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CREATING SLOWLY OSCILLATING SEQUENCES AND SLOWLY OSCILLATING CONTINUOUS FUNCTIONS

R. W. VALLIN

Abstract. This paper is a further study of slowly oscillating convergence and slowly oscillating continuity as introduced by Çakalli in [**3**]. We answer an open question from that paper and relate the slowly oscillating continuous functions to metric preserving functions, thus infusing this subject with a ready made collection of pathological examples.

1. Introduction

As per usual, let $\mathbb{R}$ denote the set of real numbers. Although many of these definitions use subsets of the real line or functions whose domain is a subset of the real line, we will state all the definitions for the real line. These first definitions come from [**3**].

Definition 1. A sequence $\{x_n\}$ of points in $\mathbb{R}$ is called *slowly oscillating* if

$$\lim_{\lambda\to 1^+}\left[\overline{\lim_{n\to\infty}}\left(\max_{n+1\le k\le[\lambda n]}|x_k - x_n|\right)\right] = 0 \tag{1}$$

where $[\lambda n]$ refers to the integer part of λn.

In this paper, there is a more usual definition using the ε's, δ's, and N's typical for generalizations of the properties of a sequence.

Definition 2. A sequence $\{x_n\}$ of points in $\mathbb{R}$ is slowly oscillating if for any $\varepsilon > 0$ there exists a positive real number $\delta = \delta(\varepsilon)$ and a natural number $N = N(\varepsilon)$ such that

$$|x_m - x_n| < \varepsilon \tag{2}$$

if $n \ge N$ and $n \le m \le (1+\delta)\,n$.

This can also be seen as a generalization of the notion of pseudo-Cauchy ([**1**]). The sequence $\{x_n\}$ is pseudo-Cauchy if for every $\varepsilon > 0$ there exists a natural number $N = N(\varepsilon)$ such that for all $n \ge N$, $|x_{n+1} - x_n| < \varepsilon$.

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In his paper, Çakalli neglects to give any examples of a slowly oscillating sequence. Obviously any Cauchy (convergent since we are staying in $\mathbb{R}$) sequence is slowly oscillating; however, it is not clear if there are sequences that are slowly oscillating yet not Cauchy. So our first result is to show that slowly oscillating is in fact different and worth studying.

Example 3. There is a sequence in $\mathbb{R}$ which is slowly oscillating, but not Cauchy.

Proof. Our sequence $\{x_n\}$ will be based on the sequence partial sums of the harmonic series. Since our sequence is unbounded, it cannot be Cauchy. Let $x_1 = 1$, $x_2 = x_3 = 1 + 1/2$, and $x_4 = x_5 = x_6 = x_7 = 1 + 1/2 + 1/3$. In general,

$$x_{2^n} = x_{2^n+1} = \cdots = x_{2^{n+1}-1} = \sum_{k=1}^{n+1} \frac{1}{k}. \tag{3}$$

Given $\varepsilon > 0$, let $\delta = 1$ and choose N such that $1/N < \varepsilon$. Then if $n > N$ and $n \leq m \leq 2n$

$$|x_n - x_m| < \frac{1}{n+2} < \frac{1}{N} < \varepsilon. \tag{4}$$

□

The key here is slowness. For increasingly long stretches the terms of the sequence are constant and, when the value does change, it does so by a very small amount. Although we use the unboundedness to quickly show non-Cauchy, it is possible to tweak this example and have one which is bounded by some positive M by sometimes subtracting values rather than adding, while continuing to make sure the sequence is not Cauchy.

2. Slowly Oscillating Continuity

The goal in Çakalli's article was to introduce the concept of a slowly oscillating continuous function. This is a sequential definition of continuity different from the G continuity given in [**5**].

Definition 4. A function $f : \mathbb{R} \to \mathbb{R}$ is *slowly oscillating continuous* if it transforms slowly oscillating sequences into slowly oscillating sequences.

Once again there are no real examples of functions with this type of behavior in the paper although there are several good results. Among them

- The sum or composition of two slowly oscillating continuous functions is slowly oscillating continuous.
- The function $f(x) = x^2$ is *not* slowly oscillating continuous.
- If a function f is uniformly continuous on a subset $E \subset \mathbb{R}$, then it is slowly oscillating continuous on E.
- The uniform limit of slowly oscillating continuous functions is slowly oscillating continuous.

In [**3**], the author asserts that every slowly oscillating continuous function is, in fact, continuous in the ordinary sense. However, in his proof he uses the "fact" that a slowly oscillating sequence must be a convergent (in the usual sense), which we have seen is not true. His idea is correct and we present a proof of that below.

Theorem 5. *If $f : \mathbb{R} \to \mathbb{R}$ is slowly oscillating continuous, then f is continuous in the ordinary sense.*

Proof. If f is not continuous in the ordinary sense, there exists a point x_0 such that either $\overline{\lim}_{x\to x_0} f(x) \neq f(x_0)$ or $\underline{\lim}_{x\to x_0} f(x) \neq f(x_0)$. Either way there exists a sequence $\{x_n\}$ with $x_n \to x_0$ while $f(x_n) \nrightarrow f(x_0)$. Then the sequence $(x_1, x_0, x_2, x_0, x_3, x_0, \ldots)$ is slowly oscillating, but

$$(f(x_1), f(x_0), f(x_2), f(x_0), \ldots)$$

is not. Hence f is not slowly oscillating continuous. □

In the talk abstract [**4**], the author writes "the purpose of this talk is to conjecture if the slowly oscillating continuity is equivalent to uniform continuity." Below we prove that the conjecture is true. We will show this by proving the contrapositive.

Theorem 6. *Let f be a continuous function defined on the real line. If f is not uniformly continuous, then f cannot be slowly oscillating continuous.*

Proof. Since f is not uniformly continuous there exists $\varepsilon > 0$ and sequences $\{x_n\}$ and $\{y_n\}$ such that for all n

$$|x_n - y_n| < 1/n \tag{5}$$

and

$$|f(x_n) - f(y_n)| \geq \varepsilon. \tag{6}$$

Case I: $\{x_n\}$ is bounded.

Then by the Bolzano-Weierstrass Theorem there exists a convergent subsequence $\{x_{n_k}\}$. But then $(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, x_{n_3}, \ldots)$ is a convergent (hence slowly oscillating) sequence while

$$(f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), f(x_{n_3}), \ldots) \tag{7}$$

cannot be slowly oscillating. Hence f is not slowly oscillating continuous.

Case II: $\{x_n\}$ is unbounded.

Without loss of generality, assume

$$x_1 < y_1 < x_2 < y_2 < \cdots . \tag{8}$$

We will then create a new sequence $\{t_n\}$ which acts as we need it to behave. Let $t_1 = x_1$ and $t_2 = y_1$. Assume the sequence of t's has been defined up to $t_k = y_n$. If $x_{n+1} - y_n \leq 1/n$, then let $t_{k+1} = x_{n+1}$ and $t_{k+2} = y_{n+1}$. If $x_{n+1} - y_n > 1/n$, then find the smallest value m such that $x_{n+1} - y_n \leq m/n$. We will then add points into our sequence very slowly to bridge the gap between y_n and x_{n+1} while making sure $\{t_n\}$ will be slowly oscillating. Let

$$
\begin{aligned}
&t_{k+1} = \cdots = t_{2k} = y_n = y_n + 0/n \\
&t_{2k+1} = \cdots = t_{4k} = y_n + 1/n \\
&\qquad\qquad\vdots \\
&t_{2^{m-1}k} = \cdots = t_{2^m k} = y_n + (m-1)/n \\
&t_{2^m k+1} = x_{n+1} \\
&t_{2^m k+2} = y_{n+1}.
\end{aligned} \tag{9}
$$

By growing so slowly we have assured ourselves that $\{t_n\}$ is a slowly oscillating sequence. However for any δ there is a K large so that $t_K, t_{K+1}, \ldots t_{(1+\delta)K}$ will contain points x_n and y_n whose image under f are guaranteed to be at least ε away from each other. Thus f cannot be slowly oscillating continuous.

By contrapositive, if such an f is slowly oscillating continuous, then it must be uniformly continuous. □

3. Metric Preserving Functions

Metric preserving functions have been studied for a long time. We will not go deeply into their history, but the interested person may consult [**7**] or [**11**]. We proceed with the definition and some pertinent properties.

Definition 7. Let $\mathbb{R}^+$ denote the non-negative real numbers. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called *metric preserving* if for every metric space (X, ρ) the composite function $f \circ \rho$ is still a metric on X.

The most well-known example of this, one found as a homework exercise in most books which introduce metric spaces, is

$$f(x) = \frac{x}{x+1} \tag{10}$$

which turns any metric into a bounded metric. Some properties of metric preserving functions include

- $f(x) = 0$ if and only if $x = 0$.
- for all $a, b > 0$, $f(a+b) \le f(a) + f(b)$ (subadditivity).
- if a metric preserving function is continuous at 0 (from the right), then it is continuous everywhere.
- the uniform limit of metric preserving functions is a metric preserving function.

It is straightforward to see that the sets of metric preserving functions and slowly oscillating continuous functions have elements in common, but neither could be made into a subset of the other. The function $f(x) = 0$ is slowly oscillating continuous, but not metric preserving when restricted to $[0, \infty)$. Metric preserving functions do not have to be continuous (and in fact can be nowhere continuous), so there is no way to extend a discontinuous metric preserving function to be slowly oscillating continuous.

There is an alternative way of looking at metric preserving functions which will be useful to us and is due to Terpe ([**9**]). He uses what are called *triangle triplets*. A triple (a, b, c) is called a triangle triplet if

$$\begin{aligned} a &\leq b + c \\ b &\leq a + c \\ c &\leq a + b \end{aligned}$$

Terpe then proves

Theorem 8. *The function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is metric preserving if (i) $f^{-1}(0) = \{0\}$, and (ii) f sends triangle triplets to triangle triplets; that is,*

$$(f(a), f(b), f(c))$$

is a triangle triplet whenever (a, b, c) is a triangle triplet.

We note the following: If $a > b > 0$, then $(a - b, b, a)$ is a triangle triplet. This means that for a metric preserving f, $f(a) \leq f(b) + f(a - b)$ or

$$f(a) - f(b) \leq f(a - b). \tag{11}$$

If we put this in terms of the slowly oscillating sequences $\{x_n\}$ we have been studying, given a continuous metric preserving f and $\varepsilon > 0$ there exists a $\delta > 0$ and a natural number N such that if $n > N$ and $n \leq m \leq (1 + \delta) n$ then (assuming without loss of generality $x_n \geq x_m > 0$)

$$f(x_n) - f(x_m) \leq f(x_n - x_m). \tag{12}$$

This will provide us with the connection we need to relate metric preserving functions with slowly oscillating continuous functions and use the literature on the former to expand what is known about the latter.

First there is some housekeeping to be done. Metric preserving functions have a domain of all non-negative real numbers while slowly oscillating continuous functions have domain $\mathbb{R}$. There are a few ways around this, but we shall go with this: with every metric preserving function f we have the corresponding function f^* with domain all real numbers via

$$f^*(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}. \tag{13}$$

This is appending the $180°$ rotation of the graph about the origin to the original graph. We chose this extension over others because if f is an increasing function, then so is f^*. This brings us to the next point, which is not yet enough to relate metric preserving and slowly oscillating continuous functions. We do have that if $\{x_n\}$ is a slowly oscillating sequence and f is a metric preserving function, then given $\varepsilon > 0$ there exists a $\delta > 0$ and a natural number N such that if $n > N$ and $n \leq m \leq (1 + \delta) n$ then

$$|f^*(x_n) - f^*(x_m)| \leq 2f^*(|x_n - x_m|). \tag{14}$$

We are finally near where we need to be. In many examples of metric preserving functions there is an interval $[0, \beta]$ on which the function is non-decreasing. We note here that this is not necessary. In [**2**] there is an example of a metric preserving function which is continuous but not increasing on any neighborhood of 0. So if f is continuous at the origin, then we can fully regulate $|f^*(x_n) - f^*(x_m)|$.

Theorem 9. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a metric preserving function which is continuous at the origin and increasing in some nondegenerate interval $[0, \beta]$. Then the function $f^* : \mathbb{R} \to \mathbb{R}$ given by*

$$f^*(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases} \tag{15}$$

is a slowly oscillating continuous function.

Proof. Let $\{x_n\}$ be a slowly oscillating sequence and suppose $\varepsilon > 0$ has been given. Since f is continuous and increasing on $[0, \beta]$ there exists a $\beta_0 > 0$ such that $f(x) < \varepsilon/2$ for all $x \in [0, \beta_0]$. Then since $\{x_n\}$ is slowly oscillating there are δ and N so that $|x_n - x_m| < \beta_0$ for $n \geq N$ and $n \leq m \leq (1+\delta)n$. This leads to

$$|f^*(x_n) - f^*(x_m)| \leq 2f^*(|x_n - x_m|) = 2f(|x_n - x_m|) < \varepsilon. \tag{16}$$

So f^* is a slowly oscillating continuous function. □

Therefore we have a new collection of slowly oscillating continuous functions: those based on metric preserving functions. Since there are already examples of metric preserving functions which meet our criteria and have unusual properties, we get some pathological examples of slowly oscillating continuous functions.

Example 10 ([**6**])**.** There exists a slowly oscillating continuous function which is nowhere differentiable.

Example 11 ([**8**])**.** There exists a slowly oscillating continuous function which is increasing yet $f'(x) = 0$ almost everywhere. (Note: This function is the so-called Cantor function with domain $[0, 1]$. However, since $f(0) = 0$ and $f(1) = 1$ it is easy enough via translation to have a function with domain $[0, \infty)$ and then our rotational extension to make the domain $\mathbb{R}$.)

Example 12 ([**10**])**.** For any measure zero, $\mathcal{G}_\delta$ set Z there exists a slowly oscillating continuous function which is everywhere differentiable (in the extended sense) such that $\{x : |f'(x)| = \infty\} = Z \cup \{0\}$. (Note: For this example we are actually using a different construction. We are taking advantage of the fact that if f_1 and f_2 both satisfy the criteria of the theorem above, then

$$f(x) = \begin{cases} f_1(x) & x \geq 0 \\ -f_2(-x) & x < 0 \end{cases} \tag{17}$$

will be a slowly oscillating continuous function.)

There are probably many other interesting examples out there of slowly oscillating continuous functions which are not derived from metric preserving functions. However this connection between the two types of functions gives us some very nice, pathological cases for free.

References see p. 78

ADDENDUM TO "CREATING SLOWLY OSCILLATING SEQUENCES AND SLOWLY OSCILLATING CONTINUOUS FUNCTIONS"

R. W. VALLIN AND H. ÇAKALLI

On page three line 16 from above, just before Theorem 6, the paragraph

"In [**3**], the author asserts that every slowly oscillating continuous function is, in fact, continuous in the ordinary sense. However, in his proof he uses the fact that a slowly oscillating sequence must be a convergent (in the usual sense), which we have seen is not true. His idea is correct and we present a proof of that below."

should be corrected as

"In [**3**], the author proves that every slowly oscillating continuous function is, in fact, continuous in the ordinary sense. However, in his proof he uses the fact that any slowly oscillating sequence must be quasi-Cauchy in the sense that forward difference of the sequence tends to 0. Now we present another proof of that below for completeness."

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R. W. Vallin, Department of Mathematics, Slippery Rock University, Slippery Rock, PA 16057 USA, *e-mail*: robert.vallin@sru.edu