Acta Math. Univ. Comenianae
Vol. LXXX, 2 (2011), pp. 237–250

UNIVERSAL BOUNDS FOR POSITIVE SOLUTIONS OF DOUBLY DEGENERATE PARABOLIC EQUATIONS WITH A SOURCE

A. F. TEDEEV

Abstract. We consider a doubly degenerate parabolic equation with a source term of the form

$$\left(u^{\beta}\right)_t = \operatorname{div}\left(|\nabla u|^{\lambda-1}\nabla u\right) + u^p \qquad \text{where } \ 0 < \beta \le \lambda < p.$$

For a positive solution of the equation we prove universal bounds and provide blow-up rate estimates under suitable assumptions on $p < p_0(\lambda,\beta,N)$. In particular, we extend some of the recent results by K. Ammar and Ph. Souplet concerning the blow-up estimates for porous media equations with a source. Our proofs are based on a generalized version of the Bochner-Weitzenbök formula and local energy estimates.

1. Introduction

We study the doubly degenerate parabolic equation with a nonlinear source of the form

$$u_t^{\beta} = \Delta_\lambda u + u^p \qquad \text{in } \ Q_T = \mathbf{R}^N \times (0,T), \quad N \ge 2, \tag{1.1}$$

where $\Delta_\lambda u = \operatorname{div}\left(|\nabla u|^{\lambda-1}\nabla u\right)$. Here and thereafter we assume that

$$0 < \beta \le \lambda < p. \tag{1.2}$$

Definition 1.1. We say that $u \ge 0$ is a weak solution of (1.1) in Q_T if it is locally bounded in Q_T, $u \in C((0,T); L^{\beta+1}_{loc}(\mathbb{R}^N))$, $|\nabla u|^{\lambda+1} \in L^1_{loc}(Q_T)$ and satisfies (1.1) in the sense of the integral identity

$$\iint_{Q_T} (-u^{\beta}\eta_t + |\nabla u|^{\lambda-1}\nabla u \nabla\eta)\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} u^p\eta\,\mathrm{d}x\,\mathrm{d}t$$

for any $\eta \in C_0^1(Q_T)$.

The existence of local solutions of (1.1) follows, for example, from [**22**], and the uniqueness of an energy solution follows from [**29**]. Moreover, weak solutions are locally Hölder continuous [**23, 31**]. We also refer to the survey [**24**], [**37**] and

Received August 28, 2010.
2010 *Mathematics Subject Classification.* Primary 35K55, 35K65, 35B33.
Key words and phrases. Degenerate parabolic equations; blow-up; universal bounds.

the books [**14, 25, 10, 38**] for various local and global properties of solutions of doubly degenerate parabolic equations.

The main purpose of the present paper is to obtain universal bounds of blow-up solutions of (1.1), that is, bounds that are independent of initial data. The paper is motivated by recent results of K. Ammar and Ph. Souplet [**3**] (see also [**33**] and earlier results [**39**]) concerning universal blow-up behaviour of a porous medium equation with a source. We extend some of these results for a solution of the equation (1.1). One of the main tools in the proof of universal estimates in [**3**] is the following Bochner-Weitzenbök formula

$$\frac{1}{2}\Delta(|\nabla v|^2) = |D^2 v|^2 + (\nabla\Delta v)\cdot\nabla v \tag{1.3}$$

with $|D^2v|^2 = \sum\limits_{i,j=1}^{N}(v_{x_ix_j})^2$. Below we use the generalized version of (1.3) (see (2.1)) in order to obtain some integral gradient estimates which together with the local $L_q - L_\infty$ estimates of [**8**] give the universal blow-up estimate of supremum norm of a solution to (1.1).

Let

$$\theta = \frac{(N-1)(1-\beta)}{N\beta}, \quad \delta = \frac{\lambda-1}{\lambda+1}, \quad \delta_1 = \frac{\delta}{\beta}, \quad A = 2(\theta+\delta_1)+(1+\delta_1)^2,$$

$$p_0(\beta,\lambda,N) = \frac{N(N+\lambda+1)}{(\lambda+1)(N-1)(2N\delta+N-1)}\left(1+\delta_1+\theta+\sqrt{A}\right).$$

The main result of the paper is as follows.

Theorem 1.1. *Let $u \geq 0$ be a weak solution of* (1.1) *in $Q_T = \mathbb{R}^N \times (0,T)$. Assume that*

$$p < p_0(\beta,\lambda,N).$$

Then there exists a constant $C = C(N,\beta,\lambda,p)$ such that

$$u(x,t) \leq C(T-t)^{-1/(p-\beta)} \tag{1.4}$$

for all $x \in \mathbb{R}^N$ and $t \in (T/2,T)$.

Remark 1.1. For the porous medium equation with a source

$$v_t = \Delta v^m + v^q,$$

(1.4) follows from the results in [**3**]. Namely, as it can be seen in this case $\beta = 1/m$ and $q = pm$, $\lambda = 1$. Thus

$$p_0 = \frac{N(N+2)}{2(N-1)^2}(1+\theta+\sqrt{A}) \text{ with } A = 1+2\theta, \quad \theta = \frac{(m-1)(N-1)}{N} \tag{1.5}$$

which coincides with the exponent found in [**3**]. While if in (1.5) $m = 1$, we get the exponent

$$p_0 = \frac{N(N+2)}{(N-1)^2},$$

which was discovered in [**11**]. Finally, if $\beta = 1$ in (1.1), that is, (1.1) is the nonstationary λ-Laplacian with a source, then

$$p_0 = p_0(\beta, \lambda, N) = \frac{2(1+\delta)N(N+\lambda+1)}{(\lambda+1)(N-1)(2N\delta+N-1)}.$$

To the best of our knowledge our result is still new in this case.

Remark 1.2. Notice that $p_0(\beta, \lambda, N)$ is less than the Sobolev exponent $p_S = (N\lambda+\lambda+1)/(N-\lambda-1)$ for $\lambda+1 < N$. However $p_0(\beta, \lambda, N)$ is bigger than the Fujita exponent p_F

$$p_0(\beta, \lambda, N) > p_F = \lambda + \beta\frac{\lambda+1}{N}.$$

Let us recall that the Fujita exponent p_F gives the threshold between the global existence and blow-up. Namely, if $1 < p \leq p_F$, then there is no positive global solution of (1.1), while if $p > p_F$, then there exist some positive global solutions (see the survey by Deng and Levine [**13**]). The Sobolev exponent p_S is known to be critical for the existence of positive steady states of the stationary solution

$$\Delta_\lambda u + u^p = 0 \quad \text{on} \quad \mathbb{R}^N$$

(see [**35**], [**12**] and references therein).

We also refer the reader for the Fujita type results for the porous medium equation and nonstationary λ-Laplacian with sources to the book [**17**], the survey [**18**] and [**4**]. For more general doubly degenerate parabolic equations with a source, the Fujita problem was recently treated in [**6, 7, 1, 2, 9, 26**], where the authors discussed dependence of the critical Fujita exponent on the geometry of the domain (see [**6, 7**]), on the behaviour of the initial data (see [**1**]), on the various forms of sources (see [**7, 9**]) and on the behaviour of the coefficients (see [**26**]). About the universal bounds near the blow-up time under the subcritical Fujita exponent we refer also to [**8**] and [**26**] for a wide class of doubly degenerate parabolic equations with a blow-up term. The problem of the optimal blow-up rate and universal bounds of both global and blow-up solutions for semilinear parabolic equations were investigated in [**5, 19, 20, 21, 27, 28, 30, 32**] (see also the book [**33**] and references therein).

The rest of the paper is devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

Turning to the proof of the theorem let us remark that since the solution to (1.1) is not regular enough, the standard way to proceed is to apply some kind of regularization to the equation before obtaining the integral estimates, and then subsequently pass to the limit with respect to the regularization parameter. This process is quite standard by now, it is described in details, for example, in [**15**]. Therefore without going into details we assume that our solution is sufficiently regular (see [**15**]).

One of the main parts in the proof of the theorem is the universal bound of the integral

$$\int_{t_1}^{t_2}\int_{B_R(x_0)} u^{2p+1-\beta}\mathrm{d}x\mathrm{d}t$$

for any $0 < t_1 < t_2 < T$, $R > 0$ and any $x_0 \in \mathbb{R}^N$. In order to do this, the starting point is the following formula

$$\begin{aligned}\left[\left(|\nabla v|^{\lambda-1} v_{x_i}\right)_{x_j} |\nabla v|^{\lambda-1} v_{x_j}\right]_{x_i} &= \left(|\nabla v|^{\lambda-1} v_{x_i}\right)_{x_j} \left(|\nabla v|^{\lambda-1} v_{x_j}\right)_{x_i} \\ &\quad + (\Delta_\lambda v)_{x_j} |\nabla v|^{\lambda-1} v_{x_j}.\end{aligned} \tag{2.1}$$

This formula is obtained by the direct differentiation and changing the order of the derivatives

$$\begin{aligned}&\left[\left(|\nabla v|^{\lambda-1} v_{x_i}\right)_{x_j} |\nabla v|^{\lambda-1} v_{x_j}\right]_{x_i} \\ &= \left(|\nabla v|^{\lambda-1} v_{x_i}\right)_{x_j} \left(|\nabla v|^{\lambda-1} v_{x_j}\right)_{x_i} + \left(|\nabla v|^{\lambda-1} v_{x_i}\right)_{x_j x_i} |\nabla v|^{\lambda-1} v_{x_j} \\ &= \left(|\nabla v|^{\lambda-1} v_{x_i}\right)_{x_j} \left(|\nabla v|^{\lambda-1} v_{x_j}\right)_{x_i} + \left[\left(|\nabla v|^{\lambda-1} v_{x_i}\right)_{x_i}\right]_{x_j} |\nabla v|^{\lambda-1} v_{x_j} \\ &= \left(|\nabla v|^{\lambda-1} v_{x_i}\right)_{x_j} \left(|\nabla v|^{\lambda-1} v_{x_j}\right)_{x_i} + (\Delta_\lambda v)_{x_j} |\nabla v|^{\lambda-1} v_{x_j}.\end{aligned}$$

Here and thereafter the summation on repeating indices is assumed and v will be smooth enough. Formula (2.1) is a natural generalization of (1.3) and coincides with the latter when $\lambda = 1$.

Next lemma is similar to [**35**, Proposition 6.2]. The proof we give here uses similar arguments to those used in [**35**]. We reproduce the proof here for the readers' convenience.

Lemma 2.1. *Let G be any domain in $\mathbb{R}^N$. Then for any sufficiently smooth function $v(x)$ and any nonnegative $\zeta \in D(G)$, for $s > 0$ large enough and any d, $\mu \in R$, it holds*

$$\begin{aligned}&-\mu\frac{2\lambda+1}{\lambda+1}\int \zeta^s v^{\mu-1}\Delta_\lambda v\, |\nabla v|^{\lambda+1} + \mu(\mu-1)\frac{\lambda}{\lambda+1}\int \zeta^s v^{\mu-2} |\nabla v|^{2(\lambda+1)} \\ &\leq \frac{N-1}{N}\int \zeta^s v^\mu (\Delta_\lambda v)^2 + 2s\int \zeta^{s-1} v^\mu \Delta_\lambda v\, |\nabla v|^{\lambda-1} v_{x_i}\zeta_{x_i} \\ &\quad + \frac{2s\mu\lambda}{\lambda+1}\int \zeta^{s-1} v^{\mu-1} |\nabla v|^{\lambda+1} |\nabla v|^{\lambda-1} v_{x_i}\zeta_{x_i} \\ &\quad + s\int \zeta^{s-1} v^\mu |\nabla v|^{\lambda-1} v_{x_i} |\nabla v|^{\lambda-1} v_{x_j}\zeta^s_{x_i x_j}.\end{aligned} \tag{2.2}$$

Proof. Multiplying both sides of (2.1) by $\zeta^s v^\mu$ and integrating by parts, we get

$$\begin{aligned}
I_1 =& \int v^\mu \zeta^s \left[\left(|\nabla v|^{\lambda-1} v_{x_i} \right)_{x_j} |\nabla v|^{\lambda-1} v_{x_j} \right]_{x_i} \\
=& \int \zeta^s v^\mu (\Delta_\lambda v)_{x_j} |\nabla v|^{\lambda-1} v_{x_j} + \int \zeta^s v^\mu \left(|\nabla v|^{\lambda-1} v_{x_i} \right)_{x_j} \left(|\nabla v|^{\lambda-1} v_{x_j} \right)_{x_i} \\
=& -\int \zeta^s v^\mu (\Delta_\lambda v)^2 - \mu \int \zeta^s v^{\mu-1} \Delta_\lambda v \, |\nabla v|^{\lambda+1} - s \int \zeta^{s-1} v^\mu \Delta_\lambda v \, |\nabla v|^{\lambda-1} v_{x_i} \zeta_{x_i} \\
&+ \int \zeta^s v^\mu \left(|\nabla v|^{\lambda-1} v_{x_i} \right)_{x_j} \left(|\nabla v|^{\lambda-1} v_{x_j} \right)_{x_i}.
\end{aligned}$$

Using the algebraic inequality (see, for instance, [**15, 16**], [**35**])

$$\left(|\nabla v|^{\lambda-1} v_{x_i} \right)_{x_j} \left(|\nabla v|^{\lambda-1} v_{x_j} \right)_{x_i} \geq \frac{1}{N} (\Delta_\lambda v)^2,$$

we obtain

$$\begin{aligned}
I_1 \geq& -\frac{N-1}{N} \int \zeta^s v^\mu (\Delta_\lambda v)^2 - \mu \int \zeta^s v^{\mu-1} \Delta_\lambda v \, |\nabla v|^{\lambda+1} \\
&- s \int \zeta^{s-1} v^\mu \Delta_\lambda v \, |\nabla v|^{\lambda-1} v_{x_i} \zeta_{x_i}.
\end{aligned} \tag{2.3}$$

On the other hand, integrating by parts twice, we have

$$\begin{aligned}
I_1 =& -\int (\zeta^s v^\mu)_{x_i} \left(|\nabla v|^{\lambda-1} v_{x_i} \right)_{x_j} |\nabla v|^{\lambda-1} v_{x_j} \\
=& \int (\zeta^s v^\mu)_{x_i} \left(|\nabla v|^{\lambda-1} v_{x_j} \right)_{x_j} |\nabla v|^{\lambda-1} v_{x_i} \\
=& \int \Delta_\lambda v (\zeta^s v^\mu)_{x_i} |\nabla v|^{\lambda-1} v_{x_i} + \int (\zeta^s v^\mu)_{x_i x_j} |\nabla v|^{\lambda-1} v_{x_j} |\nabla v|^{\lambda-1} v_{x_i} \\
=& \mu \int \zeta^s v^{\mu-1} \Delta_\lambda v \, |\nabla v|^{\lambda+1} + s \int \zeta^{s-1} v^\mu \Delta_\lambda v \, |\nabla v|^{\lambda-1} v_{x_i} \zeta_{x_i} \\
&+ \mu(\mu-1) \int \zeta^s v^{\mu-2} |\nabla v|^{2(\lambda+1)} \\
&+ 2s\mu \int \zeta^{s-1} v^{\mu-1} |\nabla v|^{\lambda+1} |\nabla v|^{\lambda-1} v_{x_j} \zeta_{x_j} \\
&+ s \int \zeta^{s-1} v^\mu |\nabla v|^{\lambda-1} v_{x_j} |\nabla v|^{\lambda-1} v_{x_i} \zeta^s_{x_i x_j} \\
&+ \mu \int \zeta^s v^{\mu-1} |\nabla v|^{\lambda-1} v_{x_j} |\nabla v|^{\lambda-1} v_{x_i} v_{x_i x_j} \\
=& \mu I_2 + s I_3 + \mu(\mu-1) I_4 + 2s\mu I_5 + s I_6 + \mu I_7.
\end{aligned} \tag{2.4}$$

We have

$$\begin{aligned}
I_7 &= \frac{1}{2} \int \zeta^s v^{\mu-1} |\nabla v|^{\lambda-1} v_{x_i} |\nabla v|^{\lambda-1} (|\nabla v|^2)_{x_i} \\
&= -\frac{1}{2} I_2 - \frac{1}{2}(\mu-1) I_4 - \frac{\lambda-1}{2} I_7 - s I_5.
\end{aligned}$$

Thus

$$I_7 = -\frac{1}{\lambda+1}I_2 - \frac{\mu-1}{\lambda+1}I_4 - \frac{2s}{\lambda+1}I_5$$

and (2.3) implies

$$I_1 = \frac{\mu\lambda}{\lambda+1}I_2 + \frac{\mu(\mu-1)\lambda}{\lambda+1}I_4 + \frac{2s\mu\lambda}{\lambda+1}I_5 + sI_3 + I_6. \tag{2.5}$$

Now combining (2.3) with (2.5), we derive

$$-\left(\frac{\mu(2\lambda+1)}{\lambda+1}I_2 + \frac{\mu(\mu-1)\lambda}{\lambda+1}I_4\right) \leq \frac{N-1}{N}\int \zeta^s v^\mu (\Delta_\lambda v)^2 + 2sI_3 + \frac{2s\mu\lambda}{\lambda+1}I_5 + sI_6 + \mu I_7.$$

Lemma 2.1 is proved. □

Denote

$$a = -\frac{2d\left\{N(d(\lambda-1)-2\lambda) + (1-\beta)(\lambda+1)\right\} + (\lambda+1)(N-1)((1-\beta)^2+d^2)}{4N(\lambda+1)},$$

$$b = \frac{d(N+\lambda+1)}{N(\lambda+1)} - p\frac{N-1}{N}.$$

Lemma 2.2. *Assume that $a > 0$ and $b > 0$. Then for a sufficiently small $\varepsilon > 0$ there holds*

$$\begin{aligned}(a-5\varepsilon)&\iint \xi^s u^{-1-\beta}|\nabla u|^{2(\lambda+1)} + (b-\varepsilon)\iint \xi^s u^{p-\beta}|\nabla u|^{\lambda+1}\\ &\leq C(\varepsilon)\left(\iint \xi^s|\nabla\xi|^{\lambda+1}u^{\beta-1}u_t^2 + \iint \xi^{s-2}\xi_t^2 u^{1+\beta}\right.\\ &\left.+\iint \xi^{s-2(\lambda+1)}|\nabla\xi|^{2(\lambda+1)}u^{1-\beta+2\lambda} + \iint \xi^{s-\lambda-1}|\nabla\xi|^{\lambda+1}u^{p+1+\lambda-\beta}\right),\end{aligned} \tag{2.6}$$

where integrals are taken over $G\times(t_1,t_2)$ with $0 < t_1 < t_2 < T$ and $\xi(x,t)$ is a smooth cutoff function of $G\times(t_1,t_2)$.

Proof. In (2.2), set $v = u^\alpha$ with some $\alpha\in\mathbb{R}$. Then

$$I_2 = \alpha^{2\lambda+1}((\alpha-1)\lambda I_8 + I_9), \qquad I_4 = \alpha^{2\lambda+1}I_8,$$

$$\int \zeta^s v^\mu(\Delta_\lambda v)^2 = \alpha^{2\lambda}((\alpha-1)^2\lambda^2 I_8 + 2(\alpha-1)\lambda I_9 + I_{10}).$$

Here

$$I_8 = \int \zeta^s u^{h-2}|\nabla u|^{2(\lambda+1)}, \qquad I_9 = \int \zeta^s u^{h-1}\Delta_\lambda u\,|\nabla u|^{\lambda+1},$$

$$I_{10} = \int \zeta^s u^h(\Delta_\lambda u)^2 \qquad h = 2(\alpha-1)\lambda + \alpha\mu.$$

Therefore, (2.2) implies that

$$-C_1I_8 - C_2I_9 \le \frac{N-1}{N}I_{10} + 2sI_{11} + 2s\lambda\left(\alpha - 1 + \frac{\mu\alpha}{\lambda+1}\right)I_{12} + I_{13}. \tag{2.7}$$

Here

$$\begin{aligned} I_{11} &= \int \zeta^{s-1}u^h \Delta_\lambda u \left|\nabla u\right|^{\lambda-1} u_{x_i}\zeta_{x_i}, \\ I_{12} &= \int \zeta^{s-1}u^{h-1} \left|\nabla u\right|^{\lambda+1} \left|\nabla u\right|^{\lambda-1} u_{x_i}\zeta_{x_i}, \\ I_{13} &= \int u^h \left|\nabla u\right|^{\lambda-1} u_{x_j} \left|\nabla u\right|^{\lambda-1} u_{x_i}\zeta^s_{x_ix_j}. \end{aligned}$$

Then replacing ζ by ξ and integrating (2.7) from t_1 to t_2, we get with $d = \alpha\mu$

$$-C_3J_8 - C_4J_9 \le \frac{N-1}{N}J_{10} + 2sJ_{11} + 2s\lambda\left(\alpha - 1 + \frac{d}{\lambda+1}\right)J_{12} + sJ_{13}, \tag{2.8}$$

where

$$\begin{aligned} J_i &= \int_{t_1}^{t_2} I_i(t)\mathrm{d}t, \\ C_3 &= \frac{2d\left[N(d(\lambda-1) - 2\lambda) + h(\lambda+1)\right] + (\lambda+1)(N-1)(h^2+d^2)}{4N(\lambda+1)}, \\ C_4 &= \frac{h(N-1)(\lambda+1) + d(N+\lambda+1)}{N(\lambda+1)}. \end{aligned}$$

By (1.1) we have

$$\begin{aligned} J_{10} &= \iint \xi^s u^h (\Delta_\lambda u)^2 = \iint \xi^s u^h \Delta_\lambda u (u_t^\beta - u^p) \\ &= \iint \xi^s u^h u_t^\beta \Delta_\lambda u - \iint \xi^s u^{h+p} \Delta_\lambda u. \end{aligned} \tag{2.9}$$

Integrating by parts, we obtain

$$\begin{aligned} -\iint \xi^s u^{h+p}\Delta_\lambda u &= (h+p)\iint \xi^s u^{h+p-1}\left|\nabla u\right|^{\lambda+1} \\ &\qquad + s\iint \xi^{s-1}u^{h+p}\left|\nabla u\right|^{\lambda-1}u_{x_i}\xi_{x_i} \\ &= (h+p)J_{14} + sJ_{15}, \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
\iint \xi^s u^h u_t^\beta \Delta_\lambda u &= \beta \iint \xi^s u^{h+\beta-1} u_t \Delta_\lambda u \\
&= -\frac{\beta}{\lambda+1} \iint \xi^s u^{h+\beta-1} \left(|\nabla u|^{\lambda+1}\right)_t \\
&\quad - \beta(h+\beta-1) \iint \xi^s u^{h+\beta-2} |\nabla u|^{\lambda+1} u_t \\
&\quad - \beta s \iint \xi^{s-1} u^{h+\beta-1} |\nabla u|^{\lambda-1} u_{x_i} \xi_{x_i} u_t \\
&= -\frac{\beta}{\lambda+1}(h+\beta-1)\lambda \iint \xi^s u^{h+\beta-2} |\nabla u|^{\lambda+1} u_t \\
&\quad + \frac{\beta s}{\lambda+1} \iint \xi^{s-1} \xi_t u^{h+\beta-1} |\nabla u|^{\lambda+1} \\
&\quad - \beta s \iint \xi^{s-1} u^{h+\beta-1} |\nabla u|^{\lambda-1} u_{x_i} \xi_{x_i} u_t \\
&= -\frac{\beta}{\lambda+1}(h+\beta-1)\lambda J_{16} + \frac{\beta s}{\lambda+1} J_{17} - \beta s J_{18}.
\end{aligned} \tag{2.11}
$$

Next, by (1.1) we have

$$
\begin{aligned}
J_9 &= \iint \xi^s u^{h-1} \Delta_\lambda u \, |\nabla u|^{\lambda+1} \\
&= \iint \xi^s u^{h-1} (u_t^\beta - u^p) \, |\nabla u|^{\lambda+1} \\
&\quad - \iint \xi^s u^{h+p-1} |\nabla u|^{\lambda+1} + \beta \iint \xi^s u^{h+p-2} |\nabla u|^{\lambda+1} u_t \\
&= -J_{14} + \beta J_{16},
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
J_{11} &= \iint \xi^{s-1} u^h \Delta_\lambda u \, |\nabla u|^{\lambda-1} u_{x_i} \xi_{x_i} \\
&= \iint \xi^{s-1} u^h (u_t^\beta - u^p) \, |\nabla u|^{\lambda-1} u_{x_i} \xi_{x_i} \\
&= -\iint \xi^{s-1} u^{p+h} |\nabla u|^{\lambda-1} u_{x_i} \xi_{x_i} \\
&\quad + \beta \iint \xi^{s-1} u^{h+\beta-1} u_t \, |\nabla u|^{\lambda-1} u_{x_i} \xi_{x_i} = -J_{15} + \beta J_{18}.
\end{aligned} \tag{2.13}
$$

Denote $E = J_8$, $F = J_{14}$. Then combining (2.9)–(2.13), from (2.8) we get

$$\begin{aligned}
&-C_3E + (C_4 - \frac{N-1}{N}(p+h))F \\
&\le -s\frac{N+1}{N}J_{15} + \beta(C_4 - \frac{(N-1)(h+\beta-1)\lambda}{N(\lambda+1)})J_{16} \\
&\quad - \frac{(N-1)\beta s}{N(\lambda+1)}J_{17} + \beta s\frac{N+1}{N}J_{18} + 2s\lambda(\alpha - 1 + d/(\lambda+1))J_{12}.
\end{aligned} \tag{2.14}$$

Let d and h be chosen as follows

$$C_3 < 0, \qquad C_4 - \frac{N-1}{N}(p+h) > 0. \tag{2.15}$$

Applying the Young inequality, we get

$$\begin{aligned}
|J_{15}| &= \left|\iint \xi^{s-1}u^{p+h}\left|\nabla u\right|^{\lambda-1}u_{x_i}\xi_{x_i}\right| \\
&\le \varepsilon F + C(\varepsilon)\iint \xi^{s-\lambda-1}u^{p+h+\lambda}\left|\nabla\xi\right|^{\lambda+1},
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
|J_{16}| &= \left|\iint \xi^{s}u^{h+\beta-2}\left|\nabla u\right|^{\lambda+1}u_t\right| \\
&\le \varepsilon E + C(\varepsilon)\iint \xi^{s}u^{h+2\beta-2}\left|\nabla\xi\right|^{\lambda+1}u_t^2,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
|J_{17}| &= \left|\iint \xi^{s-1}\xi_t u^{h+\beta-1}\left|\nabla u\right|^{\lambda+1}\right| \\
&\le \varepsilon E + C(\varepsilon)\iint \xi^{s-2}\xi_t^2 u^{h+2\beta}
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
|J_{18}| = &\left|\iint \xi^{s-1}u^{h+\beta-1}\left|\nabla u\right|^{\lambda-1}u_{x_i}\xi_{x_i}u_t\right| \\
&\le \varepsilon\iint \xi^{s-2}u^{h}\left|\nabla u\right|^{2\lambda}\left|\nabla\xi\right|^{2} + C(\varepsilon)\iint \xi^{s}u^{h+2\beta-2}u_t^2 \\
&\le \varepsilon E + C(\varepsilon)\iint \xi^{s}u^{h+2\beta-2}u_t^2 \\
&\quad + C(\varepsilon)\iint \xi^{s-2(\lambda+1)}u^{h+2\lambda}\left|\nabla\xi\right|^{2(\lambda+1)}.
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
|J_{12}| &= \left|\iint \xi^{s-1}u^{h-1}\left|\nabla u\right|^{2\lambda}u_{x_i}\xi_{x_i}\right| \\
&\le \varepsilon E + C(\varepsilon)\iint \xi^{s-2(\lambda+1)}u^{h+2\lambda}\left|\nabla\xi\right|^{2(\lambda+1)}.
\end{aligned} \tag{2.20}$$

Now we choose $h = 1 - \beta$. Then (2.15) holds true if a and b are positive which is the case. Lemma 2.2 is proved. $\square$

Notice that assumptions $a > 0$ and $b > 0$ are equivalent to

$$d^2 - 2d\frac{N\delta + N - 1 + \beta}{2N\delta + N - 1} + \frac{(N-1)(1-\beta)^2}{2N\delta + N - 1} < 0,$$

$$d > \frac{p(\lambda+1)(N-1)}{N+\lambda+1}.$$

The first of these inequalities is satisfied if

$$\frac{N\beta}{2N\delta + N - 1}(1 + \delta_1 + \theta - \sqrt{A}) < d < \frac{N\beta}{2N\delta + N - 1}(1 + \delta_1 + \theta + \sqrt{A}).$$

Therefore, both inequalities hold if $p < p_0(\beta, \lambda, N)$ which coincides with our assumption.

Next we need to bound the integral

$$J_{19} = \iint \xi^s u^{\beta-1} u_t^2.$$

Lemma 2.3. *The following inequality holds true*

$$\begin{aligned} J_{19} \le\ & 4\varepsilon E + C(\varepsilon) \iint \xi^{s-2(\lambda+1)} u^{2\lambda+1-\beta} |\nabla\xi|^{2(\lambda+1)} \\ & + C(\varepsilon) \iint \xi^{s-2} u^{1+\beta} \xi_t^2 + C(\varepsilon) \iint \xi^{s-1} u^{p+1} |\xi_t|. \end{aligned} \tag{2.21}$$

Proof. Multiply both sides of (1.1) by $u_t \xi^s$ and integrate by parts to get

$$\begin{aligned} \beta J_{19} = & -\frac{1}{\lambda+1} \iint \xi^s \left(|\nabla u|^{\lambda+1}\right)_t + \frac{1}{p+1} \iint \xi^s (u^{p+1})_t \\ & - s \iint \xi^{s-1} |\nabla u|^{\lambda-1} u_t u_{x_i} \xi_{x_i}. \end{aligned} \tag{2.22}$$

The right-hand side is equal to

$$\frac{s}{\lambda+1} \iint \xi^{s-1} \xi_t |\nabla u|^{\lambda+1} - \frac{s}{p+1} \iint \xi^{s-1} \xi_t u^{p+1} - s \iint \xi^{s-1} |\nabla u|^{\lambda-1} u_t u_{x_i} \xi_{x_i}.$$

By Young's inequality we have

$$\frac{s}{\lambda+1} \iint \xi^{s-1} \xi_t |\nabla u|^{\lambda+1} \le \varepsilon E + C(\varepsilon) \iint \xi^{s-2} u^{1+\beta} \xi_t^2,$$

$$\begin{aligned} s\left|\iint \xi^{s-1} |\nabla u|^{\lambda-1} u_t u_{x_i} \xi_{x_i}\right| & \le \frac{1}{2} J_{19} + \frac{1}{2} \iint u^{1-\beta} \xi^{s-2} |\nabla\xi|^2 |\nabla u|^{2\lambda} \\ & \le \frac{1}{2} J_{19} + \varepsilon E + C(\varepsilon) \iint \xi^{s-2(\lambda+1)} u^{2\lambda+1-\beta} |\nabla\xi|^{2(\lambda+1)}. \end{aligned}$$

Therefore, from (2.22) we arrive at the desired result. □

We continue the proof of Theorem 1.1. From Lemma 2.2 and (2.15)–(2.21), one gets

$$\begin{aligned}(a-9\varepsilon)E+(b-2\varepsilon)F \le \gamma(\varepsilon)&\iint \xi^{s-\lambda-1}u^{p+\lambda+1-\beta}\left|\nabla\xi\right|^{\lambda+1}\\ &+\gamma(\varepsilon)\iint \xi^{s-2(\lambda+1)}u^{2\lambda+1-\beta}\left|\nabla\xi\right|^{2(\lambda+1)}\\ &+\gamma(\varepsilon)\iint \xi^{s-2}u^{1+\beta}\xi_t^2+\iint \xi^{s-1}u^{p+1}\left|\xi_t\right|.\end{aligned}\tag{2.23}$$

Denote

$$M_1=\iint \xi^{s-(\lambda+1)\frac{2p+1-\beta}{p-\lambda}}\left|\nabla\xi\right|^{(\lambda+1)\frac{2p+1-\beta}{p-\lambda}},$$
$$M_2=\iint \xi^{s-\frac{2p+1-\beta}{p-\beta}}\left|\xi_t\right|^{\frac{2p+1-\beta}{p-\beta}}.$$

Applying the Young inequality, we have

$$\begin{aligned}\iint \xi^{s-\lambda-1}u^{p+\lambda+1-\beta}\left|\nabla\xi\right|^{\lambda+1} &\le \frac{p+\lambda+1-\beta}{2p+1-\beta}L+\frac{p-\lambda}{2p+1-\beta}M_1,\\ \iint \xi^{s-2(\lambda+1)}u^{2\lambda+1-\beta}\left|\nabla\xi\right|^{2(\lambda+1)} &\le \frac{2\lambda+1-\beta}{2p+1-\beta}L+\frac{2(p-\lambda)}{2p+1-\beta}M_1,\\ \iint \xi^{s-2}u^{1+\beta}\xi_t^2 &\le \frac{1+\beta}{2p+1-\beta}L+\frac{2(p-\beta)}{2p+1-\beta}M_2,\\ \iint \xi^{s-1}u^{p+1}\left|\xi_t\right| &\le \frac{p+1}{2p+1-\beta}L+\frac{p-\beta}{2p+1-\beta}M_2,\end{aligned}\tag{2.24}$$

where

$$L=\iint \xi^s u^{2p+1-\beta}.$$

In order to estimate the last integral we multiply both sides of (1.1) by $u^{p+1-\beta}\xi^s$ and integrate by parts, apply also Young's inequality to get

$$\begin{aligned}L&=\frac{\beta}{\lambda+1}\iint \xi^s(u^{p+1})_t+(p+1-\beta)F+s\iint \xi^{s-1}u^{p+1-\beta}\left|\nabla u\right|^{\lambda-1}u_{x_i}\xi_{x_i}\\ &\le \frac{s\beta}{p+1}\iint \xi^{s-1}\left|\xi_t\right|u^{p+1}+(p+1-\beta+\frac{s^{(\lambda+1)/\lambda}\lambda}{\lambda+1})F\\ &\quad+\frac{1}{\lambda+1}\iint \xi^{s-\lambda-1}u^{p+\lambda+1-\beta}\left|\nabla\xi\right|^{\lambda+1}\\ &\le \frac{s\beta}{p+1}\varepsilon_1 L+\left(p+1-\beta+\frac{s^{(\lambda+1)/\lambda}\lambda}{\lambda+1}\right)F\\ &\quad+\frac{1}{\lambda+1}\varepsilon_1 L+(\frac{s\beta}{p+1}+\frac{1}{\lambda+1})C(\varepsilon_1)(M_1+M_2).\end{aligned}$$

Therefore for a sufficiently small ε_1, we get

$$L \leq \gamma(p,\beta,\lambda)(F + M_1 + M_2)$$

and together with (2.23) and (2.24) with a suitable ε this gives

$$L + F \leq \gamma(M_1 + M_2). \tag{2.25}$$

Let $G = B_R(x_0)$ for any fixed $x_0 \in \mathbb{R}^N$, $t_1 = T_1$, $t_2 = t$ and for $0 < T_1 < T_2 < t$, ξ is so that $|\nabla\xi| \leq cR^{-1}$, $|\xi_\tau| \leq c(T_2 - T_1)^{-1}$ for $0 < T_1 < \tau < t < T$ and any $R > 0$. Then

$$M_1 + M_2 \leq cR^N \left((T_2 - T_1)R^{-\frac{(\lambda+1)(2p+1-\beta)}{p-\lambda}} + (T_2 - T_1)^{-\frac{2p+1-\beta}{p-\beta}}\right). \tag{2.26}$$

We have

$$\sup_{T_1<\tau<t} \int\limits_{B_R(x_0)} u^{p+1} \mathrm{d}x \leq c(L + F).$$

Indeed, this follows from Lemma 2.3, (2.24) and (2.25) observing that

$$\begin{aligned}
\int\limits_{B_R(x_0)} \xi^s(x,\tau)u^{p+1}(x,\tau) &= (p+1)\int\limits_{T_1}^{\tau}\int\limits_{B_R(x_0)} \xi^s u^p u_t + s\int\limits_{T_1}^{\tau}\int\limits_{B_R(x_0)} \xi^{s-1}\xi_t u^{p+1} \\
&\leq (p+1)\left(\int\limits_{T_1}^{\tau}\int\limits_{B_R(x_0)} \xi^s u^{\beta-1}u_t^2\right)^{1/2}\left(\int\limits_{T_1}^{\tau}\int\limits_{B_R(x_0)} \xi^s u^{2p+1-\beta}\right)^{1/2} \\
&\quad + s\int\limits_{T_1}^{\tau}\int\limits_{B_R(x_0)} \xi^{s-1}\left|\xi_t\right| u^{p+1}.
\end{aligned}$$

Therefore from (2.25) and (2.26), we have

$$\begin{aligned}
&\sup_{T_1<\tau<t} \int\limits_{B_R(x_0)} u^{p+1} \mathrm{d}x \\
&\leq cR^N \left((T_2 - T_1)R^{-\frac{(\lambda+1)(2p+1-\beta)}{p-\lambda}} + (T_2 - T_1)^{-\frac{2p+1-\beta}{p-\beta}}\right).
\end{aligned} \tag{2.27}$$

Now we are in a position to complete the proof of Theorem 1.1. The final step of the proof is to utilize the local estimate of Lemma 3.3 from [**8**] which we write in the suitable form

$$\begin{aligned}
\|u\|_{\infty,B_{R/2}(x_0)\times(T_2,t)} \leq\ & (T_2 - T_1)^{-1/(p-\beta)} + (R/2)^{-(\lambda+1)/(p-\lambda)} \\
& + c(t - T_1)^{1/(\omega-p)}\left(\sup_{T_1<\tau<t}\int\limits_{B_R(x_0)} u^{p+1}\mathrm{d}x\right)^{\mu/(\omega-p)}
\end{aligned} \tag{2.28}$$

for all $0 < T_1 < T_2 < t < T$ provided $p < p_S$ and $\omega > p + 1$ is a free parameter,

$$\mu = \frac{(\lambda+1)(\omega - p - 1)\beta + p}{(p+1)\beta(\lambda+1) - (p-\lambda)N}.$$

Finally, in (2.27) and (2.28) choosing $T_2 = t - (T - t)/2$, $T_1 = t - (T - t)$, $R = (T - t)^{(p-\lambda)/(\lambda+1)(p-\beta)}$ with $t \in (T/2, T)$ and noting that x_0 is an arbitrary point of $\mathbb{R}^N$, we arrive at the desired result.

The proof of Theorem 1.1 is complete. $\square$

References

1. Afanas'eva N. V. and Tedeev A. F., *Fujita type theorems for quasilinear parabolic equations with initial data slowly decaying to zero,* Sbornik: Mathematics, 195:4 (2004), 459–478.

2. ______, *Theorems on the existence and nonexistence of solutions of the Cauchy problem for degenerate parabolic equations with non local sources,* Ukrainian Mathematical Journal **57(11)** (2005), 1687–1711.

3. Ammar K. and Souplet Ph., *Liouville-type theorems and universal bounds for positive solutions of the porous medium equation with source*, Discrete and Continuous Dynamical Systems **26** (2010), 665–689.

4. Andreucci D. and DiBenedetto E., *On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources*, Annali Sc. Normale Sup. Pisa, **18** (1991), 363–441.

5. Andreucci D., Herrero M. and Velazquez J .J., *Liouville theorems and blow-up behaviour in semilinear reaction diffusion systems*, Annales Non Linéaire. I'Inst. H. Poincaré **14** (1997), 1–53.

6. Andreucci D. and Tedeev A. F., *A Fujita type result for a degenerate Neumann problem in domains with non compact boundary*, J. Math. Analysis and Appl. **231** (1999), 543–567.

7. ______, *Optimal bounds and blow up phenomena for parabolic problems in narrowing domains*, Proc. Roy. Soc. Edinburgh Sect. A **128(6)** (1998), 1163–1180.

8. ______, *Universal bounds at the blow-up time for nonlinear parabolic equations*, Adv. Differential Equations **10(1)** (2005), 89–120.

9. Andreucci D., Tedeev A. F., and Ughi M., *The Cauchy problem for degenerate parabolic equations with source and damping*, Ukrainian Mathematical Bulletin, **1(1)** (2004), 1–23.

10. Antontsev S. N., Diaz J. I. and Shmarev S., 2002, *Energy methods for the free boundary problems. Application to nonlinear PDEs and fluid mechanics*, Boston-Birkhäuser, 2002.

11. Bidaut-Véron M.-F., *Initial blow-up for the solutions of a semilinear parabolic equation with source term, Equations aux dérivées partielles et applications, articles dédiés à Jacques-Louis Lions*, Gauthier-Villars, Paris, 1998, 189–198.

12. ______, *The p-Laplace heat equation with a source term: self-similar solutions revisited*, Advanced Nonlinear Studies **5** (2005), 1–41.

13. Deng K. and Levine H. A., *The role of critical exponents in blow-up theorems, the sequel*, J. Math. Anal. Appl. **243** (2000), 85–126.

14. DiBenedetto E., *Degenerate parabolic equations*, Springer-Verlag, New-York, NY, 1993.

15. Esteban J. R. and Vázquez J. L., *Régularité des solutions positives de l'équation parabolique p-Laplacienne*, C.R. Acad. Sci. Paris **310**, Serie I (1990), 105–110.

16. Fabricant A., Marinov M., and Rangelov Ts., *Some properties of nonlinear degenerate parabolic equations*, Mathematica Balkanica. New series, **8(1)** (1994), 59–73.

17. Galaktionov V. A., Kurdyumov S., Mikhailov A., and Samarskii A., *Blow-up in quasilinear parabolic equations,* Nauka, Moscow, 1987, English translation: Gruyter Expositions in Mathematics, no. 19, Walter de Gruyter, Berlin, 1995.

18. Galaktionov V. A. and Vázquez J. L., *The problem of blow-up in nonlinear parabolic equations*, Discrete and Continuous Dynamical Systems **8** (2002), 399–433.

19. Giga Y. and Kohn R., *Asymptotically self similar blow-up of semilinear heat equation*, Coom. Pure Appl. Math., **38** (1985), 297–319.

20. ______, *Characterizing blow-up using similarity variables*, Indiana Univ. Math. J. **36** (1987), 1–40.

21. ———, *Nondegeneracy of blow-up for semilinear heat equations*, Commun. Pure Appl. Math., **42** (1989), 845–884.

22. Ishige K., *On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equations*, SIAM J. Math. Anal. , **27(5)** (1996), 1235–1260.

23. Ivanov A. V., *Hölder estimates near the boundary for generalized solutions of quasilinear parabolic equations that admit double degeneration*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov, **188** (1991), 45–69.

24. Kalashnikov A. S., *Some problems on the qualitative theory of nonlinear degenerate second-order parabolic equations*, Uspekhi Mat. Nauk, **42** (1987), 135–176, English translation in Russian Math. Surveys, **42** (1987), 169–222.

25. Ladyzhenskaya O. A., Solonnikov V. A., and Ural'ceva N. N., *Linear and quasilinear equations of parabolic type*, vol. 23 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1968.

26. Martynenko A. V. and Tedeev A. F., *The Cauchy problem for a quasilinear parabolic equation with a source and nonhomogeneous density*, (Russian) Zh. Vychisl. Mat. Mat. Fiz. **47(2)** (2007), 245–255; translation in Comput. Math. Math. Phys. **47(2)** (2007), 238–248.

27. Matos J. and Souplet Ph., *Universal blow-up rates for semilinear heat equation and applications*, Adv. Differential Equations, **8** (2003), 615–639.

28. Merle F. and Zaag H., *Refined uniform estimates at blow-up and applications for nonlinear heat equations*, Comm. pure appl. Math., **51** (1998), 139-196.

29. Otto F., L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations, J. Differential Equations, **131** (1996), 20–38.

30. Poláčik P., Quittner P., and Souplet Ph., *Singularity and decay estimates in superlinear problems via Liouville-type theorems*, Part 2. Parabolic Equations, Indiana Univ. Math. J., **56** (2007), 879–908.

31. Porzio M. and Vespri V., *Hölder estimates for local solutions of some doubly nonlinear parabolic equations*, J. Differential Equations, **103** (1993), 141–178.

32. Quittner P., *Universal bounds for global solutions of a superlinear parabolic problems*, Math. Ann., **320** (2001), 299–305.

33. Quittner P., Souplet Ph., *Superlinear Parabolic Problems, Blow-up, global existence and steady states*, Birkhäuser Advanced Text, 2007.

34. Quittner P., Souplet Ph., and Winkler M., *Initial blow-up rates and universal bounds for nonlinear heat equations*, J. Differential Equations, **196** (2004), 316-339.

35. Serrin J. and Zou H., *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math., **189** (2002), 79–142.

36. Souplet Ph., *An optimal Liouville-type theorem for radial entire solutions of the porous medium equation with source*, J. Differential Equations, **246** (2009), 3980–4005.

37. Tedeev A. F., *The interface blow-up phenomenon and local estimates for doubly degenerate parabolic equations*, Appl. Anal, V. **86(6)** (2007), 755–782.

38. Vazquez J. L., *The Porous Medium Equation. Mathematical theory.* Oxford Mathematical Monographs. Clarendon Press, Oxford, 2007.

39. Winkler M., *Universal bounds for global solutions of a forced porous medium equation*, Nonlinear Anal. **57** (2004), 349–362.

A. F. Tedeev, Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, str. R. Luksemburg 74, Donetsk, 340114, Ukraine, *e-mail*: `tedeev@iamm.ac.donetsk.ua`