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ANALYSIS OF A CLASS OF THERMAL FRICTIONAL CONTACT PROBLEM FOR THE NORTON-HOFF FLUID

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Abstract. We consider a mathematical model which describes the static flow of a Norton-Hoff fluid whose viscosity depends on the temperature, and with mixed boundary conditions, including friction. The latter is modelled by a general velocity dependent dissipation functional and the temperature. We derive a weak formulation of the coupled system of the equation of motion and the energy equation, consisting of a variational inequality for the velocity field. We prove the existence of a weak solution of the model using compactness, monotonicity, L^1-Data theory and a fixed point argument. In the asymptotic limit case of a high thermal conductivity, the temperature becomes a constant solving an implicit total energy equation involving the viscosity function and the subdifferential friction. Finally, we describe a number of concrete thermal friction conditions.

1. Introduction

The model of Norton-Hoff fluid has been used in various publications in order to model the flow of metals and viscoplastic solids. The literature concerning this topic is extensive, see, e.g., [**2**], [**27**] and references therein. An intrinsic inclusion leads in a natural way to variational equations which justify the study of problems involving the incompressible viscoplastic Norton-Hoff fluid using arguments of the variational analysis.

In this paper we consider a mathematical model which describes the static flow of Norton-Hoff fluid whose viscosity depends on the temperature. Such problem can describe the flow of metals in a die as well as the transfer heat in the non-Newtonian Norton-Hoff fluid. The flow is governed by the coupled system of motion equation and energy conservation equation. Moreover, we assume that the contact is modeled with a subdifferential boundary condition depending on the temperature in the form

$$\varphi\left(\theta, \mathbf{v}\right) - \varphi\left(\theta, \mathbf{u}\right) \geq -\sigma\nu \cdot \left(\mathbf{v} - \mathbf{u}\right), \tag{1.1}$$

where $\mathbf{u}$ represents the velocity field, θ the temperature, ν the unit outward normal vector, $\sigma\nu$ the Cauchy stress vector and φ is a given convex function. The inequality in (1.1) holds almost everywhere on the contact zone. Examples and detailed

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explanations of inequality problems in contact mechanics which lead to boundary conditions in the form (1.1), without taking into account thermal effects, can be found in [**3**], [**18**], [**12**], [**24**], [**25**], [**26**] and references therein. The new feature in the model is due to the choice of particular forms of the function φ, which can be written as the sum of two contact functions, corresponding to the normal and tangential components of the Cauchy stress vector. Furthermore, to describe the energy dissipation due to the contact, we use a Fourier type boundary condition depending on the contact function φ.

The differential coupled system containing the dissipative function leads us to prove the existence of a solution of an elliptic equation with L^1-Data. Our main idea in this context is to adapt the Kakutani-Glicksberg fixed point theorem and use the L^1-Data theory.

The paper is organized as follows. In Section 2 we present the mechanical problem of the thermal flow of a Norton-Hoff fluid and introduce some notations and preliminaries. In Section 3 we derive the variational formulation of the problem. In Section 4 we prove the existence of solutions as well as an existence result of the nonlocal Norton-Hoff problem, which can be obtained as an asymptotic limit case of a very large thermal conductivity. In Section 5 we describe a number of concrete thermal frictional conditions which may be cast in the abstract form (1.1) and to which our main results apply.

2. Problem Statement

We consider a mathematical problem modelling the static flow of a Norton-Hoff fluid in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ with the boundary Γ of class C^1, partitioned into three disjoint measurable parts Γ_0, Γ_1 and Γ_2 such that meas $(\Gamma_0) > 0$. The fluid is supposed to be incompressible and the viscosity depends on the temperature. The fluid is acted upon by given volume forces of density $\mathbf{f}$ and by given surface tractions of density $\mathbf{g}$. In addition, we admit a possible external heat source proportional to the temperature. On Γ_0 we suppose that the velocity is known. The temperature is given by a Neumann boundary condition on $\Gamma_0 \cup \Gamma_1$. We impose on Γ_2 a frictional contact described by a subdifferential type boundary condition which also depends on the temperature as well as a Fourier boundary condition.

We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$, respectively, by

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i \quad \forall\, \mathbf{u},\ \ \mathbf{v} \in \mathbb{R}^n \qquad \text{and} \quad \sigma \cdot \tau = \sigma_{ij}\tau_{ij} \quad \forall\, \sigma, \tau \in \mathbb{S}_n.$$

$$|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{\frac{1}{2}} \quad \forall \mathbf{u} \in \mathbb{R}^n \qquad \text{and} \quad |\sigma| = (\sigma \cdot \sigma)^{\frac{1}{2}} \quad \forall\, \sigma \in \mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used.

Let $1 < p < 2$. We consider the rate of deformation operator by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))_{1 \le i,j \le n}, \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

defined for all $\mathbf{u} \in W^{1,p}(\Omega)^n$. We denote by ν the unit outward normal vector on the boundary Γ. For each vector field $\mathbf{v} \in W^{1,p}(\Omega)^n$, we also write $\mathbf{v}$ for its trace on Γ. The normal and the tangential components of $\mathbf{v}$ on the boundary are given by

$$v_\nu = \mathbf{v} \cdot \nu, \qquad \mathbf{v}_\tau = \mathbf{v} - v_\nu \nu.$$

Similarly, for a regular tensor field σ, we denote by σ_ν and σ_τ the normal and tangential components of σ on the boundary given by

$$\sigma_\nu = \sigma\nu \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

We consider the following mechanical coupled problem.

Problem 1. Find a velocity field $\mathbf{u} = (u_i)_{1\le i\le n} : \Omega \longrightarrow \mathbb{R}^n$, stress field $\sigma = (\sigma_{ij})_{1\le i,j\le n} : \Omega \longrightarrow S_n$ and a temperature $\theta : \Omega \longrightarrow \mathbb{R}$ such that

$$\operatorname{Div}(\sigma) + \mathbf{f} = 0 \quad \text{in } \Omega \tag{2.1}$$

$$\sigma = \mu(\theta)\, |\varepsilon(\mathbf{u})|^{p-2}\, \varepsilon(\mathbf{u}) + P\delta \quad \text{in } \Omega \tag{2.2}$$

$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega \tag{2.3}$$

$$-k\Delta\theta = \sigma \cdot \varepsilon(\mathbf{u}) - \alpha\theta \quad \text{in } \Omega \tag{2.4}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_0 \tag{2.5}$$

$$\sigma\nu = \mathbf{g} \quad \text{on } \Gamma_1 \tag{2.6}$$

$$\varphi(\theta, \mathbf{v}) - \varphi(\theta, \mathbf{u}) \ge -\sigma\nu \cdot (\mathbf{v} - \mathbf{u}) \quad \text{on } \Gamma_2 \tag{2.7}$$

$$\frac{\partial\theta}{\partial\nu} = 0 \quad \text{on } \Gamma_0 \cup \Gamma_1 \tag{2.8}$$

$$k\frac{\partial\theta}{\partial\nu} + \beta\theta = -\sigma\nu \cdot \mathbf{u} \quad \text{on } \Gamma_2 \tag{2.9}$$

where $\operatorname{Div}(\sigma) = (\sigma_{ij,j})$ and $\operatorname{div}(\mathbf{u}) = u_{i,i}$. The flow is given by the equation (2.1) where the density is assumed equal to one. Equation (2.2) represents the constitutive law of a Norton-Hoff fluid whose viscosity μ depends on the temperature, P represents the hydrostatic pressure, $1 < p < 2$ is the sensibility coefficient of the material to the rate of the deformation tensor and δ is the identity tensor. (2.3) represents the incompressibility condition. Equation (2.4) represents the energy conservation where the specific heat is assumed to be equal to one, $k > 0$ is the thermal conductivity and the term $\alpha\theta$ represents the external heat source with $\alpha > 0$. (2.5) gives the velocity on Γ_0 and (2.6) is the surface traction on Γ_1. Condition (2.7) represents a subdifferential boundary condition on Γ_2 and $\varphi : \mathbb{R} \times \Gamma_2 \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a measurable convex function. (2.8) is a homogeneous Neumann boundary condition on $\Gamma_0 \cup \Gamma_1$. Finally, (2.9) represents a Fourier boundary condition on Γ_2, where $\beta \ge 0$ represents the Robin coefficient.

Remark 2.1.

1. The viscosity function can be given by the Arrhenius law

$$\mu(\theta) = \mu_c \exp\left(\frac{1}{\theta + \theta_0}\right),$$

where $\mu_c \in L^\infty(\Omega)$ represents the consistency of the material and θ_0 is a positive function.

2. The linear external heat source of the form $\alpha\theta$ has been introduced to guarantee the thermodynamical consistence of the irreversible processes. Moreover, the use of such external heat source in fluid mechanics permits to obtain non-local existence results.

We denote by V the set

$$V = \left\{\mathbf{v} \in W^{1,p}(\Omega)^n : \operatorname{div}(\mathbf{v}) = 0 \ \text{ in } \Omega \ \text{ and } \ \mathbf{v} = 0 \text{ on } \Gamma_0\right\}.$$

V is a Banach space equipped with the following norm

$$\|\mathbf{v}\|_V = \|\mathbf{v}\|_{W^{1,p}(\Omega)^n}.$$

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem.

Denote the conjugate of p, q, respectively, by p', q' where $1 < q < \frac{n}{n-1}$. We introduce the following functionals

$\phi : W^{1,1}(\Omega) \times V \longrightarrow \mathbb{R} \cup \{+\infty\}$,

$$\phi(\theta, \mathbf{v}) = \begin{cases} \int_{\Gamma_2} \varphi(\theta, \mathbf{v}) \mathrm{d}\gamma & \text{if } \ \varphi(\theta, \mathbf{v}) \in L^1(\Gamma_2), \\ +\infty & \text{otherwise,} \end{cases}$$

$\tilde{f} : V \longrightarrow \mathbb{R}$,

$$\tilde{f}(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \mathrm{d}x + \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{v} \mathrm{d}\gamma$$

where $\mathrm{d}\gamma$ represents the surface element. We assume

$$\forall x \in \Omega, \quad \mu(.,x) \in C^0(\mathbb{R}) : \exists \mu_1, \mu_2 > 0, \quad \mu_1 \le \mu(y,x) \le \mu_2 \quad \forall y \in \mathbb{R}, \ \forall x \in \Omega. \tag{2.10}$$

The function φ is the sum of two contact functions, corresponding to the normal and tangential components of the stress tensor on the boundary, respectively.

$$\varphi(\theta, \mathbf{v}) = \upsilon_1(\theta)\psi_1(\mathbf{v}) + \upsilon_2(\theta)\psi_2(\mathbf{v}),$$

where

$$\forall x \in \Gamma_2, \quad \upsilon_i(.,x) \in C^0(\mathbb{R}) : \exists \upsilon_0 > 0, \quad 0 \le \upsilon_i(y,x) \le \upsilon_0 \quad \forall y \in \mathbb{R}, \ \forall x \in \Gamma_2, \ i = 1,2, \tag{2.11}$$

and the function $\psi_i : \Gamma_2 \times \mathbb{R}^n \longrightarrow \mathbb{R}$ $(i = 1,2)$ is measurable, positive, convex on Γ_2 and continuous on V and verifying the following hypothesis

$$\exists \omega_i \in [1,2], \ \exists C > 0 : \ \|\psi_i(\mathbf{v})\|_{L^2(\Gamma_2)} \le C \|\mathbf{v}\|^{\omega_i}_{L^{2\omega_i}(\Gamma_2)^n} \quad \forall \mathbf{v} \in V, \quad i = 1,2. \tag{2.12}$$

We can easily prove that the Fourier boundary condition holds, using the sub-differential condition (2.7)

$$k\frac{\partial\theta}{\partial\nu}+\beta\theta\geq\varphi(\theta,\mathbf{u})\qquad\text{on }\Gamma_2. \tag{2.13}$$

Since the term $\sigma\nu\cdot\mathbf{u}$ in the Fourier boundary condition (2.9) represents the energy dissipated due to the contact, we can suppose in this paper that the condition in the abstract (2.13) can be written as sum of two dissipative contributions (see [**7**])

$$k\frac{\partial\theta}{\partial\nu}+\beta\theta=\omega_1\upsilon_1(\theta)\psi_1(\mathbf{u})+\omega_2\upsilon_2(\theta)\psi_2(\mathbf{u})\qquad\text{on }\Gamma_2, \tag{2.14}$$

here, ω_1, ω_2 may be interpreted as the powers of the contact functions $\upsilon_1(\theta)\psi_1(\mathbf{u})$, $\upsilon_2(\theta)\psi_2(\mathbf{u})$, respectively, (see examples in Section 5). Then, in the mechanical Problem 1 we can replace the condition (2.9) by the Fourier condition (2.14).

3. Variational Formulation

The aim of this section is to derive a variational formulation to the Problem 1. To do so we need the following lemma.

Lemma 3.1. *Assume that* $\mathbf{f}\in V'$ *and* $\mathbf{g}\in W^{-\frac{1}{p},p'}(\Gamma)^n$. *If* $\{\mathbf{u},\sigma,\theta\}$ *are regular functions satisfying* (2.1)–(2.9)*, then*

$$\begin{aligned}&\int_\Omega(\mu(\theta)\left|\varepsilon(\mathbf{u})\right|^{p-2}\varepsilon(\mathbf{u}))\cdot(\varepsilon(\mathbf{v})-\varepsilon(\mathbf{u}))\mathrm{d}x+\phi(\theta,\mathbf{v})-\phi(\theta,\mathbf{u})\\&\geq\tilde{f}(\mathbf{v}-\mathbf{u})\qquad\qquad\forall\mathbf{v}\in V,\end{aligned} \tag{3.1}$$

$$\begin{aligned}&k\int_\Omega\nabla\theta\cdot\nabla\tau\mathrm{d}x+\alpha\int_\Omega\theta\tau\mathrm{d}x+\beta\int_{\Gamma_2}\theta\tau\mathrm{d}\gamma\\&=\int_\Omega F(\theta,\mathbf{u})\tau\mathrm{d}x+\int_{\Gamma_2}(\omega_1\upsilon_1(\theta)\psi_1(\mathbf{u})+\omega_2\upsilon_2(\theta)\psi_2(\mathbf{u}))\tau\mathrm{d}\gamma\\&\qquad\qquad\forall\tau\in W^{1,q'}(\Omega),\end{aligned} \tag{3.2}$$

where

$$F(\theta,\mathbf{u})=\mu(\theta)\left|\varepsilon(\mathbf{u})\right|^p. \tag{3.3}$$

Proof. Let us start by the proof of variational inequality (3.1). Let $\{\mathbf{u},\sigma,\theta\}$ be regular functions satisfying (2.1)–(2.9) and let $\mathbf{v}\in V$.

Using Green's formula and (2.1), (2.2), (2.3), (2.5) and (2.6), we obtain

$$\begin{aligned}&\int_\Omega(\mu(\theta)\left|\varepsilon(\mathbf{u})\right|^{p-2}\varepsilon(\mathbf{u}))\cdot(\varepsilon(\mathbf{v})-\varepsilon(\mathbf{u}))\mathrm{d}x\\&=\int_\Omega\mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\mathrm{d}x+\int_{\Gamma_1}\mathbf{g}\cdot(\mathbf{v}-\mathbf{u})\mathrm{d}\gamma+\int_{\Gamma_2}\sigma\nu\cdot(\mathbf{v}-\mathbf{u})\mathrm{d}\gamma.\end{aligned}$$

On the other hand, by (2.7)

$$\int_{\Gamma_2} \sigma\nu \cdot (\mathbf{v} - \mathbf{u}) \mathrm{d}\gamma \geq \int_{\Gamma_2} \varphi(\theta, \mathbf{u}) \mathrm{d}\gamma - \int_{\Gamma_2} \varphi(\theta, \mathbf{v}) \mathrm{d}\gamma.$$

Then (3.1) holds. Now, to prove the variational equation (3.2) we proceed as follows. Applying Green's formula and (2.4), (2.8) and (2.14), after a simple calculation we get

$$\begin{aligned} & k \int_{\Omega} \nabla\theta \cdot \nabla\tau \mathrm{d}x + \alpha \int_{\Omega} \theta\tau \mathrm{d}x + \beta \int_{\Gamma_2} \theta\tau \mathrm{d}\gamma \\ & = (\sigma \cdot \varepsilon(\mathbf{u}), \tau) + \int_{\Gamma_2} (\omega_1 \upsilon_1(\theta)\psi_1(\mathbf{u}) + \omega_2 \upsilon_2(\theta)\psi_2(\mathbf{u}))\tau \mathrm{d}\gamma \\ & \qquad \forall \tau \in W^{1,q'}(\Omega). \end{aligned} \tag{3.4}$$

By definition of σ, using (2.2) and (2.3), we can infer

$$(\sigma \cdot \varepsilon(\mathbf{u}), \tau) = \int_{\Omega} \mu(\theta) \left|\varepsilon(\mathbf{u})\right|^p \tau \mathrm{d}x.$$

According to (3.3), we eventually obtain (3.2). □

Remark 3.2. In (3.2), the first and second terms on the right-hand side make sense since $\tau \in W^{1,q'}(\Omega) \hookrightarrow C^0(\Omega)$ for $q' > n$, that is, $q < \dfrac{n}{n-1}$.

Lemma 3.1 leads us to consider the following variational system.

Problem 2. For prescribed data $\mathbf{f} \in V'$ and $\mathbf{g} \in W^{-\frac{1}{p},p'}(\Gamma)^n$. Find $\mathbf{u} \in V$ and $\theta \in W^{1,q}(\Omega)$, satisfying the variational system

$$\begin{aligned} & \int_{\Omega} (\mu(\theta) \left|\varepsilon(\mathbf{u})\right|^{p-2} \varepsilon(\mathbf{u})) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u})) \mathrm{d}x + \phi(\theta, \mathbf{v}) - \phi(\theta, \mathbf{u}) \\ & \geq \tilde{f}(\mathbf{v} - \mathbf{u}) \qquad \forall \mathbf{v} \in V, \end{aligned} \tag{3.5}$$

$$\begin{aligned} & k \int_{\Omega} \nabla\theta \cdot \nabla\tau \mathrm{d}x + \alpha \int_{\Omega} \theta\tau \mathrm{d}x + \beta \int_{\Gamma_2} \theta\tau \mathrm{d}\gamma \\ & = \int_{\Omega} F(\theta, \mathbf{u})\tau \mathrm{d}x + \int_{\Gamma_2} (\omega_1 \upsilon_1(\theta)\psi_1(\mathbf{u}) + \omega_2 \upsilon_2(\theta)\psi_2(\mathbf{u}))\tau \mathrm{d}\gamma \\ & \qquad \forall \tau \in W^{1,q'}(\Omega). \end{aligned} \tag{3.6}$$

Now, we consider the weak nonlocal formulation to the mechanical problem (2.1)–(2.3) and (2.5)–(2.7) corresponding formally to the limit model $k = \infty$ (modelling the flow of a Norton-Hoff fluid with temperature dependent nonlocal viscosity and subdifferential friction).

Problem 3. For prescribed data $\mathbf{f} \in V'$ and $\mathbf{g} \in W^{-\frac{1}{p},p'}(\Gamma)^n$, find $\mathbf{u} \in V$ and $\Theta \in \mathbb{R}_+$ satisfying the variational inequality

$$\begin{aligned} &\mu(\Theta) \int_\Omega |\varepsilon(\mathbf{u})|^{p-2} \varepsilon(\mathbf{u}) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}))\mathrm{d}x \\ &+ \upsilon_1(\Theta) \int_{\Gamma_2} (\psi_1(\mathbf{v}) - \psi_1(\mathbf{u}))\mathrm{d}\gamma + \upsilon_2(\Theta) \int_{\Gamma_2} (\psi_2(\mathbf{v}) - \psi_2(\mathbf{u}))\mathrm{d}\gamma \\ &\geq \tilde{f}(\mathbf{v} - \mathbf{u}) \qquad \forall \mathbf{v} \in V, \end{aligned} \tag{3.7}$$

where Θ is a solution to the implicit scalar equation

$$\begin{aligned} &(\alpha meas(\Omega) + \beta meas(\Gamma_2))\Theta \\ &= \mu(\Theta) \int_\Omega |\varepsilon(\mathbf{u})|^p \,\mathrm{d}x + \omega_1 \upsilon_1(\Theta) \int_{\Gamma_2} \psi_1(\mathbf{u})\mathrm{d}\gamma + \omega_2 \upsilon_2(\Theta) \int_{\Gamma_2} \psi_2(\mathbf{u})\mathrm{d}\gamma. \end{aligned} \tag{3.8}$$

4. Existence Results

In this section we establish two existence theorems to the Problems 2 and 3.

Theorem 4.1. *The Problem 2 has a solution* $(\mathbf{u}, \theta)$ *satisfying*

$$\mathbf{u} \in V, \tag{4.1}$$

$$\theta \in W^{1,q}(\Omega). \tag{4.2}$$

Theorem 4.2. *There exists* $(\mathbf{u}, \Theta) \in V \times \mathbb{R}_+$, *a solution to the nonlocal Problem 3, which can be obtained as a limit of solutions* $(\mathbf{u}_k, \theta_k)$ *of Problem 2 in* $V \times W^{1,q}(\Omega)$ *as* $k \longrightarrow \infty$.

The proof of Theorem 4.1 is based on the application of the Kakutani-Glicksberg fixed point theorem for multivalued mappings using two auxiliary existence results. The first one results from the classical theory for inequalities with a monotone operator. The second one results from the elliptic equations theory and L^1-Data theory. Finally, compactness arguments are used to conclude the proofs. For reader's convenience, let us recall the fixed point theorem (see [**14**]).

Theorem 4.3 (Kakutani-Glicksberg)**.** *Let* X *be a locally convex Hausdorff topological vector space and* K *be a nonempty, convex, and compact subset of* X. *If* $\mathcal{L} : K \longrightarrow P(K)$ *is an upper semicontinuous mapping and* $\mathcal{L}(z) \neq \emptyset$ *is a convex and closed subset in* K *for every* $z \in K$, *then there exists at least one fixed point,* $z \in \mathcal{L}(z)$.

The first auxiliary existence result is given by the next proposition.

Proposition 4.4. *For every* $\lambda \in W^{1,1}(\Omega)$, *there exists a unique solution* $\mathbf{u} = \mathbf{u}(\lambda) \in V$ *to the problem*

$$\begin{aligned} &\int_\Omega (\mu(\lambda) |\varepsilon(\mathbf{u})|^{p-2} \varepsilon(\mathbf{u})) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}))\mathrm{d}x + \phi(\lambda, \mathbf{v}) - \phi(\lambda, \mathbf{u}) \\ &\geq \tilde{f}(\mathbf{v} - \mathbf{u}) \qquad \forall \mathbf{v} \in V, \end{aligned} \tag{4.3}$$

and it satisfies the estimate

$$\|\mathbf{u}\|_V \le c\left(\frac{\|\mathbf{f}\|_{V'}+\|\mathbf{g}\|_{W^{-\frac{1}{p},p'}(\Gamma)^n}}{\mu_1}\right)^{\frac{1}{p-1}}. \tag{4.4}$$

Proof of Proposition 4.4. Introducing the following functional

$$J\colon L^p(\Omega)_s^{n\times n}\subset S_n \longrightarrow \mathbb{R},\qquad \sigma\longmapsto J(\sigma)=\int_\Omega \frac{\mu}{p}\,|\sigma|^p\,\mathrm{d}x.$$

The functional is convex, lower semi-continuous on $L^p(\mathbf{\Omega})_s^{n\times n}$ and Gâteaux differentiable. Its Gâteaux derivate at any point σ is

$$(DJ(\sigma),\eta)_{L^{p'}(\Omega)_s^{n\times n}\times L^p(\Omega)_s^{n\times n}}=\int_\Omega \mu\,|\sigma|^{p-2}\,\sigma\cdot\eta\mathrm{d}x\qquad \forall\eta\in L^p(\Omega)_s^{n\times n}.$$

Consequently, DJ is hemi-continuous and monotone. Moreover, DJ is strictly monotone and bounded, and we have

$$\begin{aligned}&(DJ(\sigma)-DJ(\eta),\sigma-\eta)_{L^{p'}(\Omega)_s^{n\times n}\times L^p(\Omega)_s^{n\times n}}\\ \ge&\int_\Omega \mu(|\sigma|-|\eta|)(|\sigma|^{p-1}-|\eta|^{p-1})\mathrm{d}x.\end{aligned}$$

Then if $\sigma\neq\eta$, we find $(DJ(\sigma)-DJ(\eta),\sigma-\eta)_{L^{p'}(\Omega)_s^{n\times n}\times L^p(\Omega)_s^{n\times n}}>0$. It means that DJ is strictly monotone. On the other hand, for every $\sigma\in L^p(\Omega)_s^{n\times n}$,

$$\int_\Omega \left|\mu\,|\sigma|^{p-2}\,\sigma\right|^{p'}\mathrm{d}x\le \mu_2^{p'}\int_\Omega |\sigma|^p\,\mathrm{d}x,$$

which proves that DJ is bounded on $L^{p'}(\Omega)_s^{n\times n}$.

Now, we consider the following differential operator

$$\left\{\begin{array}{l} A\colon V\longrightarrow V',\qquad \mathbf{u}\longmapsto A\mathbf{u}\\ (A\mathbf{u},\mathbf{v})_{V'\times V}=(DJ(\varepsilon(\mathbf{u})),\varepsilon(\mathbf{v}))_{L^{p'}(\Omega)_s^{n\times n}\times L^p(\Omega)_s^{n\times n}}\qquad \forall\mathbf{v}\in V.\end{array}\right. \tag{4.5}$$

We deduce that A is hemi-continuous, strictly monotone and bounded on V.

Therefore, for every $\mathbf{u}\in V$, we have

$$\frac{(A\mathbf{u},\mathbf{u})_{V'\times V}}{\|\mathbf{u}\|_V}\ge \mu_1\frac{\int_\Omega|\varepsilon(\mathbf{u})|^p\,\mathrm{d}x}{\|\mathbf{u}\|_V}.$$

Applying the generalized Korn inequality, we find

$$\frac{(A\mathbf{u},\mathbf{u})_{V'\times V}}{\|\mathbf{u}\|_V}\ge \mu_1\,\|\mathbf{u}\|_V^{p-1}.$$

It follows that the operator A is coercive on V.

Furthermore, the function ψ_i $(i=1,2)$ is measurable, positive, convex and continuous on V. It follows that ϕ is positive, proper, convex and lower semi-continuous on V. Consequently, the existence and uniqueness of the solution result from the classical theorems (see [**9**]) on variational inequalities with monotone operators and convex functionals.

To prove the estimate (4.4) we proceed as follows, by choosing $\mathbf{v} = 0$ as test function in (4.3) and using (2.12), we obtain

$$\int_\Omega \mu(\lambda) |\varepsilon(\mathbf{u})|^p \,\mathrm{d}x \le \|\mathbf{f}\|_{V'} \|\mathbf{u}\|_V + \|\mathbf{g}\|_{W^{-\frac{1}{p},p'}(\mathbf{\Gamma})^n} \|\mathbf{u}\|_{W^{1-\frac{1}{p},p}(\Gamma)^n} .$$

Hence, Korn's inequality combined with the Sobolev trace inequality allow us to conclude that (4.4) holds true. □

The second auxiliary existence result is given by the next proposition.

Proposition 4.5. *Let* $\mathbf{u} = \mathbf{u}(\lambda)$ *be the solution of problem* (4.3) *given by Proposition 4.4. Then there exists* $\theta = \theta(\mathbf{u}, \lambda) \in W^{1,q}(\Omega)$, $1 < q < \dfrac{n}{n-1}$, *a solution to the problem*

$$\begin{aligned} & k \int_\Omega \nabla\theta \cdot \nabla\tau \mathrm{d}x + \alpha \int_\Omega \theta\tau \mathrm{d}x + \beta \int_{\Gamma_2} \theta\tau \mathrm{d}\gamma \\ & = \int_\Omega \mu(\lambda) |\varepsilon(\mathbf{u})|^p \tau \mathrm{d}x + \int_{\Gamma_2} (\omega_1 \upsilon_1(\lambda)\psi_1(\mathbf{u}) + \omega_2\upsilon_2(\lambda)\psi_2(\mathbf{u}))\tau \mathrm{d}\gamma \\ & \qquad \forall \tau \in W^{1,q'}(\Omega), \end{aligned} \tag{4.6}$$

and it satisfies the estimate

$$\begin{aligned} & \|\theta\|_{L^q(\Omega)} + \beta \|\theta\|_{L^q(\Gamma)} + \sqrt{k} \|\nabla\theta\|_{L^q(\Omega)^n} \\ & \quad \le \mathcal{F}(\upsilon_0, \mu_1, c, \|\mathbf{f}\|_{V'}, \|\mathbf{g}\|_{W^{-\frac{1}{p},p'}(\Gamma)^n}), \end{aligned} \tag{4.7}$$

where $\mathcal{F}$ *is a positive function.*

Proof of Proposition 4.5. Technically, it is difficult to obtain a solution of such a problem. To end this we introduce the following approximate problem

$$\begin{aligned} & k \int_\Omega \nabla\theta_m \cdot \nabla\tau \mathrm{d}x + \alpha \int_\Omega \theta_m\tau \mathrm{d}x + \beta \int_{\Gamma_2} \theta_m\tau \mathrm{d}\gamma \\ & \qquad = \int_\Omega F_m \tau \mathrm{d}x + \int_{\Gamma_2} (\omega_1 \upsilon_1(\lambda)\psi_1(\mathbf{u}) + \omega_2\upsilon_2(\lambda)\psi_2(\mathbf{u}))\tau \mathrm{d}\gamma \\ & \qquad\qquad \forall \tau \in H^1(\Omega), \end{aligned} \tag{4.8}$$

where $m \in \mathbb{N}$ and

$$F_m = \frac{m\mu(\lambda) |\varepsilon(\mathbf{u})|^p}{m + \mu(\lambda) |\varepsilon(\mathbf{u})|^p} \in L^\infty(\Omega). \tag{4.9}$$

By the Hölder inequality, it hods

$$\begin{aligned} & \left| \int_{\Gamma_2} (\omega_1 \upsilon_1(\lambda)\psi_1(\mathbf{u}) + \omega_2\upsilon_2(\lambda)\psi_2(\mathbf{u}))\tau \mathrm{d}\gamma \right| \\ & \le \upsilon_0 (\|\psi_1(\mathbf{u})\|_{L^2(\Gamma_2)} + \|\psi_2(\mathbf{u})\|_{L^2(\Gamma_2)}) \|\tau\|_{L^2(\Gamma_2)} . \end{aligned}$$

By virtue of (2.12), using the Sobolev trace inequality and the estimate (4.4), we find

$$\left|\int_{\Gamma_2} (\omega_1\upsilon_1(\lambda)\psi_1(\mathbf{u}) + \omega_2\upsilon_2(\lambda)\psi_2(\mathbf{u}))\tau \mathrm{d}\gamma\right| \leq c\,\|\tau\|_{H^1(\Omega)}\,.$$

Consequently, from the Lax-Milgram theorem, there exists a unique solution $\theta_m \in H^1(\Omega)$ to the problem (4.8).

Now, we test the approximate equation (4.8) with the function

$$\tau = \operatorname{sign}(\theta_m)\left[1 - \frac{1}{(1+|\theta_m|)^\xi}\right] \in H^1(\Omega)\cap L^\infty(\Omega), \qquad \xi > 0. \tag{4.10}$$

By using integration by parts (see for instance [**10**]), we find

$$\xi k \int_\Omega \frac{|\nabla\theta_m|^2}{(1+|\theta_m|)^{\xi+1}}\mathrm{d}x + \beta C(\xi)\int_{\Gamma_2} |\theta_m|\,\mathrm{d}\gamma \leq M, \tag{4.11}$$

where $M = M\left(\upsilon_0, \mu_1, c, \|\mathbf{f}\|_{V'}\,, \|\mathbf{g}\|_{W^{-\frac{1}{p},p'}(\Omega)^n}\right)$ is a positive function.

Particularly

$$\int_\Omega \frac{|\nabla\theta_m|^2}{(1+|\theta_m|)^{\xi+1}}\mathrm{d}x \leq \frac{M}{\xi k}. \tag{4.12}$$

The function γ is denoted

$$\gamma(r) = \int_0^r \frac{dt}{(1+|t|)^{\frac{\xi+1}{2}}}.$$

Then

$$\nabla\gamma(\theta_m) = \frac{\nabla\theta_m}{(1+|\theta_m|)^{\frac{\xi+1}{2}}}.$$

We deduce from (4.12) that $\nabla\gamma(\theta_m)$ is bounded in $L^2(\Omega)$, hence $\gamma(\theta_m)$ is bounded in $H^1(\Omega)$. Sobolev's imbedding asserts that $H^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$.

Keeping in mind that $\gamma(r) \sim r^{\frac{1-\xi}{2}}$ as $r \longrightarrow +\infty$. Then $|\theta_m|^{\frac{1-\xi}{2}}$ is bounded in $L^{\frac{2n}{n-2}}(\Omega)$. Consequently

$$|\theta_m|^{\frac{n(1-\xi)}{n-2}} \quad \text{is bounded in} \;\; L^1(\Omega). \tag{4.13}$$

Moreover, by Hölder's inequality, we get

$$\int_\Omega |\nabla\theta_m|^q\,\mathrm{d}x \leq \left(\int_\Omega \frac{|\nabla\theta_m|^2}{(1+|\theta_m|)^{\xi+1}}\mathrm{d}x\right)^{\frac{q}{2}} \left(\int_\Omega (1+|\theta_m|)^{\frac{(\xi+1)q}{2-q}}\mathrm{d}x\right)^{\frac{2-q}{2}}.$$

Hence, from (4.12), we find

$$\int_\Omega |\nabla\theta_m|^q\,\mathrm{d}x \leq \left(\frac{M}{k\xi}\right)^{\frac{q}{2}} \left(\int_\Omega (1+|\theta_m|)^{\frac{(\xi+1)q}{2-q}}\mathrm{d}x\right)^{\frac{2-q}{2}}. \tag{4.14}$$

Let us choose the couple (ξ, q) such that $\dfrac{n(1-\xi)}{n-2} = \dfrac{(\xi+1)q}{2-q}$. It means that $q = \dfrac{n(1-\xi)}{n-\xi-1}$. Then if $0 < \xi < \dfrac{1}{n-1}$, we find the condition $1 < q < \dfrac{n}{n-1}$.

Consequently, by using (4.13) and (4.14), the following estimate holds

$$\theta_m \text{ is bounded in } W^{1,q}(\Omega). \tag{4.15}$$

Combining with the estimate (4.11), we can extract a subsequence $(\theta_m)_m$ satisfying

$$\theta_m \longrightarrow \theta \text{ in } W^{1,q}(\Omega) \text{ weakly,} \tag{4.16}$$

$$\theta_m \longrightarrow \theta \text{ in } L^2(\Gamma) \text{ weakly.} \tag{4.17}$$

Rellich-Kondrachof's theorem affirms the compactness of the imbedding $W^{1,q}(\Omega) \longrightarrow L^1(\Omega)$. It then follows that we can extract a subsequence of θ_m, still denoted by θ_m, such that

$$\theta_m \longrightarrow \theta \text{ in } L^1(\Omega) \text{ strongly,} \tag{4.18}$$

We recall that by the Sobolev theorem the trace of θ_m belongs to $W^{1-\frac{1}{p},q}(\Gamma)$. Via the compactness of the imbedding $W^{1-\frac{1}{p},q}(\Gamma) \longrightarrow L^1(\Gamma)$ (see [**1**]), after a new extraction, still denoted by θ_m we can obtain,

$$\theta_m \longrightarrow \theta \text{ in } L^1(\Gamma) \text{ strongly.} \tag{4.19}$$

We conclude that the problem (4.6) admits a solution $\theta = \theta(\mathbf{u}, \lambda) \in W^{1,q}(\Omega)$.

Using (4.11), (4.13) and (4.14), the estimate (4.7) follows immediately. □

Proof of Theorem 4.1. In order to apply the Kakutani-Glicksberg fixed point theorem, let us consider the closed convex ball

$$K = \left\{ \lambda \in W^{1,q}(\Omega) : \|\lambda\|_{W^{1,q}(\Omega)} \le R_1 \right\}, \tag{4.20}$$

where $R_1 \ge \left(\dfrac{M}{k\xi}\right)^{\frac{q}{2}}$. The ball K is compact when the topological vector space is provided by the weak topology. Let us built the operator $\mathcal{L} : K \longrightarrow P(K)$ as follows

$$\lambda \longmapsto \mathcal{L}(\lambda) = \{\theta\} \subset K.$$

For all $\lambda \in K$, equation (4.6) is linear with respect to the solution θ, and the solution $\mathbf{u}$ is unique in the space V. Consequently, the set $\mathcal{L}(\lambda)$ is convex. To conclude the proof it remains to verify the closeness in $K \times K$ of the graph set

$$G(\mathcal{L}) = \{(\lambda, \theta) \in K \times K : \theta \in \mathcal{L}(\lambda)\}.$$

To do so, we consider a sequence $\lambda_n \in K$ such that $\lambda_n \longrightarrow \lambda$ in $W^{1,q}(\Omega)$ weakly and $\theta_n \in \mathcal{L}(\lambda_n)$. Let us remember that θ_n is the solution to the following variational equation

$$
\begin{aligned}
& k \int_\Omega \nabla\theta_n \cdot \nabla\tau \mathrm{d}x + \alpha \int_\Omega \theta_n \tau \mathrm{d}x + \beta \int_{\Gamma_2} \theta_n \tau \mathrm{d}\gamma \\
(4.21) \quad & = \int_\Omega \mu(\lambda_n) \left|\varepsilon(\mathbf{u}_n)\right|^p \tau \mathrm{d}x + \int_{\Gamma_2} (\omega_1 \upsilon_1(\lambda_n)\psi_1(\mathbf{u}_n) + \omega_2 \upsilon_2(\lambda_n)\psi_2(\mathbf{u}_n))\tau \mathrm{d}\gamma \\
& \qquad \forall \tau \in W^{1,p'}(\Omega),
\end{aligned}
$$

where $\mathbf{u}_n$ is the unique solution of the following variational inequality

$$
(4.22) \quad \begin{aligned}
& \int_\Omega (\mu(\lambda_n) \left|\varepsilon(\mathbf{u}_n)\right|^{p-2} \varepsilon(\mathbf{u}_n)) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}_n))\mathrm{d}x + \phi(\lambda_n, \mathbf{v}) - \phi(\lambda_n, \mathbf{u}_n) \\
& \geq \tilde{f}(\mathbf{v} - \mathbf{u}_n) \qquad \forall \mathbf{v} \in V.
\end{aligned}
$$

Then, from Propositions 4.4 and 4.5, we have

$$
\|\mathbf{u}_n\|_V \leq R_2 \qquad \text{and} \qquad \|\theta_n\|_{W^{1,q}(\Omega)} \leq R_1,
$$

where $R_2 \geq c \left(\dfrac{\|\mathbf{f}\|_{V'} + \|\mathbf{g}\|_{W^{-\frac{1}{p},p'}(\Gamma)^n}}{\mu_1} \right)^{\frac{1}{p-1}}$. Thus, we can extract subsequences $\mathbf{u}_m$ and θ_m such that

$$
(4.23) \qquad \mathbf{u}_m \longrightarrow \mathbf{u} \text{ in } V \text{ weakly,}
$$

$$
(4.24) \qquad \theta_m \longrightarrow \theta \text{ in } W^{1,q}(\Omega) \text{ weakly.}
$$

It follows from Rellich-Kondrachov's theorem and Sobolev's trace theorem that we can extract subsequences of $(\lambda_m, \mathbf{u}_m, \theta_m)$, still denoted by $(\lambda_m, \mathbf{u}_m, \theta_m)$, such that

$$
(4.25) \qquad \lambda_m \longrightarrow \lambda \text{ in } L^1(\Omega) \text{ strongly,}
$$

$$
(4.26) \qquad \mathbf{u}_m \longrightarrow \mathbf{u} \text{ in } L^1(\Omega)^n \text{ strongly,}
$$

$$
(4.27) \qquad \theta_m \longrightarrow \theta \text{ in } L^1(\Omega) \text{ strongly,}
$$

$$
(4.28) \qquad \mathbf{u}_m \longrightarrow \mathbf{u} \text{ in } L^1(\Gamma)^n \text{ strongly,}
$$

$$
(4.29) \qquad \theta_m \longrightarrow \theta \text{ in } L^1(\Gamma) \text{ strongly.}
$$

We prove now that $\varepsilon(\mathbf{u}_m) \longrightarrow \varepsilon(\mathbf{u})$ a.e. in Ω. To do so, we proceed as follows. Introducing the positive function

$$
(4.30) \quad \begin{aligned}
h_m(x) = \{&\mu(\lambda_m(x)) \left|\varepsilon(\mathbf{u}_m(x))\right|^{p-2} \varepsilon(\mathbf{u}_m(x)) \\
&- \mu(\lambda_m(x)) \left|\varepsilon(\mathbf{u}(x))\right|^{p-2} \varepsilon(\mathbf{u}(x))\} \cdot (\varepsilon(\mathbf{u}_m(x)) - \varepsilon(\mathbf{u}(x))).
\end{aligned}
$$

We get

$$\begin{aligned}\int_\Omega h_m(x)\mathrm{d}x = &\int_\Omega \mu\left(\lambda_m\left(x\right)\right)\left|\varepsilon\left(\mathbf{u}_m\left(x\right)\right)\right|^{p-2}\varepsilon\left(\mathbf{u}_m\left(x\right)\right)\cdot\left(\varepsilon\left(\mathbf{u}_m\left(x\right)\right)-\varepsilon\left(\mathbf{u}\left(x\right)\right)\right)\mathrm{d}x\\ &-\int_\Omega \mu(\lambda_m(x))\left|\varepsilon(\mathbf{u}(x))\right|^{p-2}\varepsilon(\mathbf{u}(x))\cdot(\varepsilon(\mathbf{u}_m(x))-\varepsilon(\mathbf{u}(x)))\mathrm{d}x\\ &\leq \tilde{f}(\mathbf{u}_m-\mathbf{u})+\phi(\lambda_m,\mathbf{u})-\phi(\lambda_m,\mathbf{u}_m)\\ &-\int_\Omega \mu(\lambda_m(x))\left|\varepsilon(\mathbf{u}(x))\right|^{p-2}\varepsilon(\mathbf{u}(x))\cdot(\varepsilon(\mathbf{u}_m(x))-\varepsilon(\mathbf{u}(x)))\mathrm{d}x.\end{aligned}\tag{4.31}$$

On the other hand, we have

$$\begin{aligned}\phi(\lambda_m,\mathbf{u}_m) = &\int_{\Gamma_2}(\omega_1\upsilon_1(\lambda_m(x))-\omega_1\upsilon_1(\lambda(x)))\psi_1(\mathbf{u}_m(x))\mathrm{d}\gamma\\ &+\int_{\Gamma_2}(\omega_2\upsilon_2(\lambda_m(x))-\omega_2\upsilon_2(\lambda(x)))\psi_2(\mathbf{u}_m(x))\mathrm{d}\gamma\\ &+\int_{\Gamma_2}\omega_2\upsilon_2(\lambda(x))\psi_1(\mathbf{u}_m(x))\mathrm{d}\gamma+\int_{\Gamma_2}\omega_1\upsilon_1(\lambda(x))\psi_2(\mathbf{u}_m(x))\mathrm{d}\gamma.\end{aligned}$$

Since $\lambda_m \longrightarrow \lambda$ a.e. in Ω and on Γ, the function υ_i is continuous and due to the weak lower semicontinuity of the continuous and convex function ψ_i $(i = 1, 2)$, combined with the hypothesis (2.12) and the convergence result (4.23), we deduce from the Lebesgue dominated convergence theorem that

$$\liminf \phi(\lambda_m,\mathbf{u}_m) \geq \phi(\lambda,\mathbf{u}),\tag{4.32}$$

$$\lim \phi(\lambda_m,\mathbf{u}) = \phi(\lambda,\mathbf{u}).\tag{4.33}$$

Moreover, since $\lambda_m \longrightarrow \lambda$ a.e. in Ω and on Γ, the function μ is continuous and due to (4.23) and the fact that $|\varepsilon(\mathbf{u}(x))|^{p-2}\,\varepsilon(\mathbf{u}(x))$ is bounded in $L^{p'}(\Omega)_s^{n\times n}$, we obtain by the Lebesgue dominated convergence theorem that

$$\int_\Omega \mu(\lambda_m(x))\left|\varepsilon(\mathbf{u}(x))\right|^{p-2}\varepsilon(\mathbf{u}(x))(\varepsilon(\mathbf{u}_m(x))-\varepsilon(\mathbf{u}(x)))\mathrm{d}x \longrightarrow 0.\tag{4.34}$$

Consequently, (4.31), (4.32), (4.33) and (4.34) give

$$\lim \|h_m\|_{L^1(\Omega)} = 0 \qquad \text{and} \qquad h_m \longrightarrow 0 \text{ a.e.}\tag{4.35}$$

Furthermore, $h_m(x)$ can be rewritten as follows

$$\begin{aligned}h_m(x) = &\,\mu(\lambda_m(x))\left|\varepsilon(\mathbf{u}_m(x))\right|^p-\mu(\lambda_m(x))\left|\varepsilon(\mathbf{u}_m(x))\right|^{p-2}\varepsilon(\mathbf{u}_m(x))\cdot\varepsilon(\mathbf{u}(x))\\ &-\mu(\lambda_m(x))\left|\varepsilon(\mathbf{u}(x))\right|^{p-2}\varepsilon(\mathbf{u}(x))\cdot(\varepsilon(\mathbf{u}_m(\mathbf{x}))-\varepsilon(\mathbf{u}(x))).\end{aligned}\tag{4.36}$$

Then

$$\begin{aligned}\mu(\lambda_m(x))\left|\varepsilon(\mathbf{u}_m(x))\right|^p \leq &\, h_m(x)+c\mu(\lambda_m(x))\left|\varepsilon(\mathbf{u}_m(x))\right|^{p-1}\\ &+c\mu(\lambda_m(x))\left|\varepsilon(\mathbf{u}_m(x))\right|+c.\end{aligned}$$

It follows that $(\varepsilon(\mathbf{u}_m(x)))_m$ is bounded in $\mathbb{R}^{n\times n}$. Then we can extract a subsequence, still denoted by $(\varepsilon(\mathbf{u}_m(x)))_m$, that converges to $\xi \in \mathbb{R}^{n\times n}$. By passage to the limit in h_m, we deduce that

$$(\mu(\lambda)\,|\xi|^{p-2}\,\xi - \mu(\lambda)\,|\varepsilon(\mathbf{u}(x))|^{p-2}\,\varepsilon(\mathbf{u}(x)))\cdot(\xi - \varepsilon(\mathbf{u}(x))) = 0,$$

which implies that $\varepsilon(\mathbf{u}(x)) = \xi$. We conclude that

$$\varepsilon(\mathbf{u}_m) \longrightarrow \varepsilon(\mathbf{u}) \text{ a.e. in } \Omega. \tag{4.37}$$

Thus, the sequence $(\mu(\lambda_m(x))\,|\varepsilon(\mathbf{u}_m(x))|^{p-2}\,\varepsilon(\mathbf{u}_m(x)))_m$ converges a.e. in Ω to $\mu(\lambda(x))\,|\varepsilon(\mathbf{u}(x))|^{p-2}\,\varepsilon(\mathbf{u}(x))$. Moreover, this sequence is bounded in $L^{p'}(\Omega)_s^{n\times n}$, then the $L^p - L^q$ compactness theorem (see [**11**] or [**13**]) entrains the convergence in $L^r(\Omega)_s^{n\times n}$ for all $1 < r < p'$.

By choosing $\varphi \in D(\Omega)^n$ as test function in inequality (4.22), we obtain

$$\begin{aligned}&\int_\Omega (\mu(\lambda)\,|\varepsilon(\mathbf{u})|^{p-2}\,\varepsilon(\mathbf{u}))\cdot\varepsilon(\varphi)\mathrm{d}x + \phi(\lambda,\varphi) - \tilde{f}(\varphi - \mathbf{u})\\ &\geq \int_\Omega \mu(\lambda_m)\,|\varepsilon(\mathbf{u}_m)|^p\,\mathrm{d}x + \phi(\lambda_m,\mathbf{u}_m).\end{aligned} \tag{4.38}$$

Using now (4.32), the fact that $\lambda_m \longrightarrow \lambda$ a.e. in Ω, the continuity of μ and the weak lower semicontinuity of the norm $\|.\|_{W^{1,p}(\Omega)^n}$. We conclude that u is a solution to the problem (4.3).

Our final goal is to show that

$$\mathbf{u}_m \longrightarrow \mathbf{u} \text{ in } V \text{ strongly.} \tag{4.39}$$

To do so, we proceed as follows. We introduce the function

$$\chi_m(x) = \mu(\lambda_m(x))\,|\varepsilon(\mathbf{u}_m(x))|^p\,. \tag{4.40}$$

From (4.37), we remark that $\chi_m \longrightarrow \chi$ a.e. in Ω, where

$$\chi(x) = \mu(\lambda(x))\,|\varepsilon(\mathbf{u}(x))|^p\,. \tag{4.41}$$

Substituting it in the inequality (4.22), taking $\mathbf{v} = \mathbf{u}$ as test function and using Lebesgue's dominated convergence theorem, the passage to limit, gives

$$\lim \int_\Omega \chi_m(x)\mathrm{d}x \leq \int_\Omega \chi(x)\mathrm{d}x. \tag{4.42}$$

On the other hand, we know from the weak lower semicontinuity of the norm $\|\cdot\|_{W^{1,p}(\Omega)^n}$ that

$$\liminf \int_\Omega \chi_m(x)\mathrm{d}x \geq \int_\Omega \chi(\mathbf{x})\mathrm{d}x. \tag{4.43}$$

(4.42) and (4.43) combined with the Lebesgue dominated convergence theorem lead to

$$\lim \int_\Omega |\varepsilon(\mathbf{u}_m(x))|^p\,\mathrm{d}x = \int_\Omega |\varepsilon(\mathbf{u}(x))|^p\,\mathrm{d}x. \tag{4.44}$$

The use of (4.37) and (4.44) combined with Vitali's theorem affirms that $(|\varepsilon(\mathbf{u}_m(x))|^p)_m$ is L^1-equi-integrable which asserts that $(|\varepsilon(\mathbf{u}_m(x))|)_m$ is L^p-equi-integrable. Vitali's theorem and the convergence result (4.37) prove that

$$\varepsilon(\mathbf{u}_m) \longrightarrow \varepsilon(\mathbf{u}) \text{ in } L^p(\Omega)_s^{n\times n} \text{ strongly.} \tag{4.45}$$

Using Korn's inequality, we obtain (4.39), which shows that

$$\int_\Omega \mu(\lambda_m)\,|\varepsilon(\mathbf{u}_m)|^p\,\tau \mathrm{d}x \longrightarrow \int_\Omega \mu(\lambda)\,|\varepsilon(\mathbf{u})|^p\,\tau \mathrm{d}x. \tag{4.46}$$

Furthermore, the continuity of ψ_i $(i=1,2)$ on V leads to $\psi_i(\mathbf{u}_m) \longrightarrow \psi_i(\mathbf{u})$. Then the use of the continuity of υ_i $(i=1,2)$ and the Lebesgue dominated convergence theorem gives

$$\begin{aligned} &\int_{\Gamma_2} (\omega_1\upsilon_1(\lambda_m)\psi_1(\mathbf{u}_m) + \omega_2\upsilon_2(\lambda_m)\psi_2(\mathbf{u}_m))\tau \mathrm{d}\gamma \\ \longrightarrow &\int_{\Gamma_2} (\omega_1\upsilon_1(\lambda)\psi_1(\mathbf{u}) + \omega_2\upsilon_2(\lambda)\psi_2(\mathbf{u}))\tau \mathrm{d}\gamma. \end{aligned} \tag{4.47}$$

Thus, we conclude that θ is the solution to the problem (4.21).

Hence, $\theta_n \longrightarrow \theta \in \mathcal{L}(\lambda)$ in $W^{1,q}(\Omega)$ weakly. By virtue of Kakutani-Glicksberg's fixed point theorem, the application admits a fixed point $\theta \in \mathcal{L}(\theta)$. Finally, $(\mathbf{u},\theta)$ solves the Problem 2. □

Remark 4.6. This proof permits also to verify the continuous dependence of the solution $\mathbf{u}(\lambda) \in V$ of problem (4.3) and the solution $\theta(\lambda) \in W^{1,q}(\Omega)$ of problem (4.6) with respect to the arbitrary function $\lambda \in W^{1,1}(\Omega)$.

Proof of Theorem 4.2. Let $(\mathbf{u}_k, \theta_k)$ be a solution to the Problem 2, corresponding to each $k > 0$ and let $k \longrightarrow +\infty$. From the estimates (4.4), (4.7) and using Rellich-Kondrachov's theorem, we can extract a subsequence of $(\mathbf{u}_k, \theta_k)$, still denoted by $(\mathbf{u}_k, \theta_k)$, satisfying

$$\begin{aligned} \mathbf{u}_k &\longrightarrow \mathbf{u} \ \text{ in } V \ \text{ weakly}, \\ \mathbf{u}_k &\longrightarrow \mathbf{u} \ \text{ in } L^1(\Omega)^n \ \text{ strongly}, \\ \nabla\theta_k &\longrightarrow 0 \ \text{ in } L^1(\Omega)^n \ \text{ strongly}, \\ \theta_k &\longrightarrow \Theta = \text{constant} \ \text{ in } L^1(\Omega) \ \text{ strongly}. \end{aligned}$$

We can proceed as in the proof of Theorem 4.1 to get a strong convergence of $\mathbf{u}_k$ to $\mathbf{u}$ in V. Then, we can pass to the limit $k \longrightarrow +\infty$ in (3.6) and take $\tau = 1$ to obtain the implicit scalar equation (3.8). Now, taking the limit $k \longrightarrow +\infty$ in (3.5), it follows that $\mathbf{u}$ solves the nonlocal inequality (3.7).

Moreover, the scalar equation (3.8) asserts that $\Theta \geq 0$. □

5. Examples of Subdifferential Contact Condition

In this section we present some examples of contact and dry friction laws which lead to an inequality of the form (2.7), (see, e.g., [**3**], [**4**], [**5**], [**18**], [22], [**25**] and [**26**]). We conclude by Theorem 4.1, the boundary value problem for each of the following examples has a solution and by Theorem 4.2, the nonlocal problem admits a solution.

Example 5.1 (*Bilateral contact with thermal Tresca's friction law)***.** It is in the form of the following boundary conditions:

$$(5.1) \qquad \begin{cases} u_\nu = 0, \quad |\sigma_\tau| \leq \upsilon_1(\theta), \\ |\sigma_\tau| < \upsilon_1(\theta) \implies \mathbf{u}_\tau = 0, \\ |\sigma_\tau| = \upsilon_1(\theta) \implies \mathbf{u}_\tau = -\lambda\sigma_\tau, \quad \lambda \geq 0. \end{cases} \qquad \text{on } \Gamma_2.$$

Here λ represents the friction bound, i.e. the magnitude of the limiting friction at which slip occurs. The contact is assumed to be bilateral, i.e. there is no loss of contact during the process. We suppose that the function υ_1 verifies the hypothesis (2.11).

It is straightforward to show that if $\{\mathbf{u}, \sigma, \theta\}$ are regular functions satisfying (5.1), then

$$\sigma\nu \cdot (\mathbf{v} - \mathbf{u}) \geq \upsilon_1(\theta)\,|\mathbf{u}_\tau| - \upsilon_1(\theta)\,|\mathbf{v}_\tau| \qquad \forall \mathbf{v} \in V, \ \ \forall \theta \in W^{1,q}(\Omega) \text{ a.e on } \Gamma_2.$$

So, (2.7) holds with the choice

$$\varphi(\theta, x, \mathbf{y}) = \upsilon_1(\theta, x)\psi_1(x, \mathbf{y}) \qquad \forall x \in \Gamma_2, \ \ \forall \theta \in \mathbb{R}, \ \ \forall \mathbf{y} \in \mathbb{R}^n,$$

where $\psi_1(x, \mathbf{y}) = |\mathbf{y}_\tau(x)|$. We obtain

$$\phi(\theta, \mathbf{v}) = \int_{\Gamma_2} \upsilon_1(\theta, x)\psi_1(x, \mathbf{v})\mathrm{d}\gamma \qquad \forall \mathbf{v} \in V, \ \ \forall \theta \in W^{1,q}(\Omega).$$

In addition, we have

$$\|\psi_1(\mathbf{v})\|_{L^2(\Gamma_2)} \leq C\,\|\mathbf{v}\|_{L^2(\Gamma_2)} \qquad \forall \mathbf{v} \in V.$$

Then, ψ verifies the assertion (2.12) with the choice $\omega_1 = 1$.

Example 5.2 (*Bilateral contact with thermal viscoelastic friction condition)***.** We consider the following boundary conditions

$$(5.2) \qquad u_\nu = 0, \quad \sigma_\tau = -\upsilon_1(\theta)\,|\mathbf{u}_\tau|^{a-1}\,\mathbf{u}_\tau \qquad \text{on } \Gamma_2,$$

where $0 < a \leq 1$ and $\upsilon_1(\theta)$ is the coefficient of friction. Here, the tangential shear is proportional to the power a of the tangential speed. In addition, we suppose that the function υ_1 verifies the hypothesis (2.11). It is straightforward to show that if $\{\mathbf{u}, \sigma, \theta\}$ are regular functions satisfying (5.2), then (2.7) holds with

$$\varphi(\theta, x, \mathbf{y}) = \upsilon_1(\theta, x)\psi_1(x, \mathbf{y}) \qquad \forall x \in \Gamma_2, \quad \forall \theta \in \mathbb{R}, \quad \forall y \in \mathbb{R}^n,$$

where, $\psi_1(x, \mathbf{y}) = \dfrac{1}{a+1}\,|\mathbf{y}_\tau(x)|^{a+1}$. We deduce that

$$\phi(\theta, \mathbf{v}) = \int_{\Gamma_2} \upsilon_1(\theta, x)\psi_1(x, \mathbf{v})\mathrm{d}\gamma \qquad \forall \mathbf{v} \in V, \ \ \forall \theta \in W^{1,q}(\Omega).$$

In addition, we have

$$\|\psi_1(\mathbf{v})\|_{L^a(\Gamma_2)} \leq C\,\|\mathbf{v}\|^{a+1}_{L^{2(a+1)}(\Gamma_2)} \qquad \forall \mathbf{v} \in V.$$

Then, ψ_1 verifies the assertion (2.12) with the choice $\omega_1 = a + 1$.

Example 5.3 (*Viscoelastic contact with thermal Tresca's friction law)*. We consider the contact problem with the boundary conditions:

$$(5.3) \qquad \begin{cases} \sigma_\nu = -\upsilon_2(\theta)\,|u_\nu|^{b-1}\,u_\nu, \quad |\sigma_\tau| \le \upsilon_1(\theta), \\ |\sigma_\tau| < \upsilon_1(\theta) \implies \mathbf{u}_\tau = 0, \\ |\sigma_\tau| = \upsilon_1(\theta) \implies \mathbf{u}_\tau = -\lambda\sigma_\tau, \quad \lambda \ge 0, \end{cases} \qquad \text{on } \Gamma_2.$$

Here $0 < b \le 1$, the normal contact stress depends on a power of the normal speed (this condition may describe, for example, the motion of a fluid, a wheel on a granular material, the sand on the beach). In addition, we suppose that the function υ_i $(i = 1, 2)$ verify the hypothesis (2.11).

We can easily verifies that

$$\varphi(\theta, x, \mathbf{y}) = \upsilon_1(\theta, x)\psi_1(x, \mathbf{y}) + \upsilon_2(\theta, x)\psi_2(x, \mathbf{y}), \quad \forall x \in \Gamma_2,\ \forall\theta \in \mathbb{R},\ \forall \mathbf{y} \in \mathbb{R}^n,$$

where

$$\psi_1(x, \mathbf{y}) = |\mathbf{y}_\tau(x)|\,, \quad \psi_2(x, \mathbf{y}) = \frac{1}{b+1}\,|\mathbf{y}_\nu(x)|^{b+1}\,,$$
$$\omega_1 = 1 \quad \text{and} \quad \omega_2 = b + 1.$$

Example 5.4 (*Viscoelastic contact with thermal friction)*. We consider the contact problem with the boundary conditions:

$$(5.4) \qquad \sigma_\tau = -\upsilon_1(\theta)\,|\mathbf{u}_\tau|^{a-1}\,\mathbf{u}_\tau, \qquad \sigma_\nu = -\upsilon_2(\theta)\,|u_\nu|^{b-1}\,u_\nu \qquad \text{on } \Gamma_2,$$

where $0 < a, b \le 1$. Here, the fluid is moving on sand or a granular material and the normal stress is proportional to a power of the normal speed while the tangential shear is proportional to a power of the tangential speed. In addition, we suppose that the function υ_i, $(i = 1, 2)$ verifies the hypothesis (2.11).

We prove that

$$\varphi(\theta, x, \mathbf{y}) = \upsilon_1(\theta, x)\psi_1(x, \mathbf{y}) + \upsilon_2(\theta, x)\psi_2(x, \mathbf{y}), \quad \forall x \in \Gamma_2,\ \forall\theta \in \mathbb{R},\ \forall \mathbf{y} \in \mathbb{R}^n,$$

where

$$\psi_1(x, \mathbf{y}) = \frac{1}{a+1}\,|\mathbf{y}_\tau(x)|^{a+1}\,, \quad \psi_2(x, \mathbf{y}) = \frac{1}{b+1}\,|\mathbf{y}_\nu(x)|^{b+1}\,,$$
$$\omega_1 = a + 1 \quad \text{and} \quad \omega_2 = b + 1.$$

Remark 5.5. In these different examples, we can easily verify that

$$-\sigma\nu \cdot \mathbf{u} = \omega_1\upsilon_1(\theta, x)\psi_1(x, \mathbf{y}) + \omega_2\upsilon_2(\theta, x)\psi_2(x, \mathbf{y}),$$

which permits to give reason to hypothesis (2.14).

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