Acta Math. Univ. Comenianae
Vol. LXXX, 2 (2011), pp. 229–235

ACTION OF GENERALIZED LIE GROUPS ON MANIFOLDS

M. R. FARHANGDOOST

Abstract. In this paper by definition of generalized action of generalized Lie groups (top spaces) on a manifold, the concept of stabilizer of the top spaces is introduced. We show that the stabilizer is a top space, moreover we find the tangent space of a stabilizer. By using of the quotient spaces, the dimension of some top spaces are fined.

1. Introduction

In Physics, Lie groups often appear as the set of transformations acting on a manifold. For example, $SO(3)$ is the group of rotations in R^3 while the *Poincaré* group is the set of transformations acting on the *Minkowski* spacetime. To study more general cases, the notion of top spaces as a generalization of Lie groups was introduced by M. R. Molaei in 1998 [**3**]. Here we would like to study the action of a top space T on a manifold M. Moreover we have encountered the intersection between generalized group theory and geometry.

In this paper we introduce generalized action of generalized Lie groups (top spaces) and the stabilizer of the top spaces. We show that stabilizer is a top space, moreover we find the tangent space of the stabilizer. Also, by using the generalized action, we find the dimension of some top spaces.

Now, we recall the definition of a generalized group of [**1**].

A generalized group is a non-empty set T admitting an operation called multiplication which satisfies the following conditions:

i) $(t_1 \cdot t_2) \cdot t_3 = t_1 \cdot (t_2 \cdot t_3)$; for all $t_1, t_2, t_3 \in T$.
ii) For each t in T there exists a unique $e(t)$ in T such that $t \cdot e(t) = e(t) \cdot t = t$.
iii) For each t in T there exists s in T such that $t \cdot s = s \cdot t = e(t)$.

It is easy to show that for each t in a generalized group T, there is a unique inverse in T, so inverse of t is denoted by t^{-1}.

Let T and S be generalized groups. A map $f : T \to S$ is called homomorphism if $f(t_1 t_2) = f(t_1)f(t_2)$ for every $t_1, t_2 \in T$.

Now, we recall top spaces [**3**].

Received August 12, 2010; revised January 28, 2011.
2010 *Mathematics Subject Classification.* Primary 22E15, 22A05.
Key words and phrases. Lie group; generalized topological group; top space.

A top space T is a Hausdorff d-dimensional differentiable manifold which is endowed with a generalized group structure such that the generalized group operations:

i) $\cdot\colon T\times T\longrightarrow T$ by $(t_1,t_2)\mapsto t_1\cdot t_2$.
ii) $^{-1}: T\longrightarrow T$ by $t\mapsto t^{-1}$;

are differentiable and it holds

iii) $e(t_1\cdot t_2)=e(t_1)\cdot e(t_2)$.

Example 1.1 ([**5**])**.** If I and Λ are smooth manifolds, G is a Lie group and $P\colon \Lambda\times I\longrightarrow G$ is a smooth mapping, then the matrix semigroup $M(G,I,\Lambda,P)$ is a top space.

Let $(T,\cdot)$ be a top space. Then a top space $(S,.)$ is called a subtop space of $(T,\cdot)$, when S is a submanifold of T.

Let T and S be top spaces, a smooth homomorphism map $f:T\to S$ is called homomorphism of top spaces.

A top generalized subgroup N of a top space T is called a top generalized normal subgroup of T if there exist a top space E and differentiable homomorphism $f\colon T\longrightarrow E$ such that $t\in T$, $N_t=\emptyset$ or $N_t=\ker f_t$, where $N_t=N\cap T_t$, $f_t=f\mid_{T_t}$, $T_t=\{s\in T\mid e(s)=e(t)\}$ and $t\in T$.

Theorem 1.2 ([**2**])**.** *Let N be a top normal generalized subgroup of T and let $e(T)$ be finite. Then $\Gamma_N=\{t\in T\mid N_t\neq\emptyset\}$ is an open top generalized subgroup of T. Moreover, there is a unique differentiable structure on T/N such that T/N is a top space with the topology $\{U\,|\,\pi^{-1}(U)$ is open in $\Gamma_N\}$, where the topology Γ_N: $\{U\,|\,U\cap N_t$ is open in N_t for all $t\in T\}\cup\{\Gamma_N\}$ and $\pi\colon\Gamma_N\to T/N$ is defined by $\pi(t)=tN_t$.*

In the previous theorem we can show that the map π is a submersion map [**2**].

2. Action of Top Spaces

We begin this section by definition of generalized action of top spaces on manifolds.

Definition 2.1. A generalized action of a top space T on a manifold M is a differentiable map $\lambda\colon T\times M\longrightarrow M$ which satisfies the following conditions:

i) For any $m\in M$, there is $e(t)$ in T such that $\lambda(e(t),m)=m$.
ii) $\lambda(t_1,\lambda(t_2,m))=\lambda(t_1t_2,m)$.

[*Note:* We often use the notation tm instead of $\lambda(t,m)$, so the second condition of Definition 2.1 is defined by $t_1(t_2m))=(t_1t_2)m$.]

Note that generalized action is a generalization of action of Lie groups on manifolds, i.e. if T is a Lie group, then λ is an action of the Lie group on manifold M.

Example 2.2. The Euclidean subspace $\mathbb{R}^*=\mathbb{R}-\{0\}$ with the product $(a,b)\mapsto a|b|$ is a top space with the identity element $e(T)=\{+1,-1\}$. Then $\lambda\colon\mathbb{R}^*\times\mathbb{R}\longrightarrow\mathbb{R}$ defined by $\lambda(a,m)=am$ is a generalized action of the top space $\mathbb{R}^*$ on Euclidean manifold R.

Example 2.3. $T = \mathbb{R} \times \mathbb{R} - \{0\}$ with product $(a, b) \cdot (e, f) = (be, bf)$ is a top space. The map $\lambda\colon T \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\lambda((a, b), c) = ac/b$ is a generalized action. [*Note:* $e((a, b)) = (a/b, 1)$ and $(a, b)^{-1} = (a/b^2, 1/b)$].

Example 2.4. $T = \mathbb{R} \times \mathbb{R} - \{0\}$ with product $(a, b) \cdot (e, f) = (be, bf)$ is a top space. Let $M = R$, then the map $\zeta\colon T \times M \longrightarrow M$ defined by $\zeta((a, b), c) = c$ is a generalized action.

Definition 2.5. Let T be a top space that acts on a manifold M. Then $H(m) = \{t \in T \mid \lambda(t, m) = m\}$ is called the stabilizer of m, where $m \in M$.

Example 2.6. Let T be the space of all real 2×2 matrices with product

$$\mathbf{Mat}(a_{11}, a_{12}, a_{21}, a_{22}) \times \mathbf{Mat}(b_{11}, b_{12}, b_{21}, b_{22}) = \mathbf{Mat}(a_{11}, b_{12}, b_{21}, a_{22}).$$

Then T is a top space.

Let $M = \mathbb{R}^4$ be the Euclidean manifold, then map $\lambda\colon T \times M \longrightarrow M$ defined by

$$\mathbf{Mat}(a_{11}, a_{12}, a_{21}, a_{22}) \times (b, c, d, e) = (a_{11}, c, d, a_{22})$$

is a generalized action of T on M and

$$H(b, c, d, e) = \{\mathbf{Mat}(b, a_{12}, a_{21}, e) \mid a_{12}, a_{21} \in R\}.$$

Theorem 2.7. *Let T be a top space with the finite identity elements. Then the stabilizer $H(m)$ is a generalized subgroup of T, where $m \in M$.*

Proof. Let

$$t \in H(m).$$

Then

$$t \cdot m = m.$$

So

$$t^{-1} \cdot (t \cdot m) = t^{-1} \cdot m.$$

By definition of generalized action, we deduce

$$t^{-1} \cdot (t \cdot m) = (t^{-1} \cdot t) \cdot m = e(t) \cdot m.$$

Therefore

$$(\star) \qquad e(t) \cdot m = t^{-1} \cdot m.$$

It is clear that $t \cdot (t \cdot m) = m$. So $t^{-1} \cdot (t \cdot (t \cdot m)) = t^{-1} \cdot m$. Thus $e(t) \cdot (t \cdot m) = t^{-1} \cdot m$. Since $e(t) \cdot t = t$. Then $t \cdot m = t^{-1} \cdot m$, and $t^{-1} \cdot m = m$, using $(\star)$ we deduce $e(t) \cdot m = m$. Thus $e(t)$, $t^{-1} \in H(m)$. Hence $H(m)$ is a generalized subgroup. □

Remark. The subset $e^{-1}(e(t)) = \{s \in T \mid e(s) = e(t)\}$ of T is a Lie group with an identity element $e(t)$, for any $t \in T$.

Let $\lambda\colon T \times M \longrightarrow M$ be a generalized action of a top space T on a manifold M. We define two functions:

$$\begin{aligned} &\tau(t)\colon M \longrightarrow M \qquad \textit{and} \qquad &&\rho(m)\colon T \longrightarrow M \\ &\tau(t)(m) = t \cdot m &&\rho(m)(t) = t \cdot m \end{aligned}$$

where $t \in T$ and $m \in M$.

It is clear that $\tau(t)$ and $\rho(m)$ are smooth maps. Moreover $H(m) = (\rho(m))^{-1}(m)$. So $H(m)$ is a closed generalized subgroup of T.

Theorem 2.8. *Let $e^{-1}(e(t_0))$ be an open subset of T, where $t_0 \in T$, $\lambda\colon T \times M \longrightarrow M$ is a generalized action of the top space T on a manifold M and for every $m \in M$ there is an open neighborhood U of m such that $\lambda(e(s), n) = n$ for all $n \in U$, where $e(s) \in T$. Then for any $m \in M$, the orbit map $\rho(m)$ has the constant rank. In particular $\rho(m)$ is a subimmersion.*

Proof. For any $s, t \in T$ we have

$$(\tau(s) \circ \rho(m))(t) = \tau(s)(t \cdot m) = (st) \cdot m = \rho(m)(st) = (\rho(m) \circ \lambda(s))(t)$$

i.e.

$$(\tau(s) \circ \rho(m)) = (\rho(m) \circ \lambda(s))$$

for any $s \in T$.

If we calculate the differential of the map at the identity $e(r)$ in T, where $\lambda(e(r), n) = n$ in an open neighborhood U of m, we get

$$d_m(\tau(s)) \circ d_{e(r)}(\rho(m)) = d_s(\rho(m)) \circ d_{e(r)}(\lambda(s))$$

for any $s \in e^{-1}(e(r))$. [*Note*: $\rho(m)(e(r)) = m$ and $\lambda(s)(e(r)) = s$.]

It is easy to show that $\tau(s)\colon U \to \tau(s)(U)$ and $\lambda(s)\colon U \to \lambda(s)(U)$ are diffeomorphisms, and so, by the inverse function theorem, their differentials $d_m(\tau(s))$ and $d_{e(r)}(\lambda(s))$ are isomorphisms of tangent spaces. This implies that

$$\operatorname{rank}(d_{e(r)}(\rho(m))) = \operatorname{rank}(d_s(\rho(m)))$$

for any $s \in e^{-1}(e(r))$. Since $e^{-1}(e(i))$ and $e^{-1}(e(j))$ are diffeomorphic for all $i, j \in T$ [**5**], the orbit map $\rho(m)$ has constant rank. Hence $\rho(m)$ is a subimmersion. □

Theorem 2.9. *For any $m \in M$, the stabilizer $H(m)$ is a subtop space of T.*

Proof. Since $H(m)$ is a generalized subgroup of T, $\rho(m)$ is a differentiable subimmersion function between two manifolds T and M, then

$$H(m) = (\rho(m))^{-1}(m)$$

is a closed submanifold of T. Thus $H(m)$ is a subtop space of T. □

Example 2.10. In the Example 2.6, the stabiliizer $H(b, c, d, e)$ is a subtop space.

Theorem 2.11. *For any stabilizer $H(m)$ of a top space T with finite identity elements (i.e. cardinality $\operatorname{Card}(e(T)) < \infty$), we get $d_{e(t)}(H(m)) = \ker d_{e(t)}(\rho(m))$.*

Proof. Since T has finite identity elements, then $e^{-1}(e(t))$ is an open subset of T for each $t \in T$ [**4**], and it is a Lie group with the identity element $e(t)$, for any $t \in T$. So $H(m) \cap e^{-1}(e(t))$ is an open Lie subgroup of $e^{-1}(e(t))$. Thus

$$d_{e(t)}(H(m) \cap e^{-1}(e(t))) = \ker(d_{e(t)}(\rho(m) \mid_{e^{-1}(e(t))})).$$

Since $e^{-1}(e(t))$ is an open subset of T, then

$$d_{e(t)}(H(m)) = \ker d_{e(t)}(\rho(m)).$$

□

Let T be a top space and $\mathrm{Card}(e(T)) < \infty$. Then $T^m = H(m) \cap e^{-1}(e(t))$ is an open subgroup of the Lie group $e^{-1}(e(t)) = T_{e(t)}$, we know the coset space $T_{e(t)}/T^m$ admits a differentiable structure such that $T_{e(t)}/T^m$ becomes a manifold, called homogeneous space. Let λ be a generalized action of a top space T on a manifold M such that $\lambda|_{e^{-1}(e(t))\times M}$ be a transitive action. We know that if $T_{e(t)}/T^m$ is compact, then $T_{e(t)}/T^m$ is homeomorphic to M. Since $T_{e(t)}/T_m$ and M are manifolds,

$$\dim(T_{e(t)}/T^m) = \dim(T_{e(t)}) - \dim(T^m),$$

then

$$\dim(T_{e(t)}) = \dim(T^m) + \dim(M).$$

Since $T_{e(t)}$ is an open subtop space of T, then

$$\dim(T) = \dim(T_{e(t)}),$$

and hence

$$\dim(T) = \dim(T^m) + \dim(M).$$

Example 2.12. In the Example 2.6 we have

$$\dim(T) = \dim(T_m) + \dim(M)$$

moreover, $\dim(T) = 4$, $\dim(M) = 4$ and $\dim(T_m) = 0$ for any $m \in M$.

Example 2.13. The Euclidean space $T = \mathbb{R}$ with the multiplication $(a, b) \mapsto a$ is a top space, and $\lambda\colon T\times\mathbb{R} \longrightarrow \mathbb{R}$ defined by $\lambda(a, m) = am$ is a generalized action, where $M = \mathbb{R}$ is an Euclidean manifold. Then $\dim(T) = \dim(T_m) + \dim(M)$. [*Note*: $e(a) = a$ for any $a \in T$].

Let $\lambda : T \times M \longrightarrow M$ be a generalized action of a top space T on a manifold M and let $e(t)m_0 = m_0$ for every $t \in T$, we show that

$$S = \{\tau(t)\colon M \to M | \tau(t)(m) = tm,\ t \in T \ \text{ and } \ m \in M\}$$

with product $\tau(t) \otimes \tau(s) = \tau(ts)$ is a top space

It is clear that S with this product is closed and associative. Now let $\tau(t)$ belong to S, it is easy to show that $e(\tau(t)) = \tau(e(t))$, $[\tau(t)]^{-1} = \tau(t^{-1})$. Moreover, we have

$$\begin{aligned} e(\tau(t) \otimes \tau(s)) &= e(\tau(ts)) = \tau(e(ts)) = \tau(e(t)e(s)) \\ &= \tau(e(t)) \otimes \tau(e(s)) = e(\tau(t))e(\tau(s)), \end{aligned}$$

where $t, s \in T$. [*Note*: Since T is a top space, $e(ts) = e(t)e(s)$.]

Also, since the product $(t, s) \mapsto ts$ is a C^∞ map, then the product $\otimes$ is a C^∞ map. Thus $(S, \otimes)$ is a top space.

Let $f\colon T \longrightarrow S$ be defined by $f(t) = \tau(t)$, where $t \in T$. Then it is easy to show that f is a homomorphism between two top spaces T and S.

Now, we claim that $\ker(f_t)$ is empty or Lie subgroup of Lie group T_t.

We know that $T_t = e^{-1}(e(t)) = \{s \in T \mid e(s) = e(t)\}$ and $f_t = f|_{T_t}$, also we have

$$\ker(f_t) = \{r \in T_t \mid f_t(r) = e(f(r))\}.$$

Since

$$e(f(r)) = e(\tau(r)) = \tau(e(r)),$$

then

$$\ker(f_t) = \{r \in T_t \mid f_t(r) = \tau(e(r))\} = \{r \in T_t \mid f_t(r)(m) = e(r)m\}.$$

It is easy to show that $ker(f_t)$ is a Lie subgroup of T_t, and so

$$\ker(f_t) = \{r \in T_t \mid rm = e(r)m\}.$$

We know $\ker(f) = \bigcup_{t \in T} \ker(f_t)$, so

$$\ker(f) = \bigcup_{t \in T} \{r \in T_t \mid rm = e(r)m\},$$

therefore $\ker(f) = \{r \in T \mid rm = e(r)m$, for every $m\}$.

One can show that $\ker(f)$ is a subtop space of T and so we have the following lemma.

Lemma 2.14. *Let $\lambda \colon T \times M \longrightarrow M$ be a generalized action of a top space T on a manifold M and $S = \{\tau(t) \colon M \to M \mid \tau(t)(m) = tm,\ t \in T$ and $m \in M\}$. Then there is a product on S such that with this product S is a top space, and $f \colon T \longrightarrow S$ is a homomorphism of top spaces, where f is defined by $f(t) = \tau(t)$. Moreover,* $\ker(f)$ *is a subtop space of* T.

Now, by the generalized actions we introduce a new equivalence relation on a manifold M. Let $\lambda : T \times M \longrightarrow M$ be a generalized action of top space T on a manifold M.

Now, we say that

"$m \sim n$ if and only if there is $t \in T$ such that $tm = n$."

We claim that this relation is an equivalence relation on M. As it is clear that this relation is reflexive, let $m \sim n$, so there is $t \in T$ such that $tm = n$. Then $t^{-1}tm = t^{-1}n$ and so $e(t)m = t^{-1}n$. Moreover, since λ is a generalized action, then there exists $e(r) \in T$ such that $e(r)m = m$. Therefore we have

$$e(r)(e(t)m) = e(r)(e(t)(e(r)m)) = (e(r)e(t)e(r))m = e(r)m = m.$$

[*Note*: We know that $e(t)e(s)e(t) = e(t)$ for every $t, s \in T$. [**5**]].

Thus

$$m = e(r)(t^{-1}n) = (e(r)t^{-1})n,$$

therefore $n \sim m$ and so $\sim$ is a symmetric relation.

Let $m \sim n$ and $n \sim p$, then there are t and s belonging to T such that $tm = n$ and $sn = p$. Therefore $stm = sn = p$. So $m \sim p$. Hence every generalized action of a top space on the manifold M induces an equivalence relation $\sim$ on M.

Moreover, since $\tau(t)\colon M \to M$ and the projection map $P\colon M \to M/\sim$ are continuous maps, then there is a unique continuous map $Q\colon M/\sim \to M$ such that $Q \circ P = \tau(t)$ and we have the following commutative diagram

$$\begin{tikzcd} M \arrow[r, "\tau(t)"] \arrow[d, "P"'] & M \\ M/\sim \arrow[ur, "Q"'] & \end{tikzcd}$$

By the relation $\sim$ we have a quotient space $M/\sim$. We know that if $\sim$ is a regular relation, then there is a unique differentiable structure such that $M/\sim$ is a quotient manifold.

Example 2.15. In the Example 2.2, the quotient space $\mathbb{R}$ is

$$\mathbb{R}/\sim = \{[0],[a]\}, \quad [0] = \{0\} \text{ and } [a] = \mathbb{R} - \{0\}, \quad \text{where } a \neq 0.$$

Conclusion: In this paper we prove the following statements:

1. For every generalized action of a top space T on a manifold M, the stabilizer $H(m)$ is a subtop space of T, moreover, $\rho(m)$ is a subimmersion.
2. For a top spaces with finite identity elements, the tangent space of a stabilizer is equal to the kernel of differential $\rho(m)$.
3. The set of all $\tau(t)$'s is a top space.

References

1. Molaei M. R., *Mathematical Structures Based on Completely Simple Semigroups*, Hadronic Press, 2005.

2. Molaei M. R., *Generalized Actions*, In: Intenational Conference on Geometry, Integrability and Quantization, September 1–10, Varna 1999.

3. Molaei M. R., *Top Spaces*, Journal of Inerdisciplinary Mathematics, **7**(2) (2004), 173–181.

4. Molaei M. R., Khadekar G. S. and Farhangdoost M. R., *On Top Spaces*, Balkan Journal of Geometry and its Applications, **11**(1) (2006), 101–106.

5. Farhangdoost M. R. and Molaei M. R., *Charactrization of Top Spaces by Diffeomorphic Lie Groups*, Differential Geometry-Dynamical Systems, **11** (2009), 130–134.

M. R. Farhangdoost, Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71457-44776, Iran, *e-mail*: `farhang@shirazu.ac.ir`, *e-mail*: `farhang858@yahoo.com`