Acta Math. Univ. Comenianae
Vol. LXXX, 1 (2011), pp. 1–29

VERTEX-TRANSITIVE MAPS ON A TORUS

ONDREJ ŠUCH

ABSTRACT. We examine FVT (free, vertex transitive) actions of wallpaper groups on semiregular tilings. By taking quotients by lattices we then obtain various families of FVT maps on a torus, and describe the presentations of groups acting on the torus. Altogether there are 29 families, 5 arising from the orientation preserving wallpaper groups and 2 from each of the remaining wallpaper groups. We prove that all vertex-transitive maps on torus admit an FVT map structure.

1. INTRODUCTION

This paper is mostly expository. It is devoted to the nice interplay between various symmetric objects arising from the following diagram:

$$\begin{tikzcd}
\text{plane } \mathbf{R}^2 \arrow[d] & \begin{array}{c}\text{semiregular tiling } \mathcal{T} \text{ with} \\ \text{FVT/vertex-transitive action of} \\ \text{wallpaper group } \Gamma\end{array} \arrow[l] \arrow[d] \\
\text{torus } \mathbf{R}^2/\Lambda & \begin{array}{c}\text{a toric map } \mathcal{T}/\Lambda \text{ with} \\ \text{FVT/vertex-transitive action of finite} \\ \text{group } G\end{array} \arrow[l]
\end{tikzcd}$$

This diagram can be constructed in two ways. Starting from the top, one may choose a lattice Λ invariant under Γ and construct quotients. Alternatively, one can start at the bottom, and construct the universal covering space of the torus. All objects involved have a geometric symmetry of continued interest in pure and applied mathematics.

Semiregular tilings [**16**] arise when studying analogues to Archimedean solids on the torus. Various definitions can be given, isometric and topological. Interestingly, on the torus the local notion of *identical local type* and the global notion of *vertex-transitivity* coincide. This can be contrasted with the case of sphere,

Received November 28, 2008; revised September 30, 2010.
2010 *Mathematics Subject Classification.* Primary 20H15; Secondary 05C62.
Key words and phrases. Torus, wallpaper group, vertex-transitive map, Cayley map, semiregular tiling.
This research was supported by VEGA grant 1/0722/08.

where there is a solid with vertices of the same local type, which is not vertex--transitive [**18**].

Wallpaper groups [**17**] classify symmetries of two-dimensional Euclidean patterns. They have played great importance in crystallography, so much so, they are also known as *plane crystallographic groups*. One way to understand the structure of wallpaper groups is to construct their *Cayley graphs*, or even better *Cayley maps*, in which relation words can be read from closed walks in the graph. This approach is shown for instance in [**4**].

Vertex-transitive maps on a torus have received attention for various reasons. The underlying maps provide examples of several classes of finite groups, as studied for instance in [**4**]. Vertex-transitive maps have also been studied in contexts related to the Babai conjecture ([**13**], [**1**]). There is also continuing interest in symmetric toric maps from chemistry (see e.g. [**10**]). Vertex-transitive maps on the torus are also used as the underlying structure for Kohonen neural networks [**15**].

Let us make a brief survey of several related works. Well-known is the work of Burnside [**3**, p. 203–209] in which he describes the orientation-preserving groups acting on a torus. Baker [**2**] in his work gives presentations of most groups with action on a torus. He omits some families of groups: for instance, for the semiregular tiling of local type 3.4.6.4 he omits families $p3m1_3$, $p31m_3$. The book of Coxeter and Moser [**4**, Section 8.3] also provides lists of presentations. However, quotients of *pmm* and *pmg* listed there are in error. Proulx in her work [**11**] made a careful study of which Cayley graphs can be drawn on the torus. The list of groups she provides, however, does not give explicit presentations. A result in a similar vein can be found in [**6**, Theorem 6.3.3]. Thomassen in his work [**13**] gives a topological description of vertex-transitive maps on the torus.

Our work is structured into three parts. In the first we describe wallpaper groups with various FVT actions on semiregular tilings. We shall prove that except for the pairs $(\mathcal{T}, \Gamma)$ described in Table 1, no other FVT group actions exist (Theorem 2).

The basis for the second part is a theorem of Tucker that allows us to explicitly enumerate all finite groups acting on a torus. This explicit enumeration allows us to show that there is no redundant family in our list.

In the third part, we list all FVT maps induced by FVT actions of wallpaper groups on semiregular tilings. We prove that all vertex-transitive maps on the torus are in fact FVT maps. This should be contrasted with the case of higher genus ([**8**], [**9**]).

The author would like to thank R. Nedela for suggesting the problem, and J. Karabáš, A. Rosa, V. Proulx, T. Kenney for their indispensable help during preparation of the paper, as well as to the referee for useful corrections.

1.1. Preliminaries

1.1.1. Lattices. By a *lattice* we mean a rank two $\mathbf{Z}$-submodule Λ of $\mathbf{C}$. If ω is a nonzero complex number then $\omega \cdot \Lambda := \{\omega \cdot z | z \in \Lambda\}$ is also a lattice.

There are two special lattices that we will often consider.

Firstly, the *square lattice* $\Lambda_\square$ corresponding to Gaussian integers, that is to complex numbers of the form $n_1 + n_2\,\mathrm{i}$, where n_1, n_2 are integers. Secondly, the *triangle lattice* $\Lambda_\triangle$ is the lattice corresponding to complex numbers of form

$$n_1 + n_2 \cdot \frac{-1+\sqrt{3}\,\mathrm{i}}{2},$$

where again n_1 and n_2 are integers. These two lattices are distiguished by the following property.

Proposition 1. *Let Λ be a lattice. Then the following are equivalent*

a) *Λ is invariant under a rotation by an angle ϕ with $0 < \phi < \pi$*

b) *$\Lambda = \omega\Lambda_\triangle$ or $\Lambda = \omega\Lambda_\square$ for some nonzero complex number ω.*

Let us now define several more classes of lattices

- a *square-like* lattice is a lattice of form $\omega \cdot \Lambda_\square$ with complex $\omega \neq 0$,
- a *triangle-like* lattice is a lattice of form $\omega \cdot \Lambda_\triangle$ with complex $\omega \neq 0$,
- a *real* lattice is a lattice invariant under a reflection,
- a *rectangular* lattice is a lattice generated by a pair of orthogonal vectors,
- a *rhombic* lattice is a lattice generated by a pair of vectors of equal length.

The latter two classes of lattices can be given other characterizations. The set $N_\Lambda := \{\|z\| \mid z \in \Lambda\}$ is a discrete subset of the real line. Order the elements of N_Λ

$$0 = c_0 < c_1 < c_2 < \dots.$$

If the norm c_1 is achieved for six elements of Λ, then Λ is triangle-like. If at least one of the norms c_1, c_2 is achieved by four elements, then Λ is rhombic. If the four shortest nonzero vectors are $\pm w_1, \pm w_2$ with w_1, w_2 orthogonal, then the lattice is rectangular.

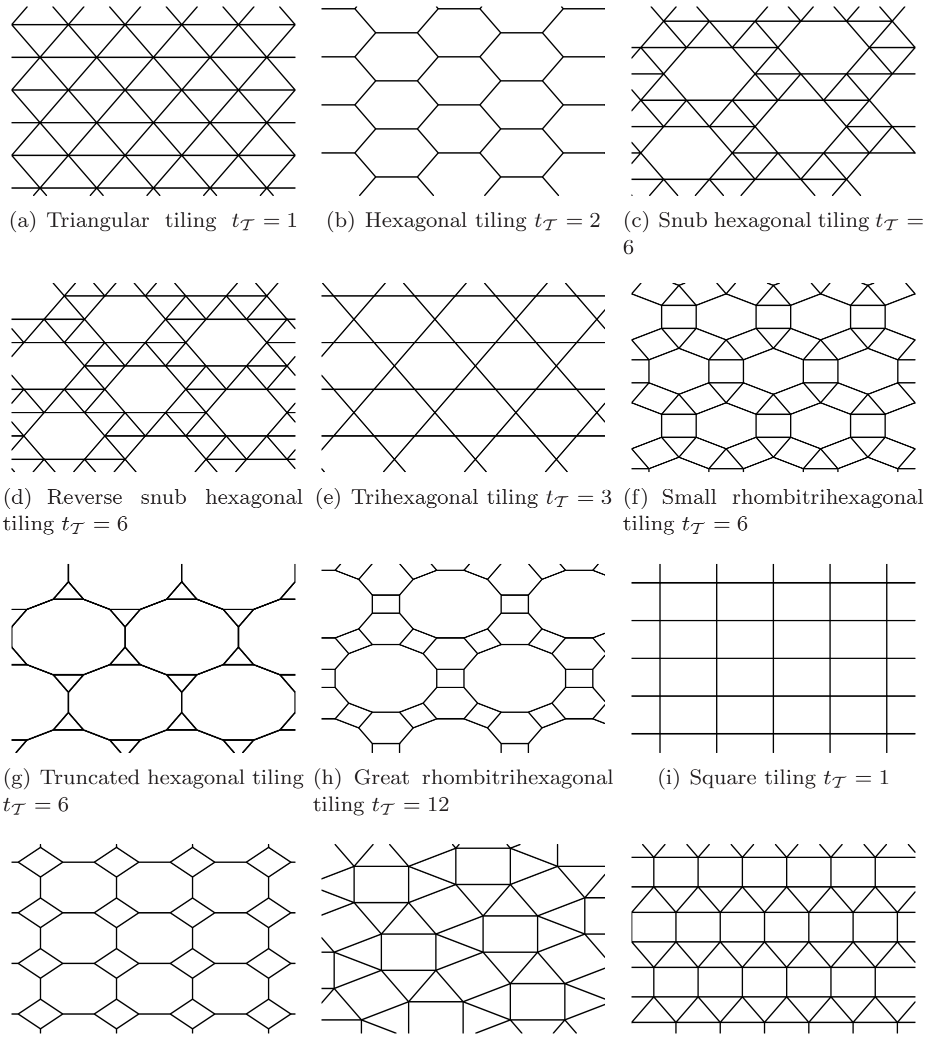
Alternatively, one can use the modular j function from the theory of elliptic curves. The function associates to any lattice Λ a complex number $j(\Lambda)$ that uniquely identifies lattices modulo the equivalence relation $\Lambda \sim \omega\Lambda$. One has equivalences:

- Λ is square-like iff $j(\Lambda) = 1728$,
- Λ is triangle-like iff $j(\Lambda) = 0$,
- Λ is real iff $j(\Lambda)$ is real,
- Λ is rectangular iff $j(\Lambda)$ is real with $j(\Lambda) \geq 1728$,
- Λ is rhombic iff $j(\Lambda)$ is real with $j(\Lambda) \leq 1728$.

1.1.2. Space groups. By a *wallpaper group* Γ we mean a group of isometries of the plane that acts discretely with a compact quotient. At present Wikipedia [**17**] provides an excellent resource on the subject of wallpaper groups. There are seventeen isomorphism classes of groups. In section 2 we will describe their presentation following the classical work of Coxeter and Moser [**4**].

The simplest one of them is the group *p1*, which is just the free abelian group on two generators. The *subgroup of translations* $T(\Gamma)$ in Γ is always isomorphic to *p1*, and its index in Γ is ≤ 12.

On the other end are the “largest” groups *p6m* and *p4m*. Any wallpaper group is a subgroup of finite index of either *p4m* or *p6m* (or both).



(a) Triangular tiling $t_{\mathcal{T}} = 1$

(b) Hexagonal tiling $t_{\mathcal{T}} = 2$

(c) Snub hexagonal tiling $t_{\mathcal{T}} = 6$

(d) Reverse snub hexagonal tiling $t_{\mathcal{T}} = 6$

(e) Trihexagonal tiling $t_{\mathcal{T}} = 3$

(f) Small rhombitrihexagonal tiling $t_{\mathcal{T}} = 6$

(g) Truncated hexagonal tiling $t_{\mathcal{T}} = 6$

(h) Great rhombitrihexagonal tiling $t_{\mathcal{T}} = 12$

(i) Square tiling $t_{\mathcal{T}} = 1$

(j) Truncated square tiling $t_{\mathcal{T}} = 4$

(k) Snub square tiling $t_{\mathcal{T}} = 4$

(l) Elongated triangular tiling $t_{\mathcal{T}} = 2$

Figure 1. Semi-regular tilings.

1.1.3. Semiregular tilings. A *semiregular tiling* in the plane is a covering of the plane by regular polygons so that the group of its symmetries acts transitively on the vertices of the tiling. There are 11 semiregular tilings of the plane [**5**], [**16**]. These are shown in Figure 1. Among them the snub hexagonal tiling (Figure 1, parts c) and d)) is the only one that is not invariant under a mirror. By a *local type* of a semiregular tiling we understand the formal product of numbers indicating numbers of vertices of polygons around a vertex. For instance the local type of the triangle tiling is 3^6.

Let $\mathcal{T}$ be a semiregular tiling. By $\mathrm{Aut}(\mathcal{T})$ we denote the group of its isometries. It is clearly a wallpaper group. We shall denote by $T_{\mathcal{T}}$ the subgroup of $\mathrm{Aut}(\mathcal{T})$ consisting of translations. It is a normal abelian subgroup and we shall denote by $t_{\mathcal{T}}$ the index $[\mathrm{Aut}(\mathcal{T}) : T_{\mathcal{T}}]$. The subgroup $T_{\mathcal{T}}$ can be viewed as a lattice. There are three possibilities for structure of this lattice

- $T_{\mathcal{T}}$ is square-like, when $\mathcal{T}$ is the square or snub square or truncated square tiling,
- $T_{\mathcal{T}}$ is rhombic, when $\mathcal{T}$ is the elongated triangular tiling,
- $T_{\mathcal{T}}$ is triangle-like for the remaining tilings.

1.1.4. Maps and FVT maps. The base framework for our work is the theory of maps as described in [**7**]. In particular, maps considered here can have multiple edges and loops.We say the action is an FVT action if Γ acts transitively on vertices of $\mathcal{T}$ and the only group element stabilizing a vertex is the identity.

Throughout the paper we will use the term *FVT map*, which is a generalization of a Cayley map. By an FVT map on a surface we mean a map and a group G acting on the map permuting its vertices freely and transitively. Thus a Cayley map is a special kind of FVT map in which there is an additional condition: the group action preserves orientation of the surface.

2. FVT actions of wallpaper groups on semiregular tilings

We have three main goals in this section. Firstly, we shall present the seventeen classes of wallpaper groups and for each of them describe its translation subgroup.

Secondly, we shall describe various FVT actions (see Figures 2–18). As is customary when dealing with wallpaper groups, actions are described graphically by describing the layout of the fundamental parallelogram of $T(\Gamma)$, mirrors of the group (in bold lines), its glides (in dashed lines) and rotations (diamond – rotation by π, square by $\pi/2$, triangle by $2\pi/3$ and hexagon for rotation by $\pi/3$).

In Section 2.4 we shall prove that there are no more FVT actions of a wallpaper group on a semiregular tiling.

2.1. Actions of non-rigid groups

These are groups whose fundamental region can be any parallelogram, not necessarily rectangular nor rhombic.

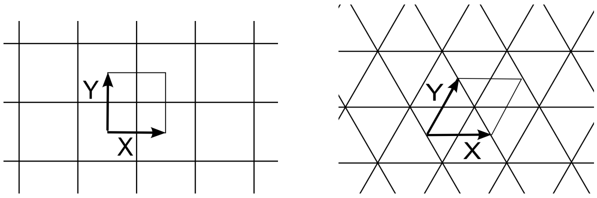
2.1.1. Group *p1*. The group *p1* is the free abelian group generated by two translations X and Y. It has presentations as follows:

$$p1 := \langle X, Y;\ XY = YX \rangle \tag{1}$$

or upon introducing $Z = -X - Y$

$$= \langle X, Y, Z;\ XYZ = ZYX = 1 \rangle. \tag{2}$$

Actions on the triangle and square tilings are shown in Figures 2(a) and 2(b).



(a) Square tiling (b) Triangle tiling

Figure 2. FVT actions of the group *p1*.

2.1.2. Group ***p2*****.** The group *p2* is obtained from the group *p1* by adding a half-rotation T that under conjugation changes X, Y to their inverses. It has presentations:

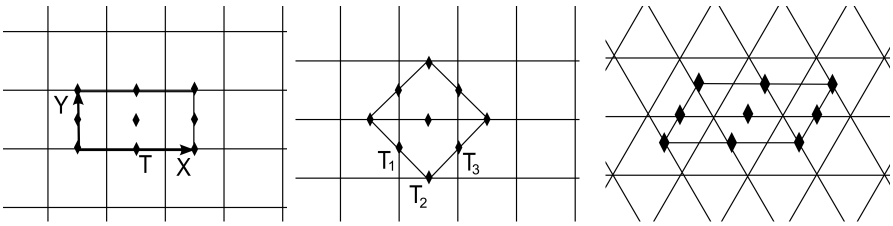
$$p2 := \langle X, Y, T;\ XY = YX, T^2 = (TX)^2 = (TY)^2 = 1\rangle \tag{3}$$

or upon introducing $T_1 = TY$, $T_2 = XT$, $T_3 = T$

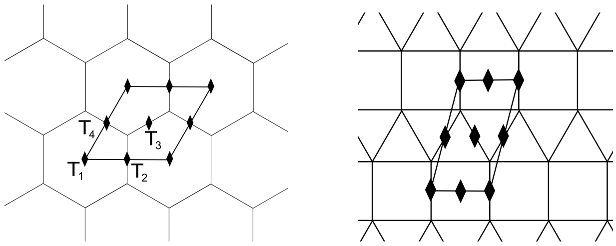
$$= \langle T_1, T_2, T_3;\ T_1^2 = T_2^2 = T_3^2 = (T_1T_2T_3)^2 = 1\rangle \tag{4}$$

or by setting $T_4 = T_1T_2T_3 = T_1X$

$$= \langle T_1, T_2, T_3, T_4;\ T_1^2 = T_2^2 = T_3^2 = T_4^2 = (T_1T_2T_3T_4)^2 = 1\rangle \tag{5}$$



(a) Square tiling (b) Triangle tiling



(c) Hexagonal tiling (d) Elongated triangular tiling

Figure 3. FVT actions of the group *p2*.

The subgroup of translations is generated by $X = T_2T_3, Y = T_3T_1$ and is of index 2. Its actions on the square, triangle, hexagonal and elongated triangular tilings can be seen in Figure 3(a), 3(b), 3(c), 3(d).

2.2. Actions of rigid subgroups of $p4m$

2.2.1. Group *pm*. The group *pm* is obtained from the group *p1* by adding a reflection R that conjugates X with X^{-1} and preserves Y. It follows that X and Y have to be orthogonal. It has presentations:

$$pm := \langle X, Y, R;\ XY = YX, R^2 = 1, RXR = X^{-1}, RYR = Y\rangle \tag{6}$$

or upon introducing $R' = RX$

$$= \langle R, R', Y;\ R^2 = {R'}^2 = 1, RY = YR, R'Y = YR'\rangle \tag{7}$$

The subgroup of translations is generated by Y and $X = RR'$.

Its FVT action on the square tiling is shown in Figure 4.

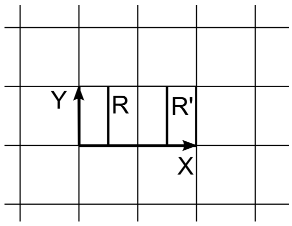


Figure 4. An FVT action of the group *pm* on the square lattice.

2.2.2. Group *pg*. The group *pg* is obtained from the group *p1* by adding a glide reflection whose square is Y and which conjugates X with X^{-1}. It follows that necessarily X and Y are orthogonal vectors.

$$pg := \langle P, X, Y;\ [X, Y] = 1, P^2 = Y, P^{-1}XP = X^{-1}\rangle$$

or by setting $Q = PX$ as in [**4**] (4.504)

$$= \langle P, Q;\ P^2 = Q^2\rangle$$

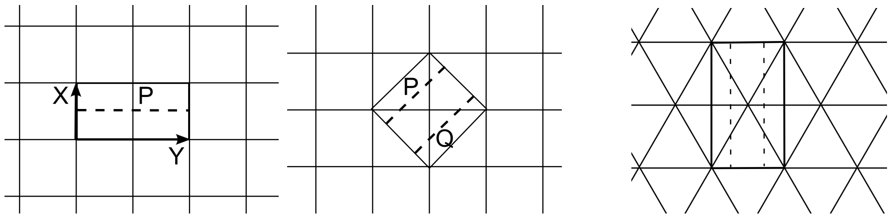
The subgroup of translations $\langle X, Y\rangle = \langle P^{-1}Q, P^2\rangle$ has index 2 in *pg*. Its FVT actions on the square and triangular tilings are shown in Figures 5(a), 5(b).

2.2.3. Group *p4*. The group *p4* can be obtained from the group *p2* by adding a rotation S which cyclically permutes four half-rotations T_1, T_2, T_3, $T := T_4$. It has the presentation

$$p4 := \langle S, T;\ S^4 = T^2 = (ST)^4 = 1\rangle$$

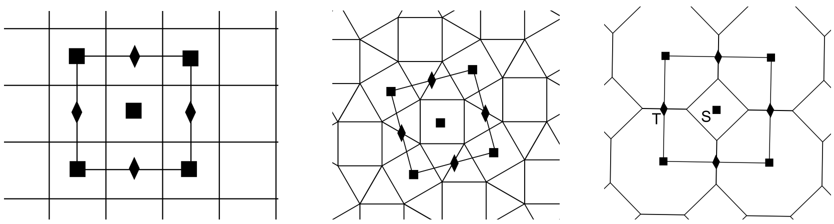
The subgroup of translations is generated by STS and TS^2, and it is of index 4.

Its FVT actions on the square, snub square and truncated square tilings are shown in Figures 6(a), 6(b), 6(c).



(a) Square tiling (b) Triangular tiling

Figure 5. FVT actions of the group *pg*.



(a) Square tiling (b) Snub square tiling (c) Truncated square tiling

Figure 6. FVT actions of the group *p4*.

2.2.4. Group pmm**.** The group pmm can be obtained from the group pm by adding a reflection R_2 that fixes reflections R and R' and conjugates Y with Y^{-1}. It has presentations

$$\begin{aligned} pmm :=&\langle R, R', R_2, Y;\ [R, Y] = [R', Y] = R^2 = R'^2 = R_2^2 = 1, \\ &R_2RR_2 = R, R_2R'R_2 = R', R_2YR_2 = Y^{-1}\rangle \end{aligned}$$

or by setting $R_1 = R$, $R_3 = R'$ and $R_4 = R_2Y$ as in [**4**] in (4.506)

$$\begin{aligned} &= \langle R_1, R_2, R_3, R_4;\ R_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_1R_2)^2 \\ &= (R_2R_3)^2 = (R_3R_4)^2 = (R_4R_1)^2 = 1\rangle \end{aligned}$$

The subgroup of translations is equal to that of pm. It is generated by the pair of orthogonal translations $RR' = R_1R_3$ and $Y = R_2R_4$.

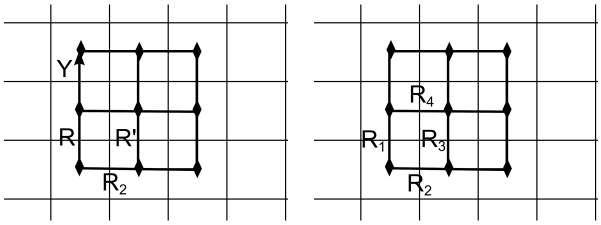
It has an FVT action on the square tiling as shown in Figure 7.

2.2.5. Group cm**.** This group can be obtained by adding a reflection R to the group pg that exchanges P and Q. It has presentations

$$cm :=\langle P, Q, R;\ P^2 = Q^2, RPR = Q, R^2 = 1\rangle$$

or by setting $S = PR$ and excluding Q as in [**4**], (4.505)

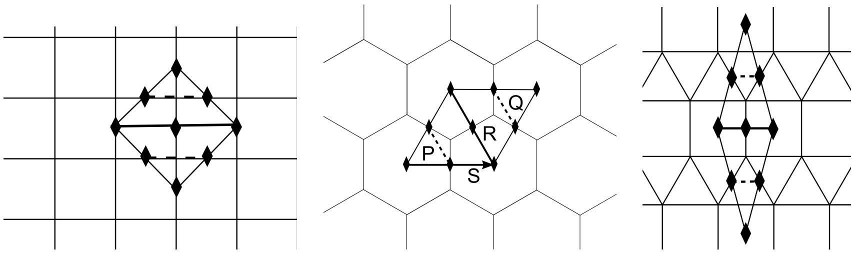
$$=\langle R, S;\ (RS)^2 = (SR)^2, R^2 = 1\rangle$$



(a) Square tiling (b) Square tiling

Figure 7. FVT actions of the group *pmm*.

Its subgroup of translations contains translations of *pg* as a subgroup of index 2. It is generated by a pair of equal length vectors S and RSR. Its FVT actions on



(a) Square tiling (b) Hexagonal tiling (c) Elongated triangular tiling

Figure 8. FVT actions of the group *cm*.

square, hexagonal and elongated triangular tilings are shown in Figures 8(a), 8(b), 8(c).

2.2.6. Group ***pmg*.** The group *pmg* can be obtained from the group *pg* by adding a reflection R that reverses both P and Q. It has presentations

$$pmg := \langle P, Q, R; P^2 = Q^2; R^2 = (RP)^2 = (RQ)^2 = 1\rangle$$

or by setting $T_1 = PR$, $T_2 = QR$ as in [**4**] in (4.507)

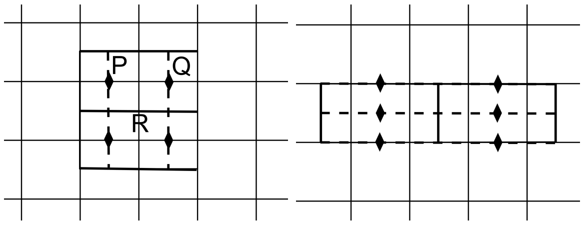
$$= \langle R, T_1, T_2; R^2 = T_1^2 = T_2^2 = 1, T_1RT_1 = T_2RT_2\rangle$$

Its group of translations coincides with those of *pg* and is generated by a pair of orthogonal translations $P^2 = (T_1R)^2$ and $P^{-1}Q = RT_1T_2R = T_1T_2$.

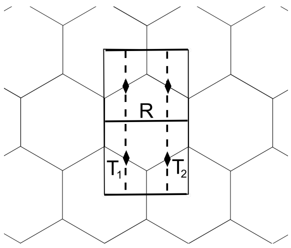
Its FVT actions on square, hexagonal and elongated triangular tilings are shown in Figure 9(a), 9(b), 9(c).

2.2.7. Group ***pgg*.** The group *pgg* can be obtained from the group *pg* by adding a half-rotation T conjugating P with Q^{-1}. It has presentations

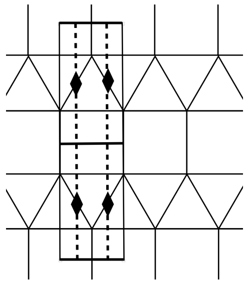
$$pgg := \langle P, Q, T;\ P^2 = Q^2, TPT^{-1} = Q^{-1}, T^2 = 1\rangle$$



(a) Square tiling



(b) Hexagonal tiling



(c) Elongated triangular tiling

Figure 9. FVT actions of the group *pmg*.

or by setting $O = PT$ as in [**4**] in (4.508)

$$=\langle P, O;\, (PO)^2 = (P^{-1}O)^2 = 1\rangle$$

Its lattice of translations equals to that of *pg*, and is generated by $P^{-1}Q = P^{-1}TP^{-1}T = (P^{-2}O)^2$ and P^2 (alternatively one can take P^2 and O^2).

FVT actions of *pgg* on the square, hexagonal, snub square, triangular and elongated triangular tilings are shown in Figure 10.

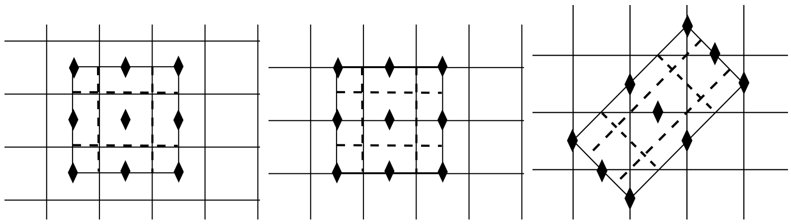
2.2.8. Group ***p4g*****.** The group *p4g* is obtained from the group *pmm* by adding a rotation S. This rotation cyclically permutes reflections R_1, R_2, R_3 and R_4. Setting $R := R_4$ and $R_i = S^{-i}RS^i$ one gets its presentation ([**4**], (4.512))

$$p4g := \langle R, S; R^2 = S^4 = (RS^{-1}RS)^2 = 1\rangle$$

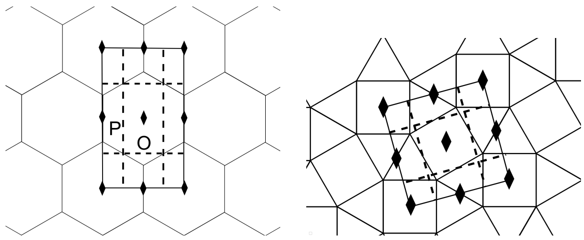
Its lattice of translations contains the lattice of translations of *pmm* as a sublattice of index 2. It is generated by orthogonal translations of equal length: $S^2R_1R_4 = S^2RS^{-1}RS$ and $S^2R_2R_1 = RSRS$.

FVT actions of this group on square and truncated square tilings are shown in Figures 11(a),11(b).

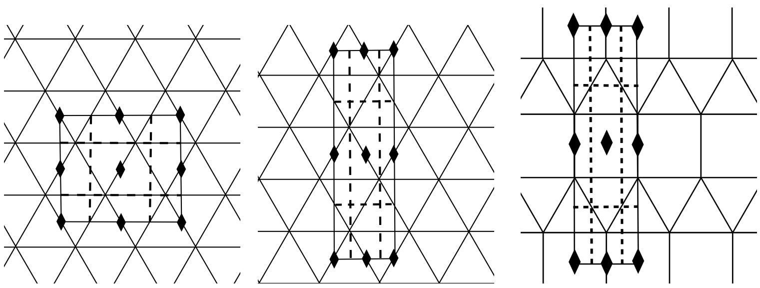
2.2.9. Group ***cmm*****.** The group *cmm* can be obtained from the group *pmm* by adding a half-rotation T that exchanges R_1 with R_3 and R_2 with R_4. It has



(a) Square tiling



(b) Hexagonal tiling (c) Snub square tiling



(d) Triangle tiling (e) Elongated triangular tiling

Figure 10. FVT actions of the group *pgg*.

presentation ([**4**], (4.509))

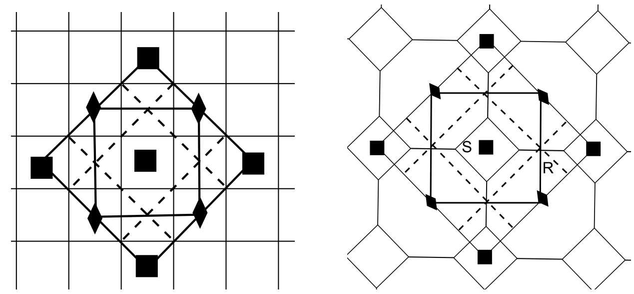
$$cmm := \langle R_1, R_2, T;\ R_1^2 = R_2^2 = T^2 = (R_1R_2)^2 = (R_1TR_2T)^2 = 1\rangle$$

Its lattice of translations contains the lattice of translations of pmm as a sublattice of index 2. It is generated by a pair of equal length translations R_1R_2T and R_2TR_1.

FVT actions of this group on the square and truncated square tilings are shown in Figures 12(a), 12(b).

2.2.10. Group $p4m$. The group $p4m$ is obtained from the group pmm by adding a reflection R. This reflection exchanges reflections R_1 with R_4 and R_2 with R_3. It has the following presentation ([**4**], (4.511)):

$$p4m := \langle R, R_1, R_2; R^2 = R_1^2 = R_2^2 = (RR_1)^4 = (R_1R_2)^2 = (R_2R)^4 = 1\rangle$$



(a) *p4g* acting on the square tiling (b) *p4g* acting on the truncated square tiling

Figure 11. FVT actions of the group *p4g*.

Its lattice of translations coincides with that of *pmm*. It is generated by a pair of orthogonal vectors of equal length: $R_1R_3 = R_1RR_2R$ and $R_2R_4 = R_2RR_1R$.

Its FVT action on truncated square tiling is shown in Figure 13.

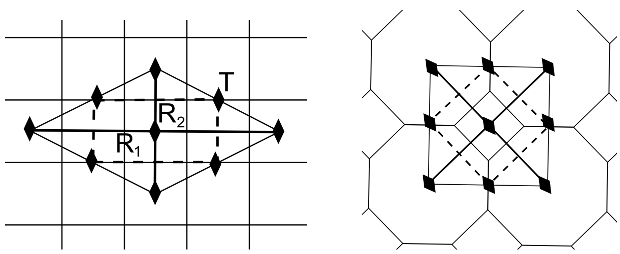
2.3. Actions of rigid subgroups of *p6m*

2.3.1. Group *p3*. Suppose the angle between generating translations X, Y of the group *p1* is $2\pi/3$. The group *p3* is then obtained by adding to *p1* a rotation S_1 by angle $2\pi/3$. The rotation cyclically permutes X, Y and $Z := -X - Y$. It has presentations:

$$\begin{aligned} p3 :=&\langle XYZ = ZYX = S_1^3 = 1, S_1^{-1}XS_1 = Y, \\ &S_1^{-1}YS_1 = Z, S_1^{-1}ZS_1 = X\rangle \end{aligned}$$

or by setting $S_2 = S_1X$ and $S_3 = X^{-1}S$ as in ([**4**], 4.513)

$$\begin{aligned} =&\langle S_1, S_2, S_3;\ S_1^3 = S_2^3 = S_3^3 = S_1S_2S_3 = 1\rangle \\ =&\langle S_1, S_2;\ S_1^3 = S_2^3 = (S_1S_2)^3 = 1\rangle \end{aligned}$$



(a) Square tiling (b) Truncated square tiling

Figure 12. FVT actions of the group *cmm*.

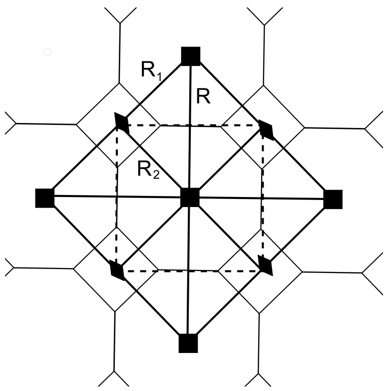
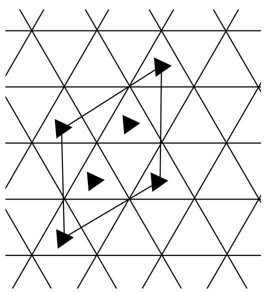


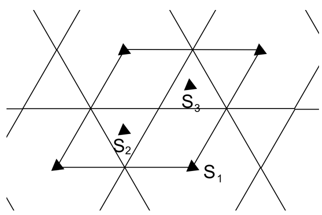
Figure 13. An FVT action of the group $p4m$ on the truncated square tiling.

The subgroup of translations is of index 3, and it is generated by a pair of equal length translations $X = S_1^{-1}S_2$ and $Y = S_1^{-1}XS_1 = S_1S_2S_1$.

Its FVT actions on the triangular and trihexagonal tilings are shown in Figures 14(a), 14(b).



(a) Triangular tiling



(b) Trihexagonal tiling

Figure 14. FVT actions of the group $p3$.

2.3.2. Group $p6$**.** The group $p6$ is obtained from the group $p3$ by adding a half-rotation T. The rotation conjugates rotation S_1 and S_2. It has presentations

$$p6 := \langle S_1, S_2, T;\, S_1^3 = S_2^3 = (S_1S_2)^3 = T^2 = 1, TS_1T = S_2\rangle$$

or by excluding S_2 and setting $S := S_1$ we get ([**4**], (4.516)):

$$= \langle S, T;\, S^3 = T^2 = (ST)^6 = 1\rangle$$

The subgroup of translations is of index 6 (it coincides with $p3$) and it is generated by a pair of equal length vectors $S_1^{-1}S_2 = S^{-1}TST$ and $S_1S_2S_1 = STSTS$.

Its FVT actions on the small rhombitrihexagonal, truncated hexagonal, hexagonal and snub hexagonal tilings are shown in Figure 15(a), 15(b), 15(c), 15(d).

2.3.3. Group *p31m***.** The group *p31m* is obtained from the group *p3* by adding a reflection R. This reflection conjugates S_1 with S_2^{-1}. It has presentations

$$p31m := \langle S_1^3 = S_2^3 = (S_1 S_2)^3 = R^2 = 1, RS_1R = S_2^{-1} \rangle \tag{8}$$

or by setting $S = S_1$ and excluding S_2 ([**4**], (4.514))

$$= \langle R, S;\ R^2 = S^3 = (RS^{-1}RS)^3 = 1 \rangle \tag{9}$$

The translation subgroup is equal to that of *p3* and is generated by $S_1^{-1}S_2 = S^{-1}RS^{-1}R$ and $S_1S_2S_1 = SRS^{-1}RS$.

It has FVT actions on the small rhombitrihexagonal tiling and truncated hexagonal tilings as shown in Figure 16(a), 16(b).

2.3.4. Group *p3m1***.** The group *p3m1* is also obtained from the group *p3* by adding a reflection R. This reflection conjugates S_1 with S_1^{-1} and S_2 with S_2^{-1} . It has presentations

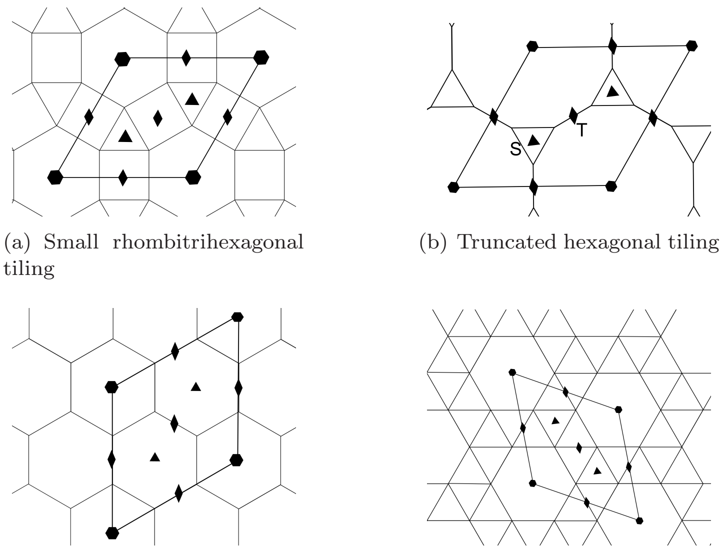
$$p3m1 := \langle S_1, S_2, R;\ S_1^3 = S_2^3 = (S_1S_2)^3 = 1, RS_1R = S_1^{-1}, RS_2R = S_2^{-1} \rangle \tag{10}$$

or by setting $R_1 = RS_2$, $R_2 = S_1R$, $R_3 = R$ ([**4**], (4.515)):

$$= \langle R_1, R_2, R_3;\ R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_1)^3 = 1 \rangle \tag{11}$$

Its translation subgroup is equal to that of *p3* and is generated by $S_1^{-1}S_2 = R_3R_2R_3R_1$ and $S_1S_2S_1 = R_2R_1R_2R_3$.

It has FVT actions on hexagonal tiling as shown in Figure 17.



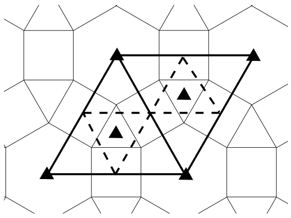
(a) Small rhombitrihexagonal tiling

(b) Truncated hexagonal tiling

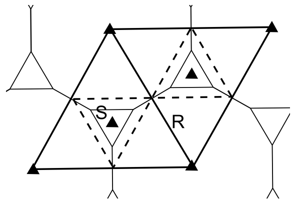
(c) Hexagonal tiling

(d) Snub hexagonal tiling

Figure 15. FVT actions of the group *p6*.



(a) Small rhombitrihexagonal tiling



(b) Truncated hexagonal tiling

Figure 16. FVT actions of the group *p31m*.

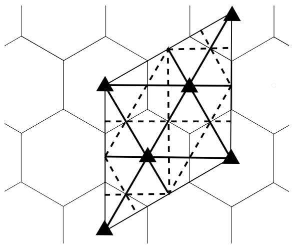


Figure 17. An FVT action of the group *p3m*1 .

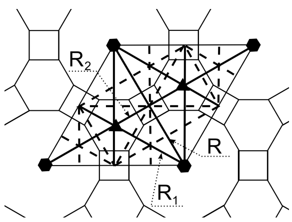


Figure 18. An FVT action of *p6m* on the great rhombitrihexagonal tiling.

2.3.5. Group *p6m***.** The group *p6m* is obtained from the group *p3m1* by adding a reflection R. This reflection conjugates mirrors R_1 and R_3 and fixes R_2 . It has presentations

$$\begin{aligned} p6m :=& \langle R, R_1, R_2, R_3;\ R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_1)^3 = 1 \\ & R^2 = 1, RR_1R = R_3, RR_2R = R_2 \rangle \end{aligned}$$

or by excluding R_3 ([**4**], (4.517))

$$= \langle R, R_1, R_2;\ R^2 = R_1^2 = R_2^2 = (R_1R_2)^3 = (R_2R)^2 = (RR_1)^6 = 1 \rangle$$

Its translation subgroup coincides with *p3m1* and is generated by $R_3R_2R_3R_1 = RR_1RR_2RR_1RR_1$ and $R_2R_1R_2R_3 = R_2R_1R_2RR_1R$.

It has an FVT action on the great rhombitrihexagonal tiling as shown in Figure 18.

2.4. Nonexistence of other actions

We have shown for every wallpaper group one or multiple FVT actions on semiregular tilings of the plane. Summary of these actions can be seen in Table 1. It is natural to ask whether there are more actions.

$t_{\mathcal{T}}$	1		2		4		3	6			12
Type	4^4	3^6	6^3	$3^3{\cdot}4^2$	$4{\cdot}8^2$	$3^2{\cdot}4{\cdot}3{\cdot}4$	$3{\cdot}6{\cdot}3{\cdot}6$	$3{\cdot}12^2$	$3{\cdot}4{\cdot}3{\cdot}6$	$3^5{\cdot}6$	$4{\cdot}6{\cdot}4{\cdot}12$
p1	2(a)	3(a)									
p2	3(a)	3(b)	3(c)	3(d)							
pm	4										
pg	5(a)	5(b)									
cm	8(a)		8(b)	8(c)							
pmm	7										
pmg	9(a)		9(b)	9(c)							
pgg	10(a)	10(d)	10(b)	10(e)		10(c)					
cmm	12(a)				12(b)						
p4	6(a)				6(c)	6(b)					
*p4*g	11(a)				11(b)						
*p4*m					13						
p3		14(a)					14(b)				
p3m1			17								
p31m								16(b)	16(a)		
p6			15(c)					15(b)	15(a)	15(d)	
*p6*m											18

Table 1. Crosstable of FVT actions.

In fact, we have have

Theorem 2. *A wallpaper group* Γ *has an FVT action on a semi-regular tiling* $\mathcal{T}$ *if and only if an action is listed in Table 1.*

Proof. The only-if part follows from Propositions 3, 4 and 5, which follow. The if part follows from direct examination of pictures referenced in Table 1. □

Suppose a semiregular plane tiling $\mathcal{T}$ is given and Γ is a wallpaper group together with action σ (by plane isometries) on $\mathcal{T}$. The key to investigation of all possible actions is the following equation relating number of vertex orbits modulo translations

$$t_{\mathcal{T}} i_\sigma = [\Gamma : T(\Gamma)]. \tag{12}$$

In this equation $t_{\mathcal{T}}$ is the number of orbits of vertices under the action of the subgroup of translations preserving $\mathcal{T}$ and i_σ is the index of the image of $T(\Gamma)$ in $T_{\mathcal{T}}$.

As an example, consider the simplest consequence of this equality. Suppose Γ is of type *p1*. Then $[\Gamma : T(\Gamma)] = 1$ hence Γ can act only on the triangular or square tiling. Moreover, i_σ has to be 1. Both square and triangle tiling in fact admit action of *p1*.

2.4.1. Square-like $T_{\mathcal{T}}$. In this section we consider only semiregular tilings for which $T_{\mathcal{T}}$ is square-like. Since any square-like lattice has no sublattices invariant under rotation by angle $2\pi/3$, it follows that FVT actions by groups *p3*, *p3m1*, *p31m*, *p6*, *p6m* are apriori not possible.

Proposition 3. *Suppose Γ is a wallpaper group and $\mathcal{T}$ a tiling with a square-like lattice of translations. If there is no action indicated in Table 1 then there is no FVT action of Γ on $\mathcal{T}$.*

Proof. We have to deal with the square, truncated square and snub square tiling. In the following we shall assume that the length of the shortest nonzero translation fixing $\mathcal{T}$ is 1.

Consider the square tiling. There is no FVT action of the group *p4m* on the square tiling, because such an action would have reflections at 45 degrees. Every line of symmetry not parallel to the edges of the square tiling however passes through a vertex, thus the resulting action would have a nontrivial vertex stabilizer.

Consider now the snub square tiling. It follows from (12) that the only remaining groups that could admit an FVT action are *pmm*, *pmg*, *cmm*, *p4m*, and *p4g*. Since every line of symmetry of the tiling passes through a vertex, the first four groups cannot act. Since its parallel lines of symmetry have distance $\sqrt{2}/2$, it cannot have FVT action by group *p4g*. (The full symmetry group of this tiling is *p4g* but it acts with non-trivial vertex stabilizers.)

Consider finally the truncated square tiling. It follows from (12) that the only remaining groups that could admit an FVT action are *pmm*, *pmg*, *pgg*. In the former two cases, that would require parallel reflections to be 1/2 apart, and for the latter two cases it would require that parallel glides be 1/2 apart. That is not the case, as is seen by looking at Figure 13 which shows the full symmetry group. There we see that parallel reflections are at minimum 1 apart, as well as parallel glides. □

2.4.2. Triangle-like $T_{\mathcal{T}}$. In this section we consider only semiregular tilings for which $T_{\mathcal{T}}$ is triangle-like. Since any triangle-like lattice has no sublattices invariant under rotation by angle $2\pi/4$, it follows that FVT actions by groups *p4*, *p4g*, *p4m* are apriori not possible.

Proposition 4. *Suppose Γ is a wallpaper group and $\mathcal{T}$ a tiling with a triangle-like lattice of translations. If there is no action indicated in Table 1 then there is no FVT action of Γ on $\mathcal{T}$.*

Proof. We have to deal with the triangle, hexagonal,trihexagonal, snub hexagonal, truncated hexagonal, small and great rhombitrihexagonal tilings. In the following we shall assume that the length of the shortest translation of $\mathcal{T}$ is 1.

Consider first the triangle tiling. Any line of symmetry passes through a vertex, thus a group can have an FVT action only if it contains no reflections. It thus remains to prove that FVT actions by group *p6* is not possible. But that follows from the fact that the triangle lattice has no sublattice of index 6 invariant under rotation by $2\pi/3$.

Consider the hexagonal tiling. From (12) it follows that we need to exclude only FVT actions by groups *pm*, *pg*, *pmm*, *cmm*, *p31m* and *p6m*. An action by first two would imply that $\sigma(T(\Gamma))$ is rectangular. But from (12) it would follow that $i_\sigma = 1$, and $\sigma(T(\Gamma))$ would be rhombic, which is a contradiction. It has no FVT action by *pmm* and *cmm* because for any pair of perpendicular lines of symmetry, one of them passes through a vertex. It cannot have an action by the group *p31m*, because parallel mirrors not containg a vertex are multiple of $\sqrt{3}/2$ apart, but an action would imply existence of parallel mirrors $3/2$ apart. Finally, an FVT action by *p6m* is not possible, since the triangle lattice has no sublattice of index 2 invariant under rotation by $2\pi/3$.

Consider the trihexagonal tiling. It follows from (12) that we need to exclude FVT actions by groups *p3m1*, *p31m*, *p6* and *p6m*. The first three are impossible because the triangle lattice has no sublattice of index 6 invariant under rotation by $2\pi/3$. If it had an action by *p6m*, the centers of rotation by angle $2\pi/6$ would have to be centered at centers of hexagons in an essentially unique way (because there is only one sublattice of index 4 invariant under rotation by $2\pi/3$). But then reflections passing through the centers would fix vertices of the tiling, thus the action would not be FVT.

Consider now the snub hexagonal tiling. This tiling is the unique tiling not invariant under reflections, thus *p6* is the full symmetry group of the tiling and its action is an FVT action.

Truncated hexagonal and small rhombitrihexagonal tilings are very similar. From (12) it follows that we need to exclude only actions of *p6m* and *p3m1*. The action by the former is impossible, because there is no index 2 sublattice of the triangle lattice having invariance under the rotation by angle $2\pi/3$. The action of the latter would imply $i_\sigma = 1$, and once easily sees that in both tilings there would be mirrors passing through a vertex.

Finally, the great rhombitrihexagonal tiling can only have an action by *p6m* as follows from (12). □

2.4.3. Elongated triangular tiling. The lattice of translations of the elongated triangular tiling is rhombic. It contains neither the square lattice nor the triangle lattice. It follows that actions by groups with rotation by angle smaller that π are impossible.

Proposition 5. *If Γ is a wallpaper group with FVT action on the elongated triangular tiling, then the action is one listed in Table 1.*

Proof. It follows from (12) that we need to exclude only groups *pm*, *pg*, *cmm* and *pmm*. The former two groups cannot have FVT action because the translation lattice $T_{\mathcal{T}}$ is not rectangular. Since $T_{\mathcal{T}}$ has no sublattice of index 2 which is rhombic, it follows it cannot have an FVT action by *cmm* either. It cannot have FVT action by *pmm* because "vertical" reflections pass through a vertex of the tiling. □

3. Families of toric groups

By a *toric* group we mean a finite group that acts faithfully on a torus (topological or real-analytic). Any such group is connected with a Cayley map on torus by the following theorem of Tucker ([**14**], see also [**6**], Theorem 6.2.4, and [**1**], Lemma 7.3):

Theorem 6. *Let the finite group G act on an orientable surface S. Then there is a Cayley graph for G cellularly embedded in S so that the natural action of G on the Cayley graph extends to the given action of G on S.*

Thus given a toric group G we can construct a Cayley map on torus. Taking its pullback to the universal cover of the torus we shall obtain an FVT action of a wallpaper group Γ on a semiregular tiling. As explained in [**4**], Section 3.6., presentations of the group G can be obtained by adding to relations of Γ the relations corresponding to the sides of a fundamental parallelogram of a lattice Λ' that induces a torus as a quotient. The precise form of relations depends on the group, and of course on the lattice. We start by describing families of lattices and then distinguish five essentially different cases.

3.1. Sublattices

Suppose a basis X, Y of a lattice Λ is given. We call $\Lambda(a; b, c)$ the sublattice generated by X^a and X^bY^c

$$\Lambda(a; b, c) := \langle X^a, X^cY^b\rangle$$

If the lattice Λ is rectangular, we can take X and Y to be orthogonal (necessarily) of minimal length. We define the following sublattices of Λ

$$\Lambda^o_o(a, b) := \Lambda(a; b, 0) = \langle X^a, Y^b\rangle$$
$$\Lambda^o_c(a, b) := \Lambda(2a; b, a) = \langle X^aY^b, X^aY^{-b}\rangle$$

If the lattice Λ is rhombic, we can take X and Y to be a linearly independent pair of minimal vectors of equal length. We define the following sublattices of Λ

$$\Lambda^c_o(a, b) := \langle X^aY^a, X^bY^{-b}\rangle$$
$$\Lambda^c_c(a, b) := \langle X^aY^b, X^bY^a\rangle$$

If the lattice Λ is square-like (and hence both rhombic and rectangular), we can take vectors X and Y to be both orthogonal and equal length. We define the following sublattices of Λ

$$\Lambda_4(b,c) := \langle X^bY^c, X^{-c}Y^b\rangle$$
$$\Lambda_4(a) := \Lambda_4(a,0) = \Lambda_o^o(a,a) = \Lambda_c^c(a,0) = \Lambda(a;a,0) = \langle X^a, Y^a\rangle$$
$$\Lambda_4'(a) := \Lambda_4(a,a) = \Lambda_c^o(a,a) = \Lambda_o^c(a,a) = \Lambda(2a;a,a) = \langle X^aY^a, X^aY^{-a}\rangle$$

If the lattice Λ is triangle-like, we can tak vectors X and Y to be minimal and forming angle $2\pi/3$. We define the following sublattices of Λ

$$\Lambda_3(b,c) := \langle X^bY^c, X^{-c}Y^{b-c}\rangle$$
$$\Lambda_3(a) := \Lambda_3(a,0) = \Lambda(a;a,0) = \langle X^a, Y^a\rangle$$
$$\Lambda_3'(a) := \Lambda_3(a,-a) = \Lambda(3a;a,-a) = \langle X^aY^{-a}, X^aY^{2a}\rangle$$

3.2. Quotients of *p1* and *p2*

These two groups are the only wallpaper groups that have no restriction on the type of the lattice of translations, meaning that any sublattice of $T(\Gamma)$ is automatically an invariant subgroup. If a basis X, Y of $T(\Gamma)$ is chosen, then any sublattice Λ' is generated by two elements $p_{11}\cdot X + p_{12}\cdot Y$ and $p_{21}\cdot X + p_{22}\cdot Y$ where p_{ij} are integers. By row reduction of the matrix $\left(\begin{smallmatrix} p_{11} & p_{12} \\ p_{21} & p_{22}\end{smallmatrix}\right)$ we can bring it into the form $\left(\begin{smallmatrix} a & 0 \\ c & b\end{smallmatrix}\right)$. Then $a\cdot X$ and $c\cdot X + b\cdot Y$ will be another basis for Λ. Corresponding to this lattice we obtain a three-parameter family of groups

$$\Gamma(a;b,c) := \Gamma/\langle X^a, Y^bX^c\rangle.$$

3.3. Quotients of *pm*, *pg*, *pmm*, *pmg* and *pgg*

The common feature of these five groups is that the lattice of translations is rectangular. Suppose a reflection or a glide r of Γ is given. Choose an orthogonal basis X and Y of $T(\Gamma)$ such that conjugation by r maps X to X^{-1} and fixes Y. Conjugation by r maps the lattice $\Lambda(a;b,c)$ to the lattice $\Lambda(a;b,-c)$. Lattices $\Lambda(a;b,c)$ and $\Lambda(a;b,-c)$ are isomorphic if and only if $c\equiv -c \pmod a$. Therefore if a lattice Λ is invariant under r it has to belong to one of the following families

$$\Lambda_o^o(a,b) = \Lambda(a;b,0)$$
$$\Lambda_c^o(a,b) = \Lambda(2a;b,a)$$

Corresponding to the former we obtain the following quotient of Γ:

$$\Gamma_1(a;b) := \Gamma/\langle X^a, Y^b\rangle$$

and corresponding to the latter the quotient

$$\Gamma_2(a;b) := \Gamma/\langle X^{2a}, X^aY^b\rangle = \Gamma/\langle Y^{2b}, X^aY^b\rangle = \Gamma/\langle X^aY^{-b}, X^aY^b\rangle.$$

Note that the definition of these groups depends on the choice of the transformation r, which dictates the choice of translations X and Y. This is not a problem for groups *pm*, *pg* because all glides and mirrors in these groups are parallel. Because an outer automorphism interchanges generating mirrors of *pmm* and glides of *pgg* this is not a problem for these groups. In the case of the group pmg we choose r to be the mirror R.

3.4. Quotients of cm and cmm

The common feature of these groups is that the lattice $\sigma(T(\Gamma))$ is rhombic.

Suppose that Λ is invariant under Γ. Let m be a reflection in Γ, and choose a pair of shortest nonzero vectors in $T(\Gamma)$ such that $m(X) = Y$. The vectors X and Y then form a basis of Λ. View Λ as a sublattice of the the rectangular lattice Λ' with the basis $E := \frac{1}{2}(X - Y)$ and $F := \frac{1}{2}(X + Y)$. If we express Λ in this basis, the invariance under m implies as above that Λ has to be one of the following lattices

$$\begin{cases} \mathbf{Z} \cdot aE \oplus \mathbf{Z} \cdot bF \\ \mathbf{Z} \cdot (cE - dF) \oplus \mathbf{Z} \cdot (cE + dF). \end{cases}$$

The condition that Λ is a sublattice of $\mathbf{Z} \cdot X \oplus \mathbf{Z} \cdot Y$ means that in former case, a and b have to be even, and in the latter case c and d have to be of the same parity. In order to obtain families dependent on pairs of integers we can reparametrize by transformations $a' = \frac{a}{2}, b' = \frac{b}{2}$ in the former case, and $c' = \frac{c-b}{2}, d' = \frac{c+b}{2}$ in the latter case.

We will thus obtain families

$$\Gamma_1(c', d') := \Gamma/\langle X^{c'}Y^{d'}, X^{d'}Y^{c'} \rangle \quad \text{corresponding to } \Lambda^c_c(c', d')$$
$$\Gamma_2(a', b') := \Gamma/\langle (XY)^{b'}, X^{a'}Y^{-a'} \rangle \quad \text{corresponding to } \Lambda^c_o(a', b')$$

3.5. Quotients of $p3, p4, p6$

The common feature of these three groups is that they contain a rotation ϕ by angle $< \pi$ and they do not contain a reflection. The first property implies that if a single vector (b, c) belongs to the lattice Λ, then also $\phi\big((b, c)\big)$ is contained in Λ. Thus lattices invariant under Γ are completely enumerated by the families $\Lambda_4(b, c)$ for the group $p4$ and $\Lambda_3(b, c)$ for groups $p3$ and $p6$.

The corresponding quotients of Γ is the group

$$\Gamma(b, c) := \Gamma/\langle X^b Y^c \rangle$$

3.6. Quotients of $p4m$ and $p4g$

The common feature of these two groups is that they contain a rotation by angle $\pi/2$ as well as a reflection or a glide. It follows that a lattice Λ is invariant only if the Λ is square-like *and* it is invariant under reflecting in the direction of the reflection (in the case of $p4m$) or the glide reflection (in the case of $p4g$). Choose a basis of $T_{\mathcal{T}}$ consisting of a pair of equal-length, mutually orthogonal vectors X and Y. There are two one-parameter families of such lattices, namely $\Lambda_4(a)$ and $\Lambda'_4(a)$The group Γ induces a FVT action on the map $\mathcal{T}/\Lambda_4(a)$ via its quotient

$$\Gamma_1(a) := \Gamma/\langle X^a \rangle$$

and on the map $\mathcal{T}/\Lambda_4'(a)$ via its quotient

$$\Gamma_2(a) := \Gamma/\langle X^a Y^a \rangle$$

3.7. Quotients of *p31m*, *p3m1* and *p6m*

The common feature of these three groups is that they contain a rotation by angle $2\pi/3$ as well as a reflection. We can choose a basis X, Y of equal length-vectors that form angle $2\pi/3$. A lattice Λ is invariant if and only if the Λ is triangle-like *and* invariant under mirroring in the direction of the reflection. There are two one-parameter families of such lattices, namely $\Lambda_3(a)$ and $\Lambda_3'(a)$. The group Γ induces an FVT action on the map $\mathcal{T}/\Lambda_3(a)$ via its quotient

$$\Gamma_1(a) := \Gamma/\langle X^a \rangle$$

and on the map $\mathcal{T}/\Lambda_3'(a)$ via its quotient

$$\Gamma_3(a) := \Gamma/\langle X^{2a} Y^a \rangle.$$

3.8. List of presentations

In this section we list explicit presentations of all 29 families of finite groups that act on torus. The list is divided into 5 parts corresponding to reasoning of the previous section.

3.8.1. Quotients of *p1* and *p2*. The group *p1* acts on the torus via its quotients

$$p1(a;b,c) := \langle X, Y;\, [X,Y] = X^a = Y^b X^c = 1 \rangle. \tag{13}$$

The group *p2* acts on the torus via its quotients

$$\begin{aligned} p2(a;b,c) = \langle T, X, Y;\, [X,Y] &= T^2 = (TX)^2 = (TY)^2 \\ &= X^a = Y^b X^c = 1 \rangle \\ \langle T_1, T_2, T_3;\, T_1^2 &= T_2^2 = T_3^2 = (T_1 T_2 T_3)^2 \\ &= (T_2T_3)^a = (T_3T_1)^b (T_2T_3)^c = 1 \rangle \\ \langle T_1, T_2, T_3, T_4;\, T_1^2 &= T_2^2 = T_3^2 = T_4^2 = T_1T_2T_3T_4^2 \\ &= (T_2T_3)^a = (T_3T_1)^b (T_2T_3)^c = 1 \rangle. \end{aligned} \tag{14}$$

3.8.2. Quotients of *pm*, *pg*, *pmm*, *pmg*, *pgg*. The group *pm* acts on the torus via its quotients

(15)
$$\begin{aligned} pm_1(a;b) :=& \langle R, X, Y;\, [X,Y] = R^2 = X^a = Y^b = 1, RXR = X^{-1}, RYR = Y \rangle \\ =& \langle R, R', Y;\, R^2 = R'^2 = (RR')^a = Y^b = 1, RY = YR, R'Y = YR' \rangle, \end{aligned}$$

(16)
$$pm_2(a,b) := \langle R, R', Y; R^2 = R'^2 = (RR')^a Y^b = Y^{2b} = 1, RY = YR, R'Y = YR' \rangle.$$

The group pg acts on the torus via its quotients

$$\begin{aligned} pg_1(a;b) :=& \langle P, X, Y;\ [X,Y] = X^a = Y^b = 1, P^2 = Y, P^{-1}XP = X^{-1}\rangle \\ =& \langle P, Q;\ P^2 = Q^2, (P^{-1}Q)^a = P^{2b} = 1\rangle \end{aligned} \tag{17}$$

and

$$pg_2(a;b) := \langle P, Q;\ P^2 = Q^2, (P^{-1}Q)^a P^{2b} = P^{4b} = 1\rangle. \tag{18}$$

The group pmm acts on the torus via its quotients

(19)

$$\begin{aligned} pmm_1(a;b) := \langle R, R', R_2, Y; [R,Y] = [R',Y] = R^2 = R'^2 = R_2^2! \ = (RR')^a = Y^b = 1, \\ R_2RR_2 = R, R_2R'R_2 = R', R_2YR_2 = Y^{-1}\rangle \end{aligned}$$

or by setting $R_1 = R$, $R_3 = R'$ and $R_4 = R_2Y$ as in [**4**] in (4.506)

$$\begin{aligned} = \langle R_1, R_2, R_3, R_4; R_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_1R_2)^2 = (R_2R_3)^2 \\ = (R_3R_4)^2 = (R_4R_1)^2 = (R_1R_3)^a = (R_2R_4)^b = 1\rangle, \end{aligned}$$

(20)

$$\begin{aligned} pmm_2(a;b) := \langle R_1, R_2, R_3, R_4; R_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_1R_2)^2 = (R_2R_3)^2 = (R_3R_4)^2 \\ = (R_4R_1)^2 = (R_1R_3)^a(R_2R_4)^b = (R_2R_4)^{2b} = 1\rangle. \end{aligned}$$

The group pmg acts on the torus via its quotients

(21)

$$\begin{aligned} pmg_1(a;b) := \langle P, Q, R; P^2 = Q^2; R^2 = (RP)^2 = (RQ)^2 = (P^{-1}Q)^a = P^{2b} = 1\rangle \\ = \langle R, T_1, T_2; R^2 = T_1^2 = T_2^2 = (T_1R)^{2a} = (T_1T_2)^b = 1\rangle, \end{aligned}$$

(22)

$$\begin{aligned} pmg_2(a;b) := \langle P, Q, R; P^2 = Q^2; R^2 = (RP)^2 = (RQ)^2 = (P^{-1}Q)^a P^{2b} = P^{4b} = 1\rangle \\ = \langle R, T_1, T_2; R^2 = T_1^2 = T_2^2 = (T_1R)^{2a}(T_1T_2)^b = (T_1T_2)^{2b} = 1\rangle. \end{aligned}$$

The group pgg acts on the torus via its quotients

$$\begin{aligned} pgg_1(a;b) := \langle P, Q, T; P^2 = Q^2; TPT^{-1} = Q^{-1}; T^2 = (P^{-1}Q)^a = P^{2b} = 1\rangle \\ = \langle P, O; (PO)^2 = (P^{-1}O)^2 = P^{2a} = (P^{-2}O)^{2b} = 1\rangle \end{aligned} \tag{23}$$

(24)

$$\begin{aligned} pgg_2(a;b) := \langle P, Q, T; P^2 = Q^2; TPT^{-1} = Q^{-1}; T^2 = (P^{-1}Q)^a P^{2b} = P^{4b} = 1\rangle \\ = \langle P, O; (PO)^2 = (P^{-1}O)^2 = P^{2a}(P^{-2}O)^{2b} = P^{4a} = 1\rangle \end{aligned}$$

3.8.3. Quotients of *cm* and *cmm*. The group *cm* acts on the torus via its quotients

$$cm_1(a,b) := \langle R, S; (RS)^2 = (SR)^2, R^2 = S^a(RSR)^b = S^b(RSR)^a = 1\rangle, \tag{25}$$

$$cm_2(a;b) := \langle R, S; (RS)^2 = (SR)^2, R^2 = (RSRS)^a = (RSRS^{-1})^b = 1\rangle. \tag{26}$$

The group *cmm* acts on the torus via its quotients

$$\begin{aligned} cmm_1(a,b) := \langle R_1, R_2, T;\ & R_1^2 = R_2^2 = T^2 = (R_1R_2)^2 = (R_1TR_2T)^2 = 1, \\ & (R_1R_2T)^a(R_2TR_1)^b = (R_1R_2T)^b(R_2TR_1)^a = 1\rangle, \end{aligned} \tag{27}$$

$$\begin{aligned} cmm_2(a,b) := \langle R_1, R_2, T;\ & R_1^2 = R_2^2 = T^2 = (R_1R_2)^2 = (R_1TR_2T)^2 = 1, \\ & (R_1R_2TR_2TR_1)^a = (R_1R_2TR_1TR_2)^b = 1\rangle. \end{aligned} \tag{28}$$

3.8.4. Quotients of *p3*, *p4* and *p6*. The group *p4* acts on the torus via its quotients

$$p4(b,c) := \langle S, T;\ S^4 = T^2 = (ST)^4 = (STS)^b(TS^2)^c = 1\rangle. \tag{29}$$

The group *p3* acts on the torus via its quotients

$$p3(b,c) := \langle S_1, S_2;\ S_1^3 = S_2^3 = (S_1S_2)^3 = (S_1^2S_2)^b(S_1S_2S_1)^c = 1\rangle. \tag{30}$$

The group *p6* acts on the torus via its quotients

$$p6(b,c) := \langle S, T;\ S^3 = T^2 = (ST)^6 = (S^{-1}TST)^b(STSTS)^c = 1\rangle. \tag{31}$$

3.8.5. Quotients of *p4g* and *p4m*. The group *p4g* acts on the torus via its quotients

$$\begin{aligned} p4g_1(a) &:= \langle R, S; R^2 = S^4 = (RS^{-1}RS)^2 = (S^2RS^{-1}RS)^a = 1\rangle \\ &= \langle R, S; R^2 = S^4 = (RS^{-1}RS)^2 = (RS)^{2a} = 1\rangle, \end{aligned} \tag{32}$$

$$p4g_2(a) := \langle R, S; R^2 = S^4 = (RS^{-1}RS)^2 = 1, (S^2RS^{-1}(RS)^3)^a\rangle \tag{33}$$

The group *p4m* acts on the torus via its quotients

$$\begin{aligned} p4m_1(a) := \langle R, R_1, R_2; R^2 = R_1^2 = R_2^2 = (RR_1)^4 = (R_1R_2)^2 = (R_2R)^4 \\ = (R_1RR_2R)^a = 1\rangle \end{aligned} \tag{34}$$

$$\begin{aligned} p4m_2(a) := \langle R, R_1, R_2; R^2 = R_1^2 = R_2^2 = (RR_1)^4 = (R_1R_2)^2 = (R_2R)^4 \\ (R_1RR_2R)^a(R_2RR_1R)^a = 1\rangle. \end{aligned} \tag{35}$$

3.8.6. Quotients of *p3m1*, *p31m* **and** *p6m*. The group *p31m* acts on the torus via its quotients

$$p31m_1(a) := \langle R, S; R^2 = S^3 = (RS^{-1}RS)^3 = (S^{-1}R)^{2a} = 1\rangle \tag{36}$$

$$\begin{aligned} p31m_2(a) := \langle R, S; R^2 = S^3 = (RS^{-1}RS)^3 = 1, \\ (S^{-1}R)^{2a}(SRS^{-1}RS)^{-a} = 1\rangle. \end{aligned} \tag{37}$$

The group *p3m1* acts on the torus via its quotients

$$\begin{aligned} p3m1_1(a) &:= \langle R_1, R_2, R_3; R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_1)^3 \\ &\qquad = (R_3R_2R_3R_1)^a = 1\rangle \\ &= \langle R_1, R_2, R_3; R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_1)^3 \\ &\qquad = (R_2R_1R_2R_3)^a = 1\rangle. \end{aligned} \tag{38}$$

$$\begin{aligned} p3m1_2(a) := \langle R_1, R_2, R_3; R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_1)^3 = 1, \\ (R_3R_2R_3R_1)^a(R_2R_1R_2R_3)^{-a} = 1\rangle. \end{aligned} \tag{39}$$

The group *p6m* acts on the torus via its quotients

$$\begin{aligned} p6m_1(a) &:= \langle R, R_1, R_2; R^2 = R_1^2 = R_2^2 = (R_1R_2)^3 = (R_2R)^2 = (RR_1)^6 = 1, \\ &\qquad (RR_1RR_2RR_1RR_1)^a = 1\rangle \\ &= \langle R, R_1, R_2; R^2 = R_1^2 = R_2^2 = (R_1R_2)^3 = (R_2R)^2 = (RR_1)^6 = 1, \\ &\qquad (R_2R_1R_2RR_1R)^a = 1\rangle, \end{aligned} \tag{40}$$

$$\begin{aligned} p6m_2(a) := \langle R, R_1, R_2; R^2 = R_1^2 = R_2^2 = (R_1R_2)^3 = (R_2R)^2 = (RR_1)^6 = 1, \\ (RR_1RR_2RR_1RR_1)^a(R_2R_1R_2RR_1R)^{-a} = 1\rangle. \end{aligned} \tag{41}$$

3.9. Non-redundancy of families

A natural question to ask is whether the groups so obtained are distinct. To that end we shall prove the following result:

Proposition 7. *None of the families above belongs entirely to the union of the rest.*

Proof. It is sufficient to exhibit for each family a group that belongs uniquely to that family. For instance for family *p1* we can take the group

$$p1(5, 1, 0) = \mathbf{Z}_5,$$

whose order is not divisible by 2 nor by 3, and thus $p1(5, 1, 0)$ belongs only to the family *p1*. Unique groups for the remaining families are listed in Table 2. □

Family	Size	Presentation	Unique group	GAP id
$p1(a,b,c)$	ab	(13)	$p1(5,1,0)$	[5,1]
$p2(a,b,c)$	$2ab$	(14)	$p2(4,4,4)$	[32,34]
$pm_1(a,b)$	$2ab$	(15)	$pm_1(2,4)$	[16,10]
$pm_2(a,b)$	$4ab$	(16)	$pm_2(2,4)$	[32,38]
$pg_1(a,b)$	$2ab$	(17)	$pg_1(4,2)$	[16,4]
$pg_2(a,b)$	$4ab$	(18)	$pg_2(2,1)$	[8,4]
$cm_1(a,b)$	$2\|a^2-b^2\|$	(25)	$cm_1(4,0)$	[32,11]
$cm_2(a,b)$	$4ab$	(26)	$cm_2(4,2)$	[32,5]
$pmm_1(a,b)$	$4ab$	(19)	$pmm_1(2,2)$	[16,14]
$pmm_2(a,b)$	$8ab$	(20)	$pmm_2(2,2)$	[32,49]
$pmg_1(a,b)$	$4ab$	(21)	$pmg_1(2,4)$	[32,28]
$pmg_2(a,b)$	$8ab$	(22)	$pmg_2(3,2)$	[48,41]
$pgg_1(a,b)$	$4ab$	(23)	$pgg_1(4,4)$	[64,12]
$pgg_2(a,b)$	$8ab$	(24)	$pgg_2(2,2)$	[32,7]
$cmm_1(a,b)$	$4\|a^2-b^2\|$	(27)	$cmm_1(4,0)$	[64,134]
$cmm_2(a,b)$	$8ab$	(28)	$cmm_2(2,4)$	[64,128]
$p4(a,b)$	$4(a^2+b^2)$	(29)	$p4(2,1)$	[20,3]
$p4g_1(a)$	$8a^2$	(32)	$p4g_1(4)$	[128,134]
$p4g_2(a)$	$16a^2$	(33)	$p4g_2(2)$	[64,32]
$p4m_1(a)$	$8a^2$	(34)	$p4m_1(4)$	[128,928]
$p4m_2(a)$	$16a^2$	(35)	$p4m_2(2)$	[64,138]
$p3(b,c)$	$3(b^2-bc+c^2)$	(30)	$p3(2,0)$	[12,3]
$p6(b,c)$	$6(b^2-bc+c^2)$	(31)	$p6(2,0)$	[24,13]
$p3m1_1(a)$	$6a^2$	(38)	$p3m1_1(3)$	[54, 8]
$p3m1_3(a)$	$18a^2$	(39)	$p3m1_3(2)$	[72,43]
$p31m_1(a)$	$6a^2$	(36)	$p31m_1(6)$	[216,92]
$p31m_3(a)$	$18a^2$	(37)	$p3m1_3(2)$	[72,42]
$p6m_1(a)$	$12a^2$	(40)	$p6m_1(2)$	[48,48]
$p6m_3(a)$	$36a^2$	(41)	$p6m_3(2)$	[144,183]

Table 2. List of 29 toric group families.

4. All vertex-transitive maps on torus admit free action

Given an FVT action of a wallpaper group Γ on a semiregular tiling $\mathcal{T}$, we can construct vertex-transitive maps on various surfaces. The general construction consists of taking a normal torsion-free subgroup H of Γ such that $\mathbf{C}/H$ is compact. Then the FVT action on the tiling gives rise to a vertex-transitive map on $\mathbf{C}/H$.

We shall consider a special case, namely vertex-transitive maps arising on the torus. These are of special interest, because of Babai's conjecture, independently proved by Babai and Thomassen ([**1**], [**13**]), that any vertex-transitive graph with "sufficiently many" vertices compared to its genus comes from a vertex-transitive map that can be drawn on either the torus or the Klein bottle. FVT maps are of course vertex-transitive and we will prove that all vertex-transitive maps on the torus are in fact FVT.

4.1. FVT maps on the torus

From Table 1 we can directly create the list of maps admitting an FVT action by a quotient of a wallpaper group. Every such map is given by a semiregular tiling of the plane, and a sublattice of translations of $\mathcal{T}$. We will use notation of 3.1 with appropriate basis X, Y of $\Lambda = T_{\mathcal{T}}$ to describe the sublattices.

- if $\mathcal{T}$ is the square or triangular tiling, then for any lattice $\Lambda = \Lambda(a; b, c)$, the quotient $\mathcal{T}/\Lambda$ is a FVT map for the group $p1(a; b, c)$,
- if $\mathcal{T}$ is the hexagonal or the elongated triangular tiling then for any lattice $\Lambda = \Lambda(a; b, c)$ the quotient $\mathcal{T}/\Lambda$ is an FVT map for the group $p2(a; b, c)$,
- if $\mathcal{T}$ is the snub hexagonal, truncated hexagonal or small rhombitrihexagonal tiling, then for any lattice $\Lambda = \Lambda_3(b, c)$ the quotient $\mathcal{T}/\Lambda$ is an FVT map for the group $p6(b, c)$,
- if $\mathcal{T}$ is the trihexagonal tiling, then for any lattice $\Lambda = \Lambda_3(b, c)$ the quotient $\mathcal{T}/\Lambda$ is an FVT map for the group $p3(b, c)$,
- if $\mathcal{T}$ is the great rhombitrihexagonal tiling, then for any lattice of the form $\Lambda = \Lambda_3(a)$ or $\Lambda = \Lambda_3'(a)$ the quotient is an FVT map for the group $p6m_1(a)$ or $p6m_3(a)$ respectively,
- if $\mathcal{T}$ is the truncated square tiling, then for a lattice Λ of any of forms
  - $\Lambda_4(b, c)$
  - $\Lambda_o^c(a, b)$
  - $\Lambda_c^c(a, b)$

  the quotient $\mathcal{T}/\Lambda$ is an FVT map for the group $p4(b, c)$, $cmm_2(a, b)$ or $cmm_1(a, b)$ respectively,
- if $\mathcal{T}$ is the snub square tiling, then for a lattice of any of the forms
  - $\Lambda_4(b, c)$
  - $\Lambda_c^o(a, b)$
  - $\Lambda_o^o(a, b)$

  the quotient $\mathcal{T}/\Lambda$ is an FVT map for the group $p4(b, c)$, $pgg_2(a, b)$ and $pgg_1(a, b)$ respectively.

4.2. All vertex-transitive toroidal maps are FVT

We are ready to prove our final result.

Theorem 8. *Every vertex-transitive map on a torus admits an FVT map structure.*

Proof. From the list in the preceding section it is clear that there are no non-FVT maps arising from the square, triangular, hexagonal or elongated triangular tilings. Proceed now by a contradiction. Suppose a map on torus is given with group G acting transitively on vertices. The pullback of the map to the universal cover must be a semiregular tiling with vertex-transitive action of a wallpaper group Γ. We have a generalization of the index equality (12):

$$[\Gamma : T(\Gamma)] = i_\sigma t_{\mathcal{T}} |\Gamma_v|, \tag{42}$$

where Γ_v is the stabilizer of any vertex (they are all conjugate).

Suppose $t_{\mathcal{T}}$ is divisible by 3. From (42) it follows that Γ can only be one of the groups *p3*, *p6*, *p3m1*, *p31m* and *p6m* . All these groups contain rotation by $2\pi/3$, thus the lattice Λ has to be triangle-like. This is in fact a sufficient condition for the map to be FVT in the case when $\mathcal{T}$ is the trihexagonal, snub hexagonal tiling, truncated hexagonal or small rhombitrihexagonal tiling. In the remaining case of the great rhombitrihexagonal tiling we have $t_{\mathcal{T}} = 12$, and since $[\Gamma : T(\Gamma)] < 12$ for all wallpaper groups other than *p6m* we have to have $\Gamma = p6m$. But then the left hand side of (42) is equal to 12, thus $|\Gamma_v| = 1$, and the action is FVT, which is a contradiction.

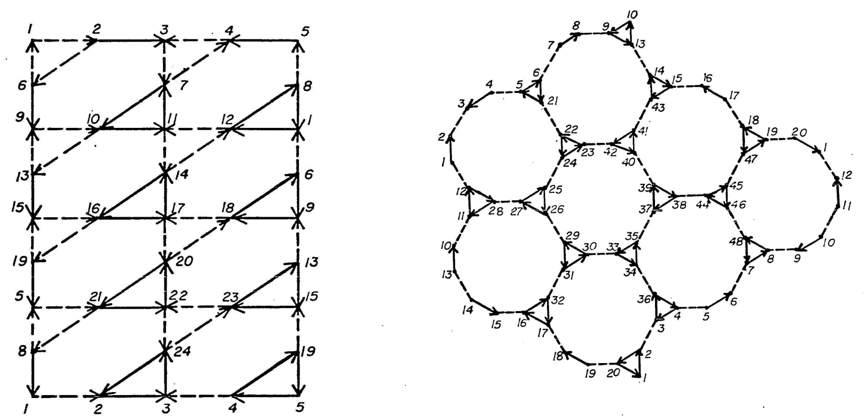
It remains to consider the case of snub square and truncated square tilings. For those tilings $t_{\mathcal{T}} = 4$ and they do not admit rotation by $2\pi/3$. For subgroups of *p4m* not containing rotation by $2\pi/4$ the index $[\Gamma : T(\Gamma)] \leq 4$. It follows that a vertex-transitive map without an FVT action would have to contain a rotation by $2\pi/4$, but for both tilings quotients by lattices invariant by such a rotation admit FVT actions by quotients of group $p4$. $\square$

4.3. Proulx examples

We conclude this section by showing examples of toric maps that are not vertex-transitive, even though the graphs they induce are vertex-transitive [**11**].

Specifically Figure 4.2 shows Cayley graphs drawn on the torus for groups

$$G_1 = \langle R, S; R^3 = S^3 = 1, RSR = SRS \rangle,$$
$$G_2 = \langle R, S; R^3 = S^2 = RSRSR^{-1}S)^2 = 1 \rangle$$



(a) Cayley graph for G_1 (b) Cayley graph for G_2

Figure 19. Proulx toroidal Cayley graphs.

Since these maps do not occur in the list of Section 4.1, they are not Cayley and from Theorem 8 it follows they are not vertex-transitive maps.

5. Open questions

An interesting group theoretic question is to classify abstract group isomorphisms between toric groups. We list some of those that we arrived at by an experiment (the GAP code is available for download on [**12**]):

$$\begin{aligned} p2(2,2k,2) &= pmg_1(k,2), \\ pgg_1(2k+1,2l) &= pmg_1(2l,2k+1), \\ p3m1_1(a) &= p31m_1(a), \quad \text{if } 3 \nmid a \end{aligned}$$

Another question of interest is to provide a list of FVT actions on vertex-transitive maps on the Klein bottle. Thirteen families of such maps have been classified by Babai [**1**], and a natural question is whether all of them admit a FVT action.

We hope to return to these questions in later papers.

References

1. Babai L., *Vertex-transitive graph.* J. Graph Theory **15(6)** (1991), 587–627.
2. Baker R. P., *Cayley Diagrams on the Anchor Ring*, American Journal of Mathematics 53(3) (1931), 645–669.
3. Burnside W., *Theory of Groups of Finite Order.* (1st ed., 1897), Cambridge Univ. Press, Cambridge, 1911.
4. Coxeter H. S. M.and Moser W. O. J., *Generators and Relations for Discrete Groups.* Springer-Verlag, Berlin and New York, 4th ed., 1980.
5. Grünbaum B.and Shephard G. C., *Tilings and Patterns.* W. H. Freeman and Company, 1987.
6. Gross J. L.and Tucker T. W., *Topological graph theory.* Dover, Toronto 2001.
7. Jones G. A. and Singerman D., *Theory of maps on orientable surfaces.* Proc. London Math. Soc. **37(3)** (1978), 273–307.
8. Karabáš J. and Nedela R., *Archimedean solids of genus two.* Electronic Notes in Discrete Mathematics **28** (2007), 331–339.
9. ———, *Archimedean solids of higher genera.* to appear in Mathematics in Computation
10. Marusic D. and Pisanski T., *Symmetries of hexagonal graphs on the torus.* Croat. Chemica Acta, **73** (2000), 969–981.
11. Proulx V., *Classification of the Toroidal Groups.* Ph.D. thesis, Columbia University, 1977.
12. `http://www.savbb.sk/~ondrejs/cry3.g`
13. Thomassen and Carsten, *Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface*, Trans. Amer. Math. Soc. **323(2)** (1991), 605–635.
14. Tucker, T.W.: Finite groups acting on surfaces and the genus of a group, J.Combin. Theory Ser. **B34**, 82–98.
15. `http://en.wikipedia.org/wiki/Self-organizing_map`
16. `http://en.wikipedia.org/wiki/Tilings_of_regular_polygons`
17. `http://en.wikipedia.org/wiki/Wallpaper_group`
18. `http://mathworld.wolfram.com/ElongatedSquareGyrobicupola.html`

Ondrej Šuch, Institute of Mathematics and Computer Science, Ďumbierska 1, 974 01 Banská Bystrica, Slovak Republic, *e-mail*: `ondrejs@savbb.sk`