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REGULAR TETRAHEDRA WHOSE VERTICES HAVE INTEGER COORDINATES

E. J. IONASCU

Abstract. In this paper we introduce theoretical arguments for constructing a procedure that allows one to find the number of all regular tetrahedra that have coordinates in the set $\{0, 1, ..., n\}$. The terms of this sequence are twice the values of the sequence A103158 in the Online Encyclopedia of Integer Sequences [**16**]. These results lead to the consideration of an infinite graph having fractal nature which is tightly connected to the set of orthogonal 3-by-3 matrices with rational coefficients. The vertices of this graph are the primitive integer solutions of the Diophantine equation $a^2 + b^2 + c^2 = 3d^2$. Our aim here is to lay down the basis of finding good estimates, if not exact formulae, for the sequence A103158.

1. Introduction

The story of regular tetrahedra having vertices with integer coordinates starts with the parametrization of some equilateral triangles in $\mathbb{Z}^3$ that began in [**9**]. There was an additional hypothesis that did not cover all the generality in the result obtained in [**9**] but it was removed successfully in [**2**]. In this note we are interested in the following problem

*How many regular tetrahedra, $T(n)$, can be found if the coordinates of its vertices must be in the set $\{0, 1, ..., n\}$? We observe that $A103158 = \frac{1}{2}T(n)$ (see [**16**]).*

This sequence starts as in the following table.

n	1	2	3	4	5	6	7	8	9	10	11
A103158	1	9	36	104	257	549	1058	1896	3199	5154	7926

n	12	13	14	15	16	17	18
A103158	11768	16967	23859	32846	44378	58977	77215

These values were computed by Hugo Pfoertner in 2005, using a brute force program. Our method of counting is based on several theoretical facts. Roughly, it is an extension of the technique described in [**10**] using the results from [**12**] about the existence of regular tetrahedrons in $\mathbb{Z}^3$. The program can be used to cover values of $T(n)$ for n quite bigger than 100, but we included here only

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the first one hundred terms just for exemplification. The rest of the terms are at the end of the paper. Our approach begins by looking first at the faces of a regular tetrahedron which must be equilateral triangles. It turns out that every equilateral triangle in $\mathbb{Z}^3$ after a translation by a vector with integer coordinates can be assumed to have the origin as one of its vertices. Then one can show that the triangle's other vertices are contained in a lattice of the form

$$\begin{aligned}\mathcal{P}_{a,b,c} := \{(\alpha,\beta,\gamma) \in \mathbb{Z}^3 \ \mid \ a\alpha + b\beta + c\gamma = 0, \ \ a^2+b^2+c^2 = 3d^2,\\ a,b,c,d \in \mathbb{Z}\}.\end{aligned} \tag{1}$$

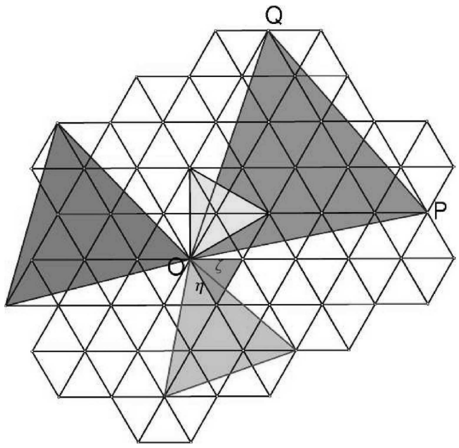

Figure 1. The lattice $\mathcal{P}_{a,b,c}$.

In general, the vertices of the equilateral triangles that dwell in $\mathcal{P}_{a,b,c}$ form a strict sub-lattice of $\mathcal{P}_{a,b,c}$ which is generated by only two vectors, $\overrightarrow{\zeta}$ and $\overrightarrow{\eta}$ (see Figure 1). These two vectors are described by Theorem 1.1 proved in [**2**].

Theorem 1.1. *Let a, b, c, d be odd integers such that $a^2 + b^2 + c^2 = 3d^2$ and $\gcd(a, b, c) = 1$. Then for every $m, n \in \mathbb{Z}$ (not both zero), the triangle OPQ determined by*

$$\overrightarrow{OP} = m\overrightarrow{\zeta} - n\overrightarrow{\eta}, \qquad \overrightarrow{OQ} = n\overrightarrow{\zeta} - (n-m)\overrightarrow{\eta}, \tag{2}$$

with $\overrightarrow{\zeta} = (\zeta_1, \zeta_1, \zeta_2)$, $\overrightarrow{\eta} = (\eta_1, \eta_2, \eta_3)$,

$$\begin{cases} \zeta_1 = -\dfrac{rac+dbs}{q}, \\ \zeta_2 = \dfrac{das-bcr}{q}, \\ \zeta_3 = r, \end{cases} \quad , \quad \begin{cases} \eta_1 = -\dfrac{db(s-3r)+ac(r+s)}{2q}, \\ \eta_2 = \dfrac{da(s-3r)-bc(r+s)}{2q}, \\ \eta_3 = \dfrac{r+s}{2}, \end{cases} \tag{3}$$

where $q = a^2 + b^2$ *and* (r, s) *is a suitable solution of* $2q = s^2 + 3r^2$ *that makes all of the numbers in* (3) *integers, forms an equilateral triangle in* $\mathbb{Z}^3$ *that is contained in the lattice* (1) *and has sides-lengths equal to* $d\sqrt{2(m^2 - mn + n^2)}$.

Conversely, if there exists a choice of the integers r *and* s *such that given an arbitrary equilateral triangle in* $\mathbb{R}^3$ *whose vertices, one at the origin and the other two in the lattice* (1)*, then there also exist integers* m *and* n *such that the two vertices not at the origin are given by* (2) *and* (3).

The Diophantine equation

$$a^2 + b^2 + c^2 = 3d^2 \tag{4}$$

has non-trivial solutions for every odd number d. As a curiosity, for $d = 2011$ one obtains 336 solutions satisfying also $0 < a \leq b \leq c$ and $\gcd(a, b, c) = 1$. We will refer to such a solution of (4) as a *positive ordered primitive* solution. For $d = 2011$, all of these solutions, except one, satisfy even a stronger condition $a < b < c$. The exception is $a = b = 913$ and $c = 3235$. Determining the exact number of solutions for (4) is certainly important if one wishes to find the number (or just an estimate) of equilateral triangles or the number of tetrahedra with vertices in $\{0, 1, 2, \ldots, n\}^3$. In the paper of Hirschhorn and Seller [**8**] from 1999, the number of solutions for (4), taking into account all permutations and changes of signs is equal to

$$8\left[\prod_{\substack{p\equiv 1 \text{ or } 7 \pmod{12} \\ p^\beta || d}} p^\beta\right]\left[\prod_{\substack{q\equiv 5 \text{ or } 11 \pmod{12} \\ q^\alpha || d}} \left(q^\alpha + 2\frac{q^\alpha - 1}{q - 1}\right) f(d)\right] \tag{5}$$

where

$$f(d) = \begin{cases} 1 & \text{if } 3|d \\ \dfrac{3^\gamma - 1}{2} & \text{if } 3^\gamma || d. \end{cases}$$

Even more important for our purpose is the calculation of the number of primitive representations of d as in (4) ($\gcd(a, b, c) = 1$) in terms of d which appeared in a more recent paper of Cooper and Hirschhorn [**3**]. One may easily check that the following is a corollary of Theorem 2 in [**3**].

Theorem 1.2 (Cooper-Hirschhorn)**.** *Given an odd number* d*, the number of primitive solutions of* (4) *taking into account all changing of signs and permutations, is equal to*

$$\Lambda(d) := 8d \prod_{p|d, p \text{ prime}} \left(1 - \frac{(\frac{-3}{p})}{p}\right), \tag{6}$$

where $(\frac{-3}{p})$ *is the Legendre symbol.*

We remind the reader that if p is prime, then

$$\left(\frac{-3}{p}\right) = \begin{cases} 0 & \text{if } p = 3 \\ 1 & \text{if } p \equiv 1 \text{ or } 7 \pmod{12} \\ -1 & \text{if } p \equiv 5 \text{ or } 11 \pmod{12} \end{cases} \tag{7}$$

We observe that the same type of prime partition is used in different calculations in both formulae (5) and (6). We have mentioned that the number of positive ordered primitive representations for $d = 2011$ was 336. This is one less than the number given by (6) modulo the number of permutations and changes of signs. Indeed, since 2011 is prime and $2011 \equiv 7 \pmod{12}$, we have $(\frac{-3}{2011}) = 1$ and then $\frac{\Lambda(2011)}{48} = \frac{8(2011)(1-\frac{1}{2011})}{48} = 335$. This happens because there is basically only one solution where we have repeating values for a, b and c as we said. We will see later how this number of positive ordered primitive representations can be obtained in general by compensating for the repeating ones.

For $k \in \mathbb{N}$, we let $\Omega(k) := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m^2 - mn + n^2 = k^2\}$. In [**9**] we showed that every regular tetrahedron with integer coordinates must have side lengths of the form $\lambda\sqrt{2}$, $\lambda \in \mathbb{N}$, and in [**12**] we found the following characterization of the regular tetrahedrons with integer coordinates.

Theorem 1.3. *Every tetrahedron whose side lengths are $\lambda\sqrt{2}$, $\lambda \in \mathbb{N}$, which has a vertex at the origin, can be obtained by taking as one of its faces an equilateral triangle having the origin as a vertex and the other two vertices given by* (2) *and* (3) *with a, b, c and d odd integers satisfying* (4) *with d, a divisor of λ, and then completing it with the fourth vertex R with coordinates*

$$\begin{aligned} \Bigg(&\frac{(2\zeta_1 - \eta_1)m - (\zeta_1 + \eta_1)n \pm 2ak}{3}, \\ &\frac{(2\zeta_2 - \eta_2)m - (\zeta_2 + \eta_2)n \pm 2bk}{3}, \\ &\frac{(2\zeta_3 - \eta_3)m - (\zeta_3 + \eta_3)n \pm 2ck}{3}\Bigg), \end{aligned} \tag{8}$$

for some $(m, n) \in \Omega(k)$, $k := \frac{\lambda}{d}$.

Conversely, if we let a, b, c and d be a primitive solution of (4)*, let $k \in \mathbb{N}$ and $(m, n) \in \Omega(k)$, then the coordinates of the point R in* (8)*, which completes the equilateral triangle OPQ given as in* (2) *and* (3)*, are*

(a) all integers if $k \equiv 0 \pmod 3$ regardless of the choice of signs or

(b) integers, precisely for only one choice of the signs if $k \not\equiv 0 \pmod 3$.

The following graph (Figure 1) is constructed on the positive ordered primitive solutions of (4), with edges defined by:

> *two vertices, say $[(a_1, b_1, c_1), d_1]$ and $[(a_2, b_2, c_2), d_2]$, that are connected if and only if*
>
> $$a_1a_2' \pm a_2b_2' \pm c_1c_2' \pm d_1d_2 = 0 \tag{9}$$
>
> *for some choice of the signs and permutation (a_2', b_2', c_2') of (a_2, b_2, c_2).*

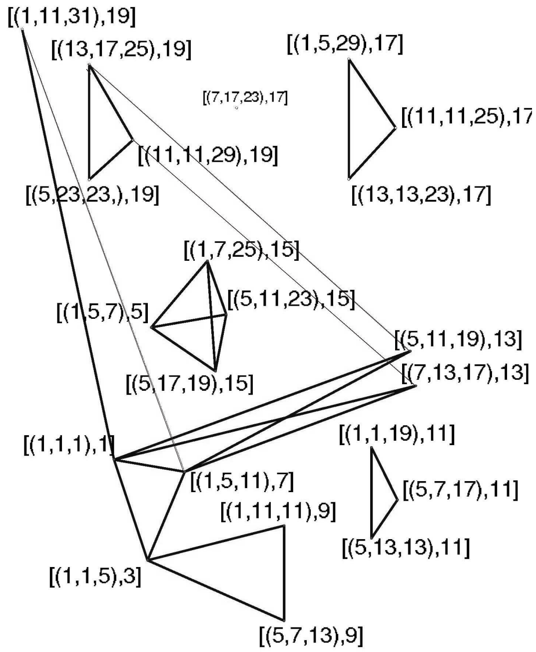

Figure 2. The graph $\mathcal{RT}$, $d \leq 19$..

Equation (9) insures basically insures that the planes $\mathcal{P}_{a_1,b_1,c_1}$ and $\mathcal{P}_{a_2',b_2',c_2'}$ associated with two faces make a dihedral angle of $\arccos(1/3) \approx 70.52878°$. In fact, this equality characterizes the existence of a regular tetrahedron having integer coordinates with one of its faces in the plane $\mathcal{P}_{a_1,b_1,c_1}$ and another contained in the plane $\mathcal{P}_{a_2',b_2',c_2'}$. For instance, $[(1,1,5),3]$ is connected to $[(1,5,11),7]$ since $1(11)+(1)5+5(1)-3(7)=0$. An example of a regular tetrahedron which has a face in $\mathcal{P}_{-5,-1,1}$ and one face in $\mathcal{P}_{-1,-5,11}$ is given by the vertices $[19,23,0]$, $[0,12,20]$, $[27,0,17]$, and $[24,27,29]$.

A few questions related to this graph appear naturally at this point. Is it connected? Is there a different characterization of the existence of an edge between two vertices in terms of only d_1 and d_2? We do not have an answer to the second question, but we have an heuristic argument that shows that the graph is disconnected. The vertices, $[(1,1,1),1]$ and $[(1,5,7),5]$, are in two different components and the component starting at $[(1,5,7),5]$ contains a copy of the whole graph.

Each edge in this graph, determined by $[(a_1,b_1,c_1),d_1]$ and $[(a_2,b_2,c_2),d_2]$, gives rise to a minimal tetrahedron whose side lengths are at most $\max\{d_1,d_2\}\sqrt{2}$. This tetrahedron is determined up to the set of isometric transformations that are generated by the symmetries of the cube in $\mathcal{C}(m)$ where m is the size of the smallest "cube" $\{0,1,\cdots,m\}^3$ containing the tetrahedron or a translation of it. In [**13**] we explained how this graph is connected with the 3-by-3 orthogonal matrices having rational entries.

2. Some preliminaries

We would like to have a good estimate of the number of primitive solutions of (4) which satisfy in addition $0 < a \le b \le c$. Let us observe that we cannot have $a = b = c$ unless $d = 1$. So, the counting in (6) via (7) would give what we want if we can count the number of positive primitive solutions of the following equation in terms of d

$$2a^2 + c^2 = 3d^2. \tag{10}$$

A description similar to the Pythagorean triples which gives the nature of the solutions of (10) is stated next.

Theorem 2.1. *For every two positive integers l and k such that* $\gcd(k, l) = 1$ *and k is odd, a, c and d given by*

$$d = 2l^2 + k^2 \quad \textit{and}$$

$$\begin{cases} a = |2l^2 + 2kl - k^2|, \ c = |k^2 + 4kl - 2l^2|, & \textit{if } k \not\equiv l \pmod 3 \\ a = |2l^2 - 2kl - k^2|, \ c = |k^2 - 4kl - 2l^2|, & \textit{if } k \not\equiv -l \pmod 3 \end{cases} \tag{11}$$

constitute a positive primitive solution for (10).

Conversely, with the exception of the trivial solution $a = c = d = 1$, every positive primitive solution for (10) *appears in the way described above for some l and k.*

Proof. First, one can check that (11) satisfies (10) for every l and k. As a result it follows that a, c and d are positive integers. Let p be a prime dividing a, c and d. Then p must divide $\pm a - d = 2k(\pm l - k)$ and so p is equal to 2, p divides k or it divides $\pm l - k$. If $p = 2$, then p must divide k, but this contradicts the assumption that k is odd.

In case p is not equal to 2 and it divides k, we see p must divide $l^2 = (d - k^2)/2$. Since we assumed $\gcd(l, k) = 1$ it follows that p must divide $\pm l - k$. By our assumptions on k and l, p cannot be equal to 3. Then p divides $\pm a + (\pm l - k)^2 = 3l^2$. Because $p \ne 3$ then p must divide l^2 and so p should divide l and then k. This contradiction shows that a, c and d cannot have prime common factors. So, we have a primitive solution in (11).

For the converse, let us assume that a, c and d represent a positive primitive solution of (10) which is different from the trivial one. We denote by $u = \frac{a}{d}$ and $v = \frac{c}{d}$. Then the point with rational coordinates (u, v) (different of $(1, 1)$) is on the ellipse $\frac{x^2}{3/2} + \frac{y^2}{3} = 1$ (Figure 3) in the first quadrant. This ellipse contains the following four points with integer coordinates: $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, 1)$. This gives the lines $y + 1 = t_1(x + 1)$, $y + 1 = t_2(x - 1)$, $y - 1 = t_3(x + 1)$ and $y - 1 = t_4(x - 1)$, passing through (u, v) and one of the points mentioned above. Hence, the slopes t_1, t_2, t_3 and t_4 are rational numbers. This gives expressions for the point (u, v) in terms of t_i $(i = 1, \ldots, 4)$. Let us assume that $t_i = \frac{k_i}{l_i}$ with

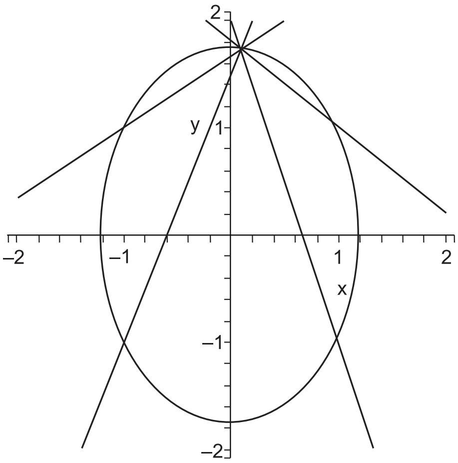

Figure 3. The ellipse $\dfrac{x^2}{3/2} + \dfrac{y^2}{3} = 1$.

$k_i, l_i \in \mathbb{Z}$, written in reduced form. Then we must have

$$u = \frac{|2 \pm 2t_i - t_i^2|}{2 + t_i^2} = \frac{|2l_i^2 \pm 2k_i l_i - k_i^2|}{2l_i^2 + k_i^2},$$
$$v = \frac{|t_i^2 \pm 4t_i - 2|}{2 + t_i^2} = \frac{|k_i^2 \pm 4k_i l_i - 2l_i^2|}{2l_i^2 + k_i^2},$$

and so, these equalities give

$$\frac{a}{d} = \frac{|2l_i^2 \pm 2k_i l_i - k_i^2|}{2l_i^2 + k_i^2}, \quad \text{and}$$
$$\frac{c}{d} = \frac{|k_i^2 \pm 4k_i l_i - 2l_i^2|}{2l_i^2 + k_i^2}, \quad i = 1, \ldots, 4. \tag{12}$$

We claim that the function $t_i \to 2l_i^2 + k_i^2$ $(i = 1, \ldots, 4)$ is injective. If for some $2l_i^2 + k_i^2 = 2l_j^2 + k_j^2$ $(i \neq j)$, that would imply that the corresponding numerators in (12) are equal. This gives enough information to conclude a contradiction. There are $\binom{4}{2} = 6$ possibilities here, but we are going to include the details only in the case $i = 1$ and $j = 2$. The rest of the cases can be done in a similar fashion. For this situation we have $2l_1^2 + 2k_1 l_1 - k_1^2 = k_2^2 + 2k_2 l_2 - 2l_2^2$ and $k_1^2 + 4k_1 l_1 - 2l_1^2 = k_2^2 - 4k_2 l_2 - 2l_2^2$. The first equality implies

$$2k_1 l_1 = k_1^2 + k_2^2 + 2k_2 l_2 - 2l_1^2 - 2l_2^2 = 2k_1^2 + 2k_2 l_2 - 4l_2^2$$

which substituted into the second equality gives

$$6k_1^2 = 2k_2^2 - 8k_2 l_2 + 8l_2^2 \Leftrightarrow 3k_1^2 = (k_2 - 2l_2)^2.$$

Because $\sqrt{3}$ is irrational, the last equality is impossible for k_1, k_2, l_2 integers and k_1 nonzero. For the other cases will get a contradiction based on the facts that $\sqrt{\frac{3}{2}}$ and $\sqrt{2}$ are irrational numbers.

A similar argument to the one in the first part of the proof shows that the fractions in the right-hand side of the equalities of (12) can be simplified only by a factor of 2, 3 or 6. Having four distinct possibilities in (12) for the denominators, exactly one of the fractions (simultaneously in the first and second equalities) must be in reduced form. This one will give the wanted representation. $\square$

Similar to Fermat's theorem about the representation of primes as a sum of two squares and the number of such representations one can show the next result.

Theorem 2.2 (Fermat [**4**])**.** *An odd prime p can be written as $2x^2 + y^2$ with $x, y \in \mathbb{Z}$ if and only if $p \equiv 1$ or $3 \pmod 8$. If $d = 2^k \prod p_i^{\alpha_i} \prod q_j^{\beta_j}$ is the prime factorization of d with q_j primes as before and p_i the rest of them, then the number of representations $d = 2x^2 + y^2$ with $x, y \in \mathbb{Z}$ is either zero if not all α_i are even and otherwise given by*

$$\left\lfloor \frac{1}{2} \prod (\beta_i + 1) \right\rfloor. \tag{13}$$

The number of positive primitive representations $d = 2x^2 + y^2$ for d odd, i.e. $x, y \in \mathbb{N}$ and $\gcd(x, y) = 1$, is equal to

$$\Gamma_2(d) = \begin{cases} 0 & \text{if } d \text{ is divisible by a prime factor} \\ & \text{of the form } 8s{+}5 \text{ or } 8s{+}7,\ s \geq 0, \\ & \\ 2^{k-1} & \text{where } k \text{ is the number of distinct prime factors} \\ & \text{of } d \text{ of the form } 8s+1, \text{ or } 8s+3\ (s \geq 0) \end{cases} \tag{14}$$

Putting the two results together (Theorem 2.2 and Theorem 1.2), we obtain the following proposition.

Proposition 2.3. *For every odd d, the number of representations of (4) which satisfy $0 < a \leq b \leq c$ and $\gcd(a, b, c) = 1$ is equal to*

$$\pi\epsilon(d) = \frac{\Lambda(d) + 24\Gamma_2(3d^2)}{48}. \tag{15}$$

Going back to the example $d = 2011$ we see that the contribution of $24\frac{\Gamma_2(3d^2)}{48}$ is exactly 1, since $2011 \equiv 3 \pmod 8$.

A regular tetrahedron whose vertices are integers is said to be *irreducible* if it cannot be obtained by an integer dilation and a translation from a smaller one also with integer coordinates. An important question at this point about irreducible tetrahedra is included next.

> *Does every irreducible tetrahedron with integer coordinates have a face with a normal vector (a, b, c) satisfying $a^2 + b^2 + c^2 = 3d^2$ such that d gives the side lengths ℓ of the tetrahedron by the formula $\ell = d\sqrt{2}$? In other words, is there a face for which $k = 1$ in Theorem 1.3?*

Unfortunately the last question is solved by the next counterexample. The following points together with the origin, [−6677, −2672, 1445], [−5940, 4143, −1167], [−3837, 2595, 5688] form a regular tetrahedron of side-lengths equal to $5187\sqrt{2}$ and the highest d for the faces is 1729. We observe that 3, 7, 13 and 19 are the first three distinct primes numbers of the form $u^2 + 3v^2$, $u, v \in \mathbb{Z}$.

3. The Code

The program is written in Maple code and it is based on Theorem 1.3. The main idea is to create a list of irreducible regular tetrahedra that can be used to generate all the others in $\{0, 1, 2, \ldots, n\}^3$ by certain transformations generating a partition for the set of all tetrahedra. Each such irreducible tetrahedron is constructed from the equation of one face using Theorem 1.3. For the interested reader we included the details in [**11**]. The result of the *calculation(100)* gives in less than a few hours of computation:

$[1, 1]$	$[2, 9]$	$[3, 36]$	$[4, 104]$	$[5, 257]$
$[6, 549]$	$[7, 1058]$	$[8, 1896]$	$[9, 3199]$	$[10, 5145]$
$[11, 7926]$	$[12, 11768]$	$[13, 16967]$	$[14, 23859]$	$[15, 32846]$
$[16, 44378]$	$[17, 58977]$	$[18, 77215]$	$[19, 99684]$	$[20, 126994]$
$[21, 159963]$	$[22, 199443]$	$[23, 246304]$	$[24, 301702]$	$[25, 366729]$
$[26, 442587]$	$[27, 530508]$	$[28, 631820]$	$[29, 748121]$	$[30, 880941]$
$[31, 1031930]$	$[32, 1202984]$	$[33, 1395927]$	$[34, 1612655]$	$[35, 1855676]$
$[36, 2127122]$	$[37, 2429577]$	$[38, 2765531]$	$[39, 3137480]$	$[40, 3548434]$
$[41, 4001071]$	$[42, 4498685]$	$[43, 5044606]$	$[44, 5641892]$	$[45, 6294195]$
$[46, 7005191]$	$[47, 7778912]$	$[48, 8620242]$	$[49, 9533105]$	$[50, 10521999]$
$[51, 11591474]$	$[52, 12746562]$	$[53, 13992107]$	$[54, 15332971]$	$[55, 16775590]$
$[56, 18324372]$	$[57, 19985523]$	$[58, 21765013]$	$[59, 23668266]$	$[60, 25702480]$
$[61, 27873699]$	$[62, 30188259]$	$[63, 32655348]$	$[64, 35281418]$	$[65, 38074085]$
$[66, 41040495]$	$[67, 44188592]$	$[68, 47525856]$	$[69, 51061295]$	$[70, 54804647]$
$[71, 58763604]$	$[72, 62949850]$	$[73, 67371219]$	$[74, 72037311]$	$[75, 76958126]$
$[76, 82143618]$	$[77, 87606245]$	$[78, 93355379]$	$[79, 99403446]$	$[80, 105762770]$
$[81, 112443331]$	$[82, 119456581]$	$[83, 126814970]$	$[84, 134532746]$	$[85, 142621185]$
$[86, 151093691]$	$[87, 159964136]$	$[88, 169245226]$	$[89, 178954039]$	$[90, 189102295]$
$[91, 199706864]$	$[92, 210781424]$	$[93, 222341631]$	$[94, 234402515]$	$[95, 246978962]$
$[96, 260093046]$	$[97, 273757925]$	$[98, 287989943]$	$[99, 302809940]$	$[100, 318235290]$

We observe a similar behavior with the sequence $\dfrac{\ln(ET(n))}{\ln(n+1)}$ in [**10**].

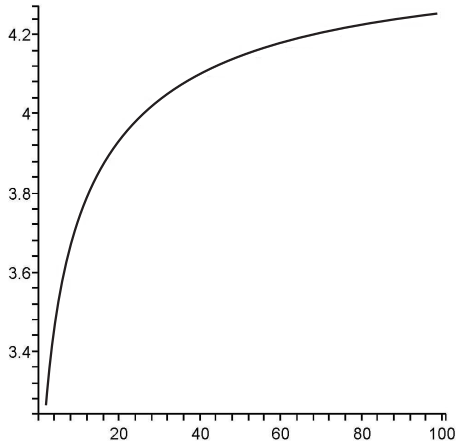

Figure 4. The graph $\dfrac{\ln(T(n)/2)}{\ln(n+1)}$, $1 \le n \le 100$.

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E. J. Ionascu, Department of Mathematics, Columbus State University, 4225 University Avenue, Columbus, GA 31907, U.S.A, *e-mail*: `ionascu@columbusstate.edu;`