Acta Math. Univ. Comenianae
Vol. LXXX, 2 (2011), pp. 251–257

THE LAPLACE-STIELTJES TRANSFORMATION ON ORDERED TOPOLOGICAL VECTOR SPACE OF GENERALIZED FUNCTIONS

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Abstract. We have combined the Laplace transform and the Stieltjes transform of the form $\hat{f}(x) = \int_0^\infty \frac{f(t)}{(x^m+t^m)^\rho} \mathrm{d}t,\ m, \rho > 0$ and applied it to an ordered vector space of generalized functions to which the topology of bounded convergence is assigned. Some of the order properties of the transform and its inverse are studied. Also we solve an initial value problem and compare different solutions of the problem.

1. Introduction

In an earlier paper [**2**] we associated the notion of order to multinormed spaces, countable union spaces and their duals. The topology of bounded convergence was assigned to the dual spaces. Some properties of these spaces were also studied. As illustrative examples the spaces $\mathscr{D}$, $\mathscr{E}$, $\mathscr{L}_{a,b}$, $\mathscr{L}(w, z)$ and their duals were also studied.

In the present paper an ordered testing function space $M^m_{a,b,c}$ is defined and the topology of bounded convergence is assigned to its dual. Some properties of the cone in $M^m_{a,b,c}$ and in $(M^m_{a,b,c})'$ are studied in Section 2.

The Laplace-Stieltjes transform and its properties were studied by Geetha K. V. and John J. K. [**1**]. We apply these results to the new context when the topology of bounded convergence is assigned to the ordered vector space $(M^m_{a,b,c})'$ in Section 3. Some of the order properties of the transform are also studied.

In Section 4, the inversion of the transform and the order properties of the inverse are studied.

An operational transform formula which can be applied to boundary value problems is established in Section 5. As the solutions are from an ordered vector space, comparison between different solutions is possible.

Received December 12, 2010.
2010 *Mathematics Subject Classification.* Primary 46F10; Secondary 46A04, 46A08.
Key words and phrases. Laplace-Stieltjes transformation; positive cone; order-bounded.

2. The test function space $M^m_{a,b,c}$ and its dual as ordered vector spaces

Let $M^m_{a,b,c}$ denote the linear space of all complex-valued smooth functions $\phi(u,t)$ defined on $(0,\infty)\times(0,\infty)$. Let (K_m) be a sequence of compact subsets of $\mathbb{R}_+\times\mathbb{R}_+$ such that $K_1 \subseteq K_2 \subseteq \ldots$ and such that each compact subset of $\mathbb{R}_+\times\mathbb{R}_+$ is contained in one K_j, $j=1,2,\ldots$. Each K_j defines

$$\mu_{a,b,c,K_j,q,l}\phi(u,t) \;=\; \sup_{(u,t)\in K_j}\left|e^{cu}u^{q+1}D_u^q t^{m(1-a+b)}(1+t^m)^{a-b}(t^{1-m}D_t^l)\phi(u,t)\right|$$

where $a,b,c\in\mathbb{R}$, $q,l=0,1,2,\ldots$, $m\in(0,\infty)$ and

$$D_u=\frac{\partial}{\partial u}D_t=\frac{\partial}{\partial t}.$$

$\{\mu^m_{a,b,c,K_j,q,l}\}^\infty_{q,l=0}$ is a multinorm on M^m_{a,b,c,K_j}, where M^m_{a,b,c,K_j} is a subspace of $M^m_{a,b,c}$ consisting of functions with support in K_j. The above multinorm generates the topology τ^m_{a,b,c,K_j} on M^m_{a,b,c,K_j}. The inductive limit topology $\tau^m_{a,b,c}$ as K_j varies over all compact sets $K_1,K_2,\ldots$ is assigned to $M^m_{a,b}$. With respect to this topology $M^m_{a,b,c}$ is complete. On M^m_{a,b,c,K_j} an equivalent multinorm is given by

$$\overline{\mu}^m_{a,b,c,K_j,q,l}(\phi)=\sup_{\substack{0\le q'\le q\\0\le l'\le l}}\mu^m_{a,b,c,K_j,q',l'}(\phi).$$

By identifying a positive cone in $M^m_{a,b,c}$, an order relation is defined on it as follows.

Definition 2.1. The positive cone of $M^m_{a,b,c}$ when $M^m_{a,b,c}$ is restricted to real-valued functions is the set of all non-negative functions in $M^m_{a,b,c}$. When the field of scalars is $\mathbb{C}$, the positive cone in $M^m_{a,b,c}$ is $C+\mathrm{i}\,C$ which is also denoted as C.

We say $\phi\le\psi$ in $M^m_{a,b,c}$ if $\psi-\phi\in C$ in $M^m_{a,b,c}$.

Theorem 2.1. *The cone C of $M^m_{a,b,c}$ is not normal.*

Proof. Suppose that $M^m_{a,b,c}$ is restricted to real-valued functions. Let j be a fixed positive integer and let (ϕ_i) be a sequence of functions in $C\cap(M^m_{a,b,c,K_j})$ such that $\lambda_i=\sup\{\phi_i(u,t):(u,t)\in K_j\}$ converges to 0 for M^m_{a,b,c,K_j}.

Define

$$\psi_i=\begin{cases}\lambda_i, & (u,t)\in K_{j+1}\\ 0, & (u,t)\notin K_{j+1}.\end{cases}$$

Let ξ_i be the regularization of ψ_i defined by

$$\xi_i(u,t)=\int_0^\infty\int_0^\infty\theta_\alpha(u-u',t-t')\psi_i(u',t')\mathrm{d}u'\,\mathrm{d}t'.$$

Then $\xi_i\in M^m_{a,b,c,K_{j+2}}$, for all i.

In [**3**, Proposition 1.3, Chapter 1] it was proved that the positive cone in an ordered vector space $E(\tau)$ is normal if and only if for any two nets $\{x_\beta:\beta\in I\}$ and $\{y_\beta:\beta\in I\}$ in $E(\tau)$ if $0\le x_\beta\le y_\beta$ for all $\beta\in I$ and if $\{y_\beta:\beta\in I\}$ converges to 0 for τ, then $\{x_\beta : \beta \in I\}$ converges to 0 for τ. So we conclude that C is not normal for the Schwartz topology. It follows that $C + \mathrm{i}\, C$ is not normal for $M^m_{a,b,c}$. $\square$

Theorem 2.2. *The cone C is a strict b-cone in $M^m_{a,b,c}$.*

Proof. Let $M^m_{a,b,c}$ be restricted to real-valued functions. Let $\mathbb{B}$ be the saturated class of all bounded circled subsets of $M^m_{a,b,c}$ for $\tau^m_{a,b,c}$. Then $M^m_{a,b,c} = \cup_{B\in\mathscr{B}}\mathscr{B}_c$ where $\mathscr{B}_c = \{(B\cap C) - (B \cap C) : B \in \mathscr{B}\}$ is a fundamental system for $\mathscr{B}$ and C is a strict b-cone since $\mathscr{B}$ is a collection of all $\tau^m_{a,b,c}$-bounded sets in $M^m_{a,b,c}$.

Let B be a bounded circled subset of $M^m_{a,b,c}$ for $\tau^m_{a,b,c}$. Then all functions in B have their support in a compact set K_{j_0} and there exists a constant $M > 0$ such that $|\phi(u,t)| \le M$, for all $\phi \in B$, $(u,t) \in K_{j_0}$. Let

$$\psi(u,t) = \begin{cases} M, & (u,t) \in K_{j_0+1} \\ 0, & (u,t) \notin K_{j_0+1}. \end{cases}$$

The regularization $\xi(u,t)$ of $\psi(u,t)$ is defined by

$$\xi(u,t) = \int_0^\infty \int_0^\infty \theta_\alpha(u-u', t-t')\psi(u',t')\mathrm{d}u'\,\mathrm{d}t'$$

and ξ has its support in K_{j_0+2}.
Also,

$$B \subseteq (B+\xi) \cap \{\xi\} \subseteq (B+\xi)\cap C - (B+\xi)\cap C.$$

It follows that C is a strict b-cone. We conclude that $C + \mathrm{i}\, C$ is a strict b-cone. $\square$

Order and Topology on the dual of $M^m_{a,b,c}$.

Let $(M^m_{a,b,c})'$ denote the linear space of all linear functionals defined on $M^m_{a,b,c}$. An order relation is defined on $(M^m_{a,b,c})'$ by identifying the positive cone of $(M^m_{a,b,c})'$ to be the dual cone C' of C in $M^m_{a,b,c}$. The class of all B^0, the polars of B as B varies over all $\sigma(M^m_{a,b,c}, (M^m_{a,b,c})')$-bounded subsets of $M^m_{a,b,c}$ is a neighbourhood basis of 0 in $(M^m_{a,b,c})'$ for the locally convex topology $\beta((M^m_{a,b,c})', M^m_{a,b,c})$. When $(M^m_{a,b,c})'$ is ordered by the dual cone C' and is equipped with the topology $\beta((M^m_{a,b,c})', M^m_{a,b,c})$ it follows that C' is a normal cone since C is a strict b-cone in $M^m_{a,b,c}$ for $\tau^m_{a,b,c}$. Note that $\beta((M^m_{a,b,c})', M^m_{a,b,c})$ is the $\mathscr{G}$-topology corresponding to the class $\mathscr{C}$ of all complete, bounded, convex, circled subsets of $(M^m_{a,b,c})'$ for the topology $\sigma(M^m_{a,b,c}, (M^m_{a,b,c})')$. Anthony L. Perissini [**3**] observed that the subset $\mathscr{L}(E,F)$, the linear space of all continuous linear mappings of E into F is bounded for the topology of pointwise convergence if and only if it is bounded for the $\mathscr{G}$-topology corresponding to the class of all complete, bounded, convex, circled subsets of $E(\tau)$.

Theorem 2.3. *The order dual $(M^m_{a,b,c})^+$ is the same as the topological dual $(M^m_{a,b,c})'$ when $(M^m_{a,b,c})'$ is assigned the topology of bounded convergence.*

Proof. Every positive linear functional is continuous for the Schwartz topology $\tau^m_{a,b,c}$ on $M^m_{a,b,c}$.

$$(M^m_{a,b,c})^+ = C(M^m_{a,b,c}, \mathbb{R}) - C(M^m_{a,b,c}, \mathbb{R})$$

where $C(M^m_{a,b,c}, \mathbb{R})$ is the linear subspace consisting of all non-negative order bounded linear functionals on $M^m_{a,b,c}$ of $L(M^m_{a,b,c}, \mathbb{R})$, the linear space of all order bounded linear functionals on $M^m_{a,b,c}$. It follows that

$$(M^m_{a,b,c})^+ \subseteq (M^m_{a,b,c})'.$$

On the other hand, the space $(M^m_{a,b,c})'$ equipped with the topology of bounded convergence $\beta((M^m_{a,b,c})', M^m_{a,b,c})$ and ordered by the dual cone C' of the cone C in $M^m_{a,b,c}$ is a reflexive space ordered by a closed normal cone. Hence if D is a directed set $(\leq)$ of distributions that is either majorized in $(M^m_{a,b,c})'$ or contains a $\beta((M^m_{a,b,c})', M^m_{a,b,c})$-bounded section, then $\sup D$ exists in $(M^m_{a,b,c})'$ and the filter $\mathscr{F}(D)$ of sections of D converges to $\sup D$ to $\beta((M^m_{a,b,c})', M^m_{a,b,c})$. Hence $(M^m_{a,b,c})^+ = (M^m_{a,b,c})'$ we conclude that $(M^m_{a,b,c})'$ is both order complete and topologically complete with respect to the topology $\beta((M^m_{a,b,c})', M^m_{a,b,c})$. □

3. The Laplace-Stieltjes Transformation

For $\phi(u,t) \in M^m_{\alpha,\beta,\gamma}$, the Laplace-Stieltjes transform is defined as

$$\begin{aligned}\mathrm{SL}^m_\rho(\phi(u,t)) &= \hat{\phi}(y,x)\\ &= \int_0^\infty \int_0^\infty \mathrm{e}^{-yu}(x^m + t^m)^{-\rho}\phi(u,t)\mathrm{d}u\,\mathrm{d}t\end{aligned}$$

for a fixed $m > 0$, $\rho \geq 1$.

With suitable integrability conditions the multiple integral

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{e}^{-yu}(x^m + t^m)^{-\rho} f(x,y)\phi(u,t)\,\mathrm{d}u\,\mathrm{d}t\,dx\,dy$$

can be evaluated in two different ways so that

$$\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle.$$

Because of this relation we apply the method of adjoints to the Laplace-Stieltjes transformation.

The following result is due to Geetha K. V. and John J. K. [**1**].

Theorem 3.1. *Let $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - 1/m$. Then the Laplace-Stieltjes transformation maps $C \cap M^m_{\alpha,\beta,\gamma}$ continuously into $C \cap M^m_{a,b,c}$ if*

$$\begin{array}{llll} a < 1, & a \leq \frac{1}{m} + \alpha - \rho & \text{and} \quad a < 1 & \text{if} \;\; \alpha = \rho + 1 - \frac{1}{m} \\ b \geq 1 - \rho, & b \geq \frac{1}{m} + \beta - \rho & \text{and} \quad b > 1 - \rho & \text{if} \; \beta = 1 - \frac{1}{m}. \end{array}$$

Definition 3.1. Let $f \in (M^m_{a,b,c})'$. The adjoint mapping $\langle \mathrm{SL}^m_\rho(f), \phi\rangle = \langle f, \mathrm{SL}^m_\rho(\phi)\rangle$ defines the Laplace-Stieltjes transformation $\mathrm{SL}^m_\rho(f) \in (M^m_{\alpha,\beta,\gamma})'$ of f.

Theorem 3.2. *The Laplace-Stieltjes transform is strictly positive and order-bounded.*

Proof. $(M^m_{a,b,c})'$ is an ordered vector space. We define an order relation on the field of complex numbers by identifying the positive cone to be the set of complex numbers $\alpha + \mathrm{i}\,\beta$, $\alpha > 0$, $\beta > 0$. If $f > 0$, $\mathrm{SL}^m_\rho(f) > 0$. So SL^m_ρ is strictly positive and hence is order bounded. □

4. Inversion

For a non-negative integer n, we define a differential operator for $\rho + n > \frac{1}{m}$ by

$$L_{n,y,x}\phi(y,x) = My^{2n+1}(D_yD_x)(D_yx^{1-m}D_x)^{n-1}x^{2mn+m\rho-m}(D_yx^{1-m}D_x)^n\phi(y,x)$$

where

$$M = \frac{m^{1-2n}\Gamma(\rho)}{\Gamma(n+1/m)\Gamma(\rho+n-1/m)\Gamma(2n+1)}.$$

The Laplace-Stieltjes transform can be inverted by the application of this differential operator. The formal adjoint of this operator is itself.

Lemma 4.1. *If $\rho + n > \frac{1}{m}$,*

$$\int_0^\infty\int_0^\infty L_{n,y,x}\, \mathrm{e}^{-yu}(x^m + t^m)^{-\rho}\mathrm{d}u\mathrm{d}t = 1.$$

Lemma 4.2. *$L_{n,u,t}$ maps $M^m_{a,b,c}$ continuously into $M^m_{\alpha,\beta,\gamma}$ provided $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta + \rho$.*

Lemma 4.3. *If L_n is the differential operator and SL^m_ρ the Laplace-Stieltjes transform operator, then either*

$$y^{-2n}x^{1-m\rho}L_n \quad \textit{and} \quad \mathrm{SL}^m_\rho\, x^{m\rho-1}y^{-2n}$$

or

$$L_nx^{1-m\rho}y^{-2n} \quad \textit{and} \quad y^{-2n}x^{m\rho-1}\,\mathrm{SL}^m_\rho \quad \textit{commute on} \quad M^m_{a,b,c}$$

where $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$, i.e.

$$\begin{aligned} y^{-2n}x^{m\rho-1}\,\mathrm{SL}^m_\rho[L_{n,u,t}(\phi)] &= y^{-2n}x^{m\rho-1}[L_{n,u,t}(\phi)]^\wedge \\ &= L_{n,y,x}\int_0^\infty\int_0^\infty \mathrm{e}^{-yu}(x^m+t^m)^{-\rho}u^{-2n}t^{m\rho-1}\phi(u,t)\mathrm{d}u\mathrm{d}t \\ &= L_{n,y,x}\,\mathrm{SL}^m_\rho(u^{-2n}t^{m\rho-1}\phi), \qquad \textit{for} \quad \phi \in M^m_{a,b,c}. \end{aligned}$$

Theorem 4.1. *Let $\alpha > 1 - \frac{1}{m}$, $\beta < \rho + 1 - \frac{1}{m}$, then the sequence $\{L_{n,y,x}\hat{\phi}(y,x)\}$ converges in $M^m_{\alpha,\beta,\gamma}$ to $\phi(y,x)$.*

Theorem 4.2. *Let $a = \frac{1}{m} + \alpha - \rho$, $b = \frac{1}{m} + \beta - \rho$, then $(L_n(\phi)^\wedge)$ converges to ϕ in $M^m_{a,b,c}$ as $n \to \infty$.*

Theorem 4.3. *Let $f \in (M^m_{a,b,c})'$. Then $f \in C'$ if and only if for every non-negative integer n, $L_{n,y,x}\,\mathrm{SL}^m_\rho(f) \in C'$, where C' is the positive cone in $(M^m_{a,b,c})'$. It follows that $L_{n,y,x}$ is strictly positive and hence is order bounded. We summarize the above results as follows:*
For $f \in (M^m_{\alpha,\beta,\gamma})'$, $\phi \in M^m_{\alpha,\beta,\gamma}$

$$\langle \mathrm{SL}^m_\rho(L_{n,y,x}(f)), \phi\rangle = \langle f, L_{n,y,x}\,\mathrm{SL}^m_\rho(\phi)\rangle \to \langle f, \phi\rangle \text{ as } n \to \infty.$$

For $f \in (M^m_{a,b,c})'$, $\phi \in M^m_{a,b,c}$

$$\langle L_{n,y,x}\,\mathrm{SL}^m_\rho(f), \phi\rangle = \langle f, \mathrm{SL}^m_\rho\, L_{n,y,x}(\phi)\rangle \to \langle f, \phi\rangle.$$

5. Operational calculus

$$\mathrm{SL}^m_\rho[D_u D_t(\phi)] = (m\rho)y\,\mathrm{SL}^m_{\rho+1}[t^{m-1}\phi(u,t)],$$

provided

$$\lim_{t\to\infty} D_u(\phi(u,t)) = 0 = \lim_{t\to 0} D_u(\phi(u,t))$$
$$\lim_{u\to\infty} \phi(u,t) = 0 = \lim_{u\to\infty} \phi(u,t).$$

Consider the differential equation

$$(D_u D_t)\phi(u,t) = f(u,t), \quad u > 0, \quad t > 0 \text{ where } f(u,t)$$

is a generalized function upon which Laplace-Stieltjes transform can be applied.

Let $F_1(u,t) = \int f(u,t)\mathrm{d}t$ be such that

$$\lim_{t\to\infty} F_1(u,t) = 0 = \lim_{t\to 0} F_1(u,t)$$

and let

$$F_2(u,t) = \int F_1(u,t)\mathrm{d}u$$

be such that

$$\lim_{u\to\infty} F_2(u,t) = 0 = \lim_{u\to 0} F_2(u,t).$$

Applying the operational calculus

$$(m\rho)y\,\mathrm{SL}^m_{\rho+1}[t^{m-1}\phi(u,t)] = (m\rho)y\,\mathrm{SL}^m_{\rho+1}[t^{m-1}F_2(u,t)].$$

Inverting using the differential operator $L_{n,y,x}$ for $\rho + 1 + n > \frac{1}{m}$.

$$t^{m-1}\phi(u,t) = t^{m-1}F_2(u,t)$$

so that $\phi(u,t) = F_2(u,t)$ where $F_2(u,t) = \int\int f(u,t)\mathrm{d}t\mathrm{d}u$. If more than one solution is obtained comparison between the solution is possible since they belong to an ordered vector space. Let $\phi_1(u,t)$, $\phi_2(u,t)$ be two solutions of the initial value problem such that $\phi_1(u,t)$ for all (u,t) lies on the left of the line, say, $\mathrm{Re}s = \alpha$ and $\phi_2(u,t)$ lies on the right for all (u,t), then $\phi_1 \le \phi_2$ by the order relation we have defined on $(M^m_{a,b,c})'$.

Acknowledgment. The authors acknowledge with gratitude the help and guidance received from Dr. T. Thrivikraman in the preparation of this paper.

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