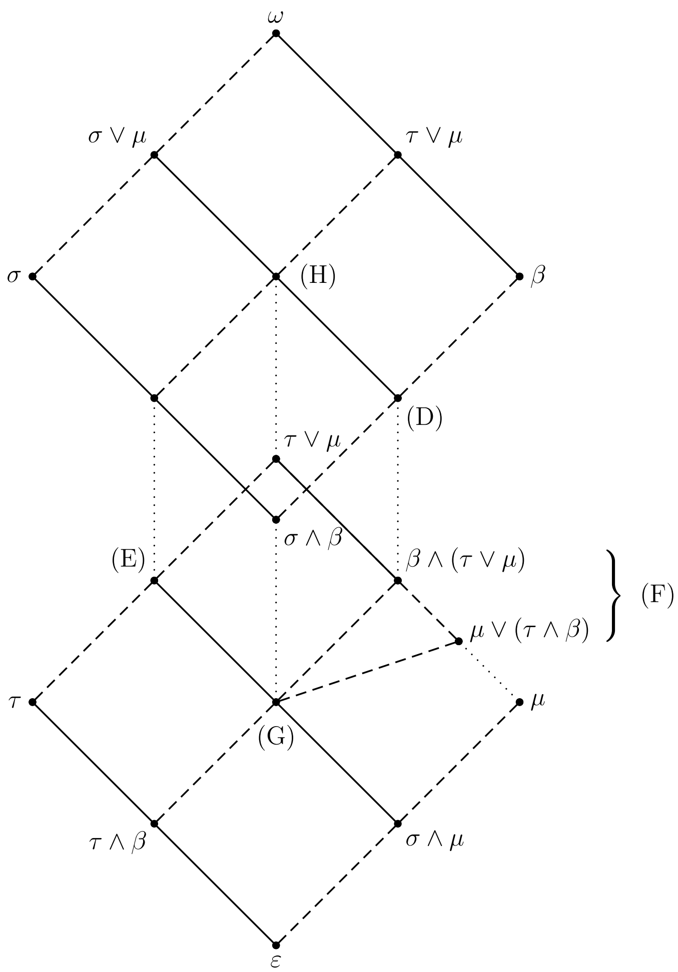

Diagram 3. (D), (E), $\sim$(F), (G), (H), $\sim$(I).

Proposition 8.2.

(i) (E), (N) $\Rightarrow$ (G).
(ii) (F), (O) $\Rightarrow$ (G).

Proof. Let

$$\xi = \sigma \wedge (\tau \vee \mu) \wedge \beta, \qquad \eta = (\sigma \wedge \mu) \vee (\tau \wedge \beta),$$

so that (G) means $\xi = \eta$. By [**7**, Lemma 3.1], we have $\xi \geq \eta$.

(i) Assuming the validity of (E) and (N), we get

$$\xi = [\tau \vee (\sigma \wedge \mu)] \wedge \beta = \tau \wedge \beta \leq (\sigma \wedge \mu) \vee (\tau \wedge \beta) = \eta.$$

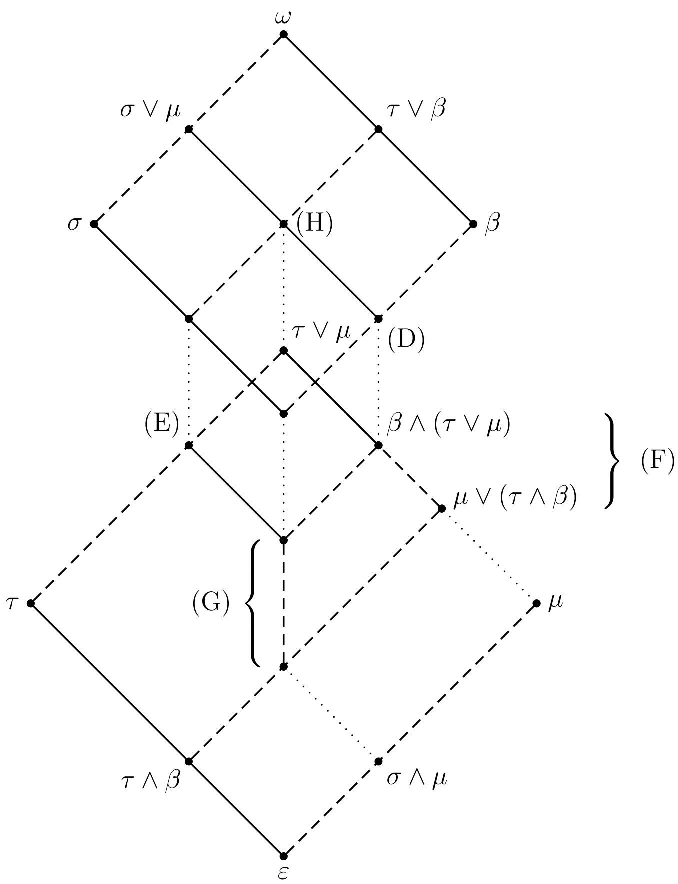

Diagram 4. (D), (E), $\sim$(F), $\sim$(G), (H), $\sim$(I).

(ii) Similarly

$$\xi = \sigma \wedge [\mu \vee (\tau \wedge \beta)] = \sigma \wedge \mu \leq (\sigma \wedge \mu) \vee (\tau \wedge \beta) = \eta.$$

$\square$

It follows that both (N) and (O) are sufficient conditions for the equivalence of (D), (E), (F) and (D), (F), (G). Since $\operatorname{tr} \mu = \varepsilon$, $\mu = \mathcal{H}^0$ (where ρ^0 is the greatest congruence contained in an equivalence relation ρ) and $\ker \beta = S$ for a regular semigroup S, we have

$$\begin{aligned} &(\mathrm{N}) \Longleftrightarrow \ker \sigma \cap \ker \mu = E(S) \iff (\sigma \cap \mathcal{H})^0 = \varepsilon, \\ &(\mathrm{O}) \Longleftrightarrow \operatorname{tr} \tau \cap \operatorname{tr} \beta = \varepsilon_{E(S)}, \end{aligned}$$

elucidating somewhat the nature of the conditions (N) and (O).

Regular semigroups in which $\mu = \varepsilon$ are usually called fundamental, those where $\sigma = \tau$ coincide with E-unitary semigroups; each of these conditions implies (N).

Regular semigroups where $\tau = \varepsilon$ are called E-disjunctive, each condition implies condition (O).

Corollary 8.3. *Let S be a regular semigroup satisfying at least one of the following conditions: completely regular, fundamental, E-unitary, E-disjunctive. If any three of the congruences σ, τ, μ, β on S generate a distributive lattice, then they generate a distributive lattice.*

Proof. This follows directly from [**7**, Theorems 4.2 and 5.2], Proposition 8.1 for completely regular semigroups, Proposition 8.2 and the above remarks. □

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