Acta Math. Univ. Comenianae
Vol. LXXX, 2 (2011), pp. 263–277

g-NATURAL METRICS ON TANGENT BUNDLES AND JACOBI OPERATORS

S. DEGLA AND L. TODJIHOUNDE

ABSTRACT. Let (M, g) be a Riemannian manifold and G a nondegenerate g-natural metric on its tangent bundle TM. In this paper we establish a relation between the Jacobi operators of (M, g) and that of (TM, G).

In the case of a Riemannian surface (M, g), we compute explicitly the spectrum of some Jacobi operators of (TM, G) and give necessary and sufficient conditions for (TM, G) to be an Osserman manifold.

0. INTRODUCTION

In [**1**] the authors introduced g-natural metrics on the tangent bundle TM of a Riemannian manifold (M, g) as metrics on TM which come from g through first order natural operators defined between the natural bundle of Riemannian metrics on M and the natural bundle of $(0, 2)$-tensors fields on the tangent bundles. Classical well-known metrics like Sasaki metric (cf. [**14**] , [**6**]) or Cheeger-Gromoll metrics (cf. [**3**] , [**11**]) are examples of natural metrics on the tangent bundle. By associating the notion of F-tensors fields they got a characterization of g-natural metrics on TM in terms of the basis metric g and of some functions defined on the set of positive real numbers and obtained necessary and sufficient conditions for g-natural metrics to be either nondegenerate or Riemannian (see [**8**] for more details on natural operators and F-tensors fields).

Some geometrical properties of g-natural metrics are inherited from the basis metric g and conversely (cf. [**1**], [**2**], [**7**], [**10**]). We will investigate in this paper the property of being Osserman which is closely related to the spectrum of Jacobi operators.

Recall that for a tangent vector $X \in T_xM$ with $x \in M$, the Jacobi operator J_X is defined as the linear self-adjoint map

$$\begin{array}{llll} J_X : & T_xM & \to & T_xM \\ & Y & \mapsto & J_X(Y) := R(X, Y)X\,, \end{array}$$

Received December 10, 2010; revised June 3, 2011.
2010 *Mathematics Subject Classification.* Primary 53B20, 53C07; Secondary 53A55, 53C25.
Key words and phrases. F-tensor fields; g-natural metrics; Jacobi operators; Osserman manifolds.

where R denotes the Riemannian curvature operator of (M, g). Osserman manifolds are defined as follows:

Definition 0.1.

1. Let $x \in M$. (M, g) is Osserman at x if, for any unit tangent vector $X \in T_xM$, the eigenvalues of the Jacobi operator J_X do not depend on X.
2. (M, g) is pointwise Osserman if it is Osserman at any point of M.
3. (M, g) is globally Osserman manifold if, for any point $x \in M$ and any unit tangent vector $X \in T_xM$, the eigenvalues of the Jacobi operator J_X depend neither on X nor on x.

Globally Osserman manifolds are obviously pointwise Osserman.

Remark 0.1. For any point $x \in M$ the map defined on T_xM by $X \longmapsto J_X$ satisfies the identity $J_{\lambda X} = \lambda^2 J_X$, $\forall\, \lambda \in \mathbb{R}$. So the spectrum of $J_{\lambda X}$ is, up to the factor $\frac{1}{\lambda^2}$, the same as that one of J_X. Thus (M, g) is Osserman at $x \in M$ if and only if for any vector $X \in T_xM$ with $X \neq 0$ and for any eigenvalue $\lambda(X)$ of J_X, the quotient $\frac{\lambda(X)}{g(X,X)}$ does not depend on X.

Flat manifolds or locally symmetric spaces of rank one are examples of globally Osserman manifolds since the local isometry group acts transitively on the unit tangent bundle, and hence the eigenvalues of the Jacobi operators are constant on the unit tangent bundle.

Osserman conjectured that the converse holds; that is all Osserman manifolds are locally symmetric of rank one. The Osserman conjecture was proved in many special cases (cf. [**4**], [**12**], [**13**], [**15**]).

Using the fact that (M, g) is totally geodesic in (TM, G) (cf. [**1**]) we show that any eigenvalue of a Jacobi operator of (M, g) is an eigenvalue of a Jacobi operator of its g-natural tangent bundle (TM, G). Furthermore, we investigate the Jacobi operators of g-natural metrics on tangent bundles of Riemannian surfaces, and we compute their spectrums explicitly. Then we establish necessary and sufficient conditions for g-natural tangent bundles of Riemannian surfaces to be Osserman manifolds.

1. Preliminaries

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection of g. The tangent space of TM at a point $(x, u) \in TM$ splits into the horizontal and vertical subspaces with respect to ∇

$$T_{(x,u)}TM = H_{(x,u)}M \oplus V_{(x,u)}M.$$

A system of local coordinates $(U\,;\, x_i,\ i = 1, \ldots, m)$ in M induces on TM a system of local coordinates $\left(\pi^{-1}(U)\,;\, x_i, u^i,\ i = 1, \ldots, m\right)$. Let $X = \sum_{i=1}^m X^i \frac{\partial}{\partial x_i}$ be the local expression in U of a vector field X on M. Then, the horizontal lift X^h and the vertical lift X^v of X are given, with respect to the induced coordinates, by:

$$X^h = \sum_i X^i \frac{\partial}{\partial x_i} - \sum_{i,j,k} \Gamma^i_{jk} u^j X^k \frac{\partial}{\partial u^i} \quad \text{and} \tag{1}$$

$$X^v = \sum_i X^i \frac{\partial}{\partial u^i}, \tag{2}$$

where the Γ^i_{jk} are the Christoffel's symbols defined by g.

Next, we introduce some notations which will be used to describe vectors obtained from lifted vectors by basic operations on TM. Let T be a tensor field of type $(1,s)$ on M. If $X_1, X_2, \ldots, X_{s-1} \in T_xM$, then $h\{T(X_1,\ldots,u,\ldots,X_{s-1})\}$ and $v\{T(X_1,\ldots,u,\ldots,X_{s-1})\})$ are horizontal and vertical vectors respectively at the point (x,u) which are defined by

$$h\{T(X_1,\ldots,u,\ldots,X_{s-1})\} = \sum u^\lambda \left(T(X_1,\ldots,\frac{\partial}{\partial x_\lambda}_{|x},\ldots,X_{s-1})\right)^h$$

$$v\{T(X_1,\ldots,u,\ldots,X_{s-1})\} = \sum u^\lambda \left(T(X_1,\ldots,\frac{\partial}{\partial x_\lambda}_{|x},\ldots,X_{s-1})\right)^v.$$

In particular, if T is the identity tensor of type $(1,1)$, then we obtain the geodesic flow vector field at (x,u), $\xi_{(x,u)} = \sum_\lambda u^\lambda \left(\frac{\partial}{\partial x_\lambda}\right)^h_{(x,u)}$, and the canonical vertical vector at (x,u), $\mathcal{U}_{(x,u)} = \sum_\lambda u^\lambda \left(\frac{\partial}{\partial x_\lambda}\right)^v_{(x,u)}$.

Also $h\{T(X_1,\ldots,u,\ldots,u,\ldots,X_{s-t})\}$ and $v\{T(X_1,\ldots,u,\ldots,u,\ldots,X_{s-t})\}$ are defined in a similar way.

Let us introduce the notations

$$h\{T(X_1,\ldots,X_s)\} =: T(X_1,\ldots,X_s)^h \tag{3}$$

and

$$v\{T(X_1,\ldots,X_s)\} =: T(X_1,\ldots,X_s)^v. \tag{4}$$

Thus $h\{X\} = X^h$ and $v\{X\} = X^v$ for each vector field X on M.

From the preceding quantities, one can define vector fields on TU in the following way: If $u = \sum_i u^i \left(\frac{\partial}{\partial x_i}\right)_x$ is a given point in TU and $X_1,\ldots,X_{s-1}$ are vector fields on U, then we denote by

$$h\{T(X_1,\ldots,u,\ldots,X_{s-1})\} \quad (\text{respectively} \quad v\{T(X_1,\ldots,u,\ldots,X_{s-1})\})$$

the horizontal (respectively vertical) vector field on TU defined by

$$h\{T(X_1,\ldots,u,\ldots,X_{s-1})\} = \sum_\lambda u^\lambda T\left(X_1,\ldots,\frac{\partial}{\partial x_\lambda},\ldots,X_{s-1}\right)^h$$

$$(\text{resp.} \quad v\{T(X_1,\ldots,u,\ldots,X_{s-1})\} = \sum_\lambda u^\lambda T\left(X_1,\ldots,\frac{\partial}{\partial x_\lambda},\ldots,X_{s-1}\right)^v).$$

Moreover, for vector fields $X_1, \dots, X_{s-t}$ on U, where $s, t \in \mathbb{N}^*$ $(s > t)$, the vector fields $h\{T(X_1, \dots, u, \dots, u, \dots, X_{s-t})\}$ and $v\{T(X_1, \dots, u, \dots, u, \dots, X_{s-t})\}$ on TU, are defined by similar way.

Now, for $(r, s) \in \mathbb{N}^2$, we denote by $\pi_M \colon TM \to M$ the natural projection and by F the natural bundle defined by

$$(5)\qquad \begin{aligned} FM &= \pi_M^*(\underbrace{T^* \otimes \ldots \otimes T^*}_{r \text{ times}} \otimes \underbrace{T \otimes \ldots \otimes T}_{s \text{ times}})M \to M, \\ Ff(X_x, S_x) &= (Tf \cdot X_x, (T^* \otimes \ldots \otimes T^* \otimes T \otimes \ldots \otimes T)f \cdot S_x) \end{aligned}$$

for $x \in M$, $X_x \in T_xM$, $S \in (T^* \otimes \ldots \otimes T^* \otimes T \otimes \ldots \otimes T)M$ and any local diffeomorphism f of M.

We call the sections of the canonical projection $FM \to M$ F-tensor fields of type (r, s). So, if we denote the product of fibered manifolds by $\oplus$, then the F-tensor fields are mappings $A : \ TM \oplus \underbrace{TM \oplus \ldots \oplus TM}_{s \text{ times}} \to \bigsqcup_{x \in M} \otimes^r T_xM$ which are linear in the last s summands and such that $\pi_2 \circ A = \pi_1$, where π_1 and π_2 are respectively the natural projections of the source and target fiber bundles of A. For $r = 0$ and $s = 2$, we obtain the classical notion of F-metrics. So, F-metrics are mappings $TM \oplus TM \oplus TM \to \mathbb{R}$ which are linear in the second and the third arguments.

Moreover, let us fix $(x, u) \in TM$ and a system of normal coordinates $S := (U; x_i\,, i = 1, \dots, m)$ of (M, g) centered at x. Then we can define on U the vector field $\mathbf{U} := \sum_i u^i \frac{\partial}{\partial x_i}$, where $(u^1, \dots, u^m)$ are the coordinates of $u \in T_xM$ with respect to its basis $(\frac{\partial}{\partial x_i}{}_{|x};\, i = 1, \dots, m)$.

Let P be an F-tensor field of type (r, s) on M. Then on U we can define an (r, s)-tensor field P_u^S (or P_u if there is no risk of confusion) associated with u and S by

$$(6)\qquad P_u(X_1, \dots, X_s) := P(\mathbf{U}_z; X_1, \dots, X_s),$$

for all $(X_1, \dots, X_s) \in T_zM$ and all $z \in U$.

On the other hand, if we fix $x \in M$ and s vectors $X_1, \dots, X_s$ in T_xM, then we can define a C^∞-mapping $P_{(X_1,\dots,X_s)} \colon T_xM \to \otimes^r T_xM$, associated with $(X_1, \dots, X_s)$ by

$$(7)\qquad P_{(X_1,\dots,X_s)}(u) := P(u;\, X_1, \dots, X_s),$$

for all $u \in T_xM$.

Let s, t where $s > t$ be two non-negative integers, T be a $(1, s)$-tensor field on M and P^T be an F-tensor field of type $(1, t)$ of the form

$$(8)\qquad P^T(u; X_1, \dots, X_t) = T(X_1, \dots, u, \dots, u, \dots, X_t),$$

for all $(u; X_1, \dots, X_t) \in TM \oplus \ldots \oplus TM$, i.e., u appears $s - t$ times at positions $i_1, \dots, i_{s-t}$ in the expression of T. Then

- P_u^T is a $(1, t)$-tensor field on a neighborhood U of x in M, for all $u \in T_xM$;
- $P_{(X_1,\dots,X_t)}^T$ is a C^∞-mapping $T_xM \to T_xM$, for all $X_1, \dots, X_t$ in T_xM.

Furthermore, it holds:

Lemma 1.1 ([**2**]).

1) *The covariant derivative of* P_u^T *with respect to the Levi-Civita connection of* (M,g) *is given by*

$$\left(\nabla_X P_u^T\right)(X_1,\dots,X_t) = (\nabla_X T)(X_1,\dots,u,\dots,u,\dots,X_t), \tag{9}$$

for all vectors $X, X_1,\dots,X_t \in T_xM$, *where* u *appears at positions* $i_1,\dots,i_{s-t}$ *in the right-hand side of the preceding formula.*

2) *The differential of* $P^T_{(X_1,\dots,X_t)}$ *at* $u \in T_xM$, *is given by*

$$\begin{aligned} d\left(P^T_{(X_1,\dots,X_t)}\right)_u (X) = {} & T(X_1,\dots,X,\dots,u,\dots,X_t) + \dots \\ & + T(X_1,\dots,u,\dots,X,\dots,X_t), \end{aligned} \tag{10}$$

for all $X \in T_xM$.

2. g-NATURAL METRICS ON TANGENT BUNDLES

Definition 2.1. Let (M,g) be a Riemannian manifold. A g-natural metric on the tangent bundle of M is a metric on TM which is the image of g by a first order natural operator defined from the natural bundle of Riemannian metrics $S^2_+T^*$ on M into the natural bundle of $(0,2)$-tensor fields $(S^2T^*)T$ on the tangent bundles (cf. [**1**] , [**2**]).

Tangent bundles equipped with g-natural metrics are called g-natural tangent bundles.

The following result gives the classical expression of g-natural metrics

Proposition 2.1 ([**1**]). *Let* (M,g) *be a Riemannian manifold and* G *a* g*-natural metric on* TM. *There exist six smooth functions* $\alpha_i,\ \beta_i : \mathbb{R}^+ \to \mathbb{R},\ i = 1,2,3$, *such that for any* $x \in M$, *all vectors* u *and* $X,\ Y \in T_xM$, *we have*

$$\begin{aligned} G_{(x,u)}\left(X^h,Y^h\right) &= (\alpha_1+\alpha_3)(t)g_x(X,Y) + (\beta_1+\beta_3)(t)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}\left(X^h,Y^v\right) &= \alpha_2(t)g_x(X,Y) + \beta_2(t)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}\left(X^v,Y^h\right) &= \alpha_2(t)g_x(X,Y) + \beta_2(t)g_x(X,u)g_x(Y,u), \\ G_{(x,u)}\left(X^v,Y^v\right) &= \alpha_1(t)g_x(X,Y) + \beta_1(t)g_x(X,u)g_x(Y,u), \end{aligned}$$

where $t = g_x(u,u)$, X^h *and* X^v *respectively, are the horizontal lift and the vertical lift of the vector* $X \in T_xM$ *at the point* $(x,u) \in TM$.

Notation 2.1.

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t), \quad i = 1,2,3,$
- $\alpha(t) = \alpha_1(t)(\alpha_1+\alpha_3)(t) - \alpha_2^2(t),$
- $\phi(t) = \phi_1(t)(\phi_1+\phi_3)(t) - \phi_2^2(t)$

for all $t \in \mathbb{R}^+$.

For a g-natural metric to be nondegenerate or Riemannian, there are some conditions to be satisfied by the functions α_i and β_i of Proposition 2.1. It holds:

Proposition 2.2 ([**1**]). *A g-natural metric G on the tangent bundle of a Riemannian manifold $(M,\ g)$ is:*

(i) *nondegenerate if and only if the functions α_i, β_i, $i = 1, 2, 3$ defining G are such that*

$$\alpha(t)\phi(t) \neq 0 \tag{11}$$

for all $t \in \mathbb{R}^+$.

(ii) *Riemannian if and only if the functions α_i, β_i, $i = 1, 2, 3$ defining G, satisfy the inequalities*

$$\begin{cases} \alpha_1(t) > 0, & \phi_1(t) > 0, \\ \alpha(t) > 0, & \phi(t) > 0, \end{cases} \tag{12}$$

for all $t \in \mathbb{R}^+$.

For $\dim M = 1$*, this system reduces to $\alpha_1(t) > 0$ and $\alpha(t) > 0$ for all $t \in \mathbb{R}^+$.*

Before giving the formulas relating both Levi-Civita connexions ∇ of (M, g) and $\overline{\nabla}$ of (TM, G), let us introduce the following notations:

Notation 2.2. For a Riemannian manifold (M, g), we set:

$$\begin{array}{ll} T^1(u; X_x, Y_x) = R(X_x, u)Y_x, & T^2(u; X_x, Y_x) = R(Y_x, u)X_x\,, \\ T^3(u; X_x, Y_x) = R(X_x, Y_x)u, & T^4(u; X_x, Y_x) = g(R(X_x, u)Y_x, u)u\,, \\ T^5(u; X_x, Y_x) = g(X_x, u)Y_x, & T^6(u; X_x, Y_x) = g(Y_x, u)X_x\,, \\ T^7(u; X_x, Y_x) = g(X_x, Y_x)u, & T^8(u; X_x, Y_x) = g(X_x, u)g(Y_x, u)u, \end{array} \tag{13}$$

where $(x, u) \in TM$, $X_x, Y_x \in T_xM$ and R is the Riemannian curvature of g.

For the g-natural metric G being defined by the functions α_i, β_i of Proposition 2.1, the following equations hold.

Proposition 2.3 ([**7**]). *Let $(x, u) \in TM$ and $X, Y \in \mathfrak{X}(M)$. We have*

$$\left(\overline{\nabla}_{X^h} Y^h\right)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} + h\{A(u; X_x, Y_x)\} + v\{B(u; X_x, Y_x)\} \tag{14}$$

$$\left(\overline{\nabla}_{X^h} Y^v\right)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + h\{C(u; X_x, Y_x)\} + v\{D(u; X_x, Y_x)\} \tag{15}$$

$$\left(\overline{\nabla}_{X^v} Y^h\right)_{(x,u)} = h\{C(u; Y_x, X_x)\} + v\{D(u; Y_x, X_x)\} \tag{16}$$

$$\left(\overline{\nabla}_{X^v} Y^v\right)_{(x,u)} = h\{E(u; Y_x, X_x)\} + v\{F(u; Y_x, X_x)\} \tag{17}$$

where $P(u; X_x, Y_x) = \sum_{i=1}^{8} f_i^P(|u|^2)T^i(u; X_x, Y_x)$ for $P = A, B, C, D, E, F$, and the functions f_i^P defined as in [**7**].

In [**1**] the authors notified that the Riemannian manifod (M, g), considered as an embedded submanifold in its g-natural tangent bundle (TM, G) by the null section, is always totally geodesic.

Indeed the null section S_0 of $\mathfrak{X}(M)$ is defined by

$$\begin{aligned} S_0 \colon M &\to TM \\ x &\mapsto (x,\ 0_x), \end{aligned} \tag{18}$$

which determines an embedding of M in TM.

Its differential at any point $x \in M$ is given by

$$\begin{aligned} dS_{0|x}\colon T_xM &\to T_{(x,\,0_x)}TM \\ X_x &\mapsto X^h_{(x,0_x)}. \end{aligned} \tag{19}$$

Then according to (14) and (19), we have

$$\overline{\nabla}_{S_{0*}X} S_{0*}Y = \overline{\nabla}_{X^h \circ S_0}(Y^h \circ S_0) = S_{0*}(\nabla_X Y), \tag{20}$$

for all $X, Y \in \mathfrak{X}(M)$.

Thus from the relation (20) we get the next proposition.

Proposition 2.4. [**1**] *Any Riemannian manifold (M, g) is totally geodesic in its tangent bundle TM equipped with a non-degenerate g-natural metric G.*

Remark 2.1. If G is nondegerate, then the orthogonal of $S_0(M) \equiv M$ in (TM, G) is given by

$$\begin{aligned} T_xM^{\perp_G} = \{&H^h_{(x,0_x)} + V^v_{(x,0_x)} \in T_{(x,\,0_x)}TM; \\ &H, V \in T_xM \text{ and } (\alpha_1 + \alpha_3)H + \alpha_2 V = 0_x\}, \end{aligned} \tag{21}$$

where the functions α_i, $i = 1, 2, 3$ are evaluated at 0.

3. Jacobi operators and Osserman g-natural tangent bundles

In the above section, we mentioned that (M, g) is totally geodesic in (TM, G). By using this observation we get the following result:

Proposition 3.1. *Assume that* $\dim M \geq 2$ *and $x \in M$. If λ is an eigenvalue of a Jacobi operator J_X for $X \in S(T_xM)$, then λ is an eigenvalue of the Jacobi operator $\bar{J}_{X^h_{(x,0_x)}}$ of G at the point $(x, 0_x) \in TM$.*

Proof. In (T_xM, g_x) let us choose an orthonormal basis $(X_1, \dots X_m)$ such as $X_1 = X$ and an orthonormal basis $(V_1, \dots V_m)$ in $T_xM^{\perp_G}$.
Then $(X^h_1|_{(x,0_x)}, \dots, X^h_m|_{(x,0_x)}, V_1, \dots, V_m)$ is an orthogonal basis of $T_{(x,0_x)}TM$. Since (M, g) is totally geodesic in (TM, G) and $\bar{J}_{X^h_{(x,0_x)}}$ is self-adjoint, the matrix of $\bar{J}_{X^h_{(x,0_x)}}$ in this basis has the form $\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$, where J_1 is the matrix of J_X in the basis $(X_1, \dots X_m)$, and J_2 is a square matrice of order m. Thus if λ is an eigenvalue of J_X, then λ is an eigenvalue of $\bar{J}_{X^h_{(x,0_x)}}$. □

We come to following corollary.

Corollary 3.1. *If (TM, G) is pointwise Osserman manifold (respectively globally Osserman manifold), then the same holds for (M, g).*

Now we shall give the explicit expression of $\bar{J}$ in terms of the Levi-Civita connexion ∇ and the curvature tensor R of (M, g) and some F-tensors on M.

Let $(x, u) \in TM$ and $\bar{X} = H^h_{(x,u)} + V^v_{(x,u)} \in T_{(x,u)}TM$ with $H \in T_xM$ and $V \in T_xM$. In the following we give an expression of the Jacobi operator $\bar{J}_{\bar{X}}$ of

(TM, G). Firstly, let us consider the following F-tensors which are defined in terms of the F-tensors A, B, C, D, E and F of Proposition 2.3 such as we have at any point $x \in M$:

$$\begin{aligned} P^1_{(A,B,C)}(u;H,Y,V) &= B(u;H,A(u;Y,H)) - B(u;Y,A(u;H,H)) \\ &\quad + B(u;H,C(u;Y,V)) - 2B(u;Y,C(u;H,V)) \end{aligned} \tag{22}$$

$$\begin{aligned} P^2_{(A,C,D,F)}(u;H,Y,V) &= D(u;A(u;Y,H),V) + D(u;C(u;Y,V),V) \\ &\quad - D(u;Y,F(u;V,V)), \end{aligned} \tag{23}$$

$$\begin{aligned} P^3_{(B,C,D)}(u;H,Y,V) &= C(u;H,B(u;Y,H)) - C(u;Y,B(u;H,H)) \\ &\quad + C(u;H,D(u;Y,V)) - 2C(u;Y,D(u;H,V)), \end{aligned} \tag{24}$$

$$\begin{aligned} P^4_{(A,B,D,E,F)}(u;H,Y,V) &= F(u;V,B(u;Y,H)) + F(u;V,D(u;Y,V)) \\ &\quad - A(u;Y,E(u;V,V)), \end{aligned} \tag{25}$$

$$P^5_{(C,E)}(u;H,Y,V) = C(u;H,R(H,Y)u) + E(u;R(H,Y)u,V), \tag{26}$$

$$P^6_{(A,C)}(u;H,Y,V) = d\left(A_{(Y,H)}\right)_u(V) + d\left(C_{(Y,V)}\right)_u(V), \tag{27}$$

$$\begin{aligned} Q^1_{(A,B,C,D,F)}(u;H,Y,V) &= A(u;H,C(u;H,Y)) + A(u;H,E(u;Y,V)) \\ &\quad - F(u;Y,B(u;H,H)) - 2F(u;Y,D(u;H,V)), \end{aligned} \tag{28}$$

$$\begin{aligned} Q^2_{(D,E,F)}(u;H,Y,V) &= E(u;V,D(u;H,Y)) + E(u;V,F(u;Y,V)) \\ &\quad - E(u;Y,F(u;V,V)), \end{aligned} \tag{29}$$

$$\begin{aligned} Q^3_{(A,B,C)}(u;H,Y,V) &= D(u;C(u;H,Y),V) - 2D(u;C(u;H,V),Y) \\ &\quad + D(u;E(u;Y,V),V) - D(u;E(u;V,V),Y), \end{aligned} \tag{30}$$

$$\begin{aligned} Q^4_{(A,C,D,F)}(u;H,Y,V) &= C(u;H,D(u;H,Y)) + C(u;H,F(u;Y,V)) \\ &\quad - C(u;A(u;H,H),Y), \end{aligned} \tag{31}$$

$$Q^5_{(C)}(u;H,Y,V) = d\left(C_{(H,Y)}\right)_u(V) - 2d\left(C_{(H,V)}\right)_u(Y), \tag{32}$$

$$\begin{aligned} Q^6_{(A,E)}(u;H,Y,V) &= d\left(E_{(Y,V)}\right)_u(V) - d\left(E_{(V,V)}\right)_u(Y) \\ &\quad - d\left(A_{(H,H)}\right)_u(Y) \end{aligned} \tag{33}$$

for all $u, H, Y, V \in T_xM$.

The Jacobi operator $\bar{J}_{\bar{X}}$ is then determined by

$$
\begin{aligned}
\bar{J}_{\bar{X}}\left(Y^h\right) = {} & h\{R(H,Y)H + [(\nabla_H A_u)(Y,H) - (\nabla_Y A_u)(H,H)] \\
& + [(\nabla_H C_u)(Y,V) - (\nabla_Y C_u)(H,V)] \\
& - (\nabla_Y C_u)(H,V) - (\nabla_Y E_u)(V,V) \\
& + P^1_{(A,A,C)}(u;H,Y,V) + P^2_{(A,C,C,F)}(u;H,Y,V) \\
& + P^3_{(B,C,D)}(u;H,Y,V) + P^4_{(A,B,D,E,E)}(u;H,Y,V) \\
& + P^5_{(C,E)}(u;H,Y,V) + P^6_{(A,C)}(u;H,Y,V)\} \\
& + \\
& v\{R(H,Y)V + [(\nabla_H B_u)(Y,H) - (\nabla_Y B_u)(H,H)] \\
& + [(\nabla_H D_u)(Y,V) - (\nabla_Y D_u)(H,V)] \\
& - (\nabla_Y D_u)(H,V) - (\nabla_Y F_u)(V,V) \\
& + P^1_{(A,B,C)}(u;H,Y,V) + P^2_{(A,C,D,F)}(u;H,Y,V) \\
& + P^3_{(B,D,D)}(u;H,Y,V) + P^4_{(B,B,D,E,F)}(u;H,Y,V) \\
& + P^5_{(D,F)}(u;H,Y,V) + P^6_{(B,D)}(u;H,Y,V)\}
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
\bar{J}_{\bar{X}}\left(Y^v\right) = {} & h\{(\nabla_H C_u)(H,Y) + (\nabla_H E_u)(Y,V) \\
& + Q^1_{(A,B,C,D,E)}(u;H,Y,V) + Q^2_{(D,E,F)}(u;H,Y,V) \\
& + Q^3_{(C,C,E)}(u;H,Y,V) + Q^4_{(A,C,D,F)}(u;H,Y,V) \\
& + Q^5_{(C)}(u;H,Y,V) + Q^6_{(A,E)}(u;H,Y,V)\} \\
& + \\
& v\{(\nabla_H D_u)(H,Y) + (\nabla_H F_u)(Y,V) \\
& + Q^1_{(B,B,C,D,F)}(u;H,Y,V) + Q^2_{(D,F,F)}(u;H,Y,V) \\
& + Q^3_{(C,D,E)}(u;H,Y,V) + Q^4_{(A,D,D,F)}(u;H,Y,V) \\
& + Q^5_{(D)}(u;H,Y,V) + Q^6_{(B,F)}(u;H,Y,V)\}
\end{aligned} \tag{35}
$$

for any $Y \in T_xM$ where the horizontal lift and vertical lift are taken at (x,u).

4. Osserman g-natural tangent bundles of Riemannian surfaces

Let (M,g) be a connected Riemannian surface, $x \in M$ and $(U,(x_1,x_2))$ a normal coordinates system on (M,g) centred at x. For any vector $X = X^1\partial_{x_1} + X^2\partial_{x_2} \in T_xM$, let us set

$$\mathbf{i}X = -X^2\partial_{x_1} + X^1\partial_{x_2}. \tag{36}$$

Then the Riemannian curvature is given by

$$R(X,Y)Z = k(x)g(\mathbf{i}X,Y)\mathbf{i}Z \tag{37}$$

for all vectors $X,Y,Z \in T_xM$, where k denotes the Gaussian curvature of (M,g).

We have the following result

Proposition 4.1. *Let $H \in T_xM$ such that $H^h_{(x,0_x)}$ is a unit tangent vector in $(T_{(x,0_x)}TM, G_{(x,0_x)})$. Then the spectrum of the Jacobi operator $\bar{J}_{H^h_{(x,0_x)}}$ is given by the set*

$$\left\{0,\ \frac{k(x)}{(\alpha_1+\alpha_3)(0)},\ -\frac{f_6^B + k(x)(f_1^B+f_2^B)(0)}{(\alpha_1+\alpha_3)(0)},\ -\frac{(f_4^B+f_5^B+f_6^B)(0)}{(\alpha_1+\alpha_3)(0)}\right\}. \tag{38}$$

Proof. Since $H \neq 0_x$, $(H^h_{(x,0_x)}, (\mathbf{i}H)^h_{(x,0_x)}, H^v_{(x,0_x)}, (\mathbf{i}H)^v_{(x,0_x)})$ is a basis in $T_{(x,0_x)}TM$ and according to (34) and (35), we have

$$\bar{J}_{H^h_{(x,0_x)}}\left(H^h_{(x,0_x)}\right) = 0_x \tag{39}$$

$$\bar{J}_{H^h_{(x,0_x)}}\left((\mathbf{i}H)^h_{(x,0_x)}\right) = \frac{k(x)}{(\alpha_1+\alpha_3)(0)}(\mathbf{i}H)^h_{(x,0_x)} \tag{40}$$

$$\begin{aligned}\bar{J}_{H^h_{(x,0_x)}}\left(H^v_{(x,0_x)}\right) = &-(f_4^A+f_5^A+f_6^A)(0)H^h_{(x,0_x)}\\ &-(f_4^B+f_5^B+f_6^B)(0)H^v_{(x,0_x)}\end{aligned} \tag{41}$$

$$\begin{aligned}\bar{J}_{H^h_{(x,0_x)}}\left((\mathbf{i}H)^v_{(x,0_x)}\right) = &-[f_6^A(0)+k(x)(f_1^A+f_2^A)(0)](\mathbf{i}H)^h_{(x,0_x)}\\ &-[f_6^B(0)+k(x)(f_1^B+f_2^B)(0)](\mathbf{i}H)^v_{(x,0_x)}.\end{aligned} \tag{42}$$

Then the matrix of the operator $\bar{J}_{H^h_{(x,0_x)}}$ in the basis $(H^h_{(x,0_x)}, (\mathbf{i}H)^h_{(x,0_x)}, H^v_{(x,0_x)}, (\mathbf{i}H)^v_{(x,0_x)})$ is

$$\begin{pmatrix} 0 & 0 & -\frac{\delta^A(0)}{(\alpha_1+\alpha_3)(0)} & 0 \\ 0 & \frac{k(x)}{(\alpha_1+\alpha_3)(0)} & 0 & -\frac{\eta^A(0)}{(\alpha_1+\alpha_3)(0)} \\ 0 & 0 & -\frac{\delta^B(0)}{(\alpha_1+\alpha_3)(0)} & 0 \\ 0 & 0 & 0 & -\frac{\eta^B(0)}{(\alpha_1+\alpha_3)(0)} \end{pmatrix} \tag{43}$$

where we set

$$\delta^P(0) = (f_4^P+f_5^P+f_6^P)(0) \tag{44}$$

$$\eta^P(0) = -f_6^P(0)+k(x)(f_1^P+f_2^P)(0) \tag{45}$$

for $P = A$, B. This is a triangular matrix and then we get the result. □

Similary arguments and Proposition 4.1 lead to the following conclusion:

Corollary 4.1. *Let $\dim M = 2$. If (TM, G) is a pointwise Riemannian Osserman manifold, then (M, g) has constant Gauss curvature.*

Proof. Let $x \in M$ and V be a vector in T_xM such that $g(V,V) = \frac{1}{\alpha_1(0)}$. Then $V^v_{(x,0_x)}$ is a unit vector in $(T_{(x,0_x)}TM, G_{(x,0_x)})$ and $(V^h_{(x,0_x)}, (\mathbf{i}V)^h_{(x,0_x)}, V^v_{(x,0_x)}, (\mathbf{i}V)^v_{(x,0_x)})$ is a basis of $T_{(x,0_x)}TM$.

By computing the matrix of the Jacobi operator $\bar{J}_{V^v_{(x,0_x)}}$ in this basis, as in the proof of Proposition 4.1 we get

$$\begin{pmatrix} \frac{\delta^C(0)}{\alpha_1(0)} & 0 & 0 & 0 \\ 0 & \frac{f_6^C(0)}{\alpha_1(0)} & 0 & \frac{(f_6^E - f_7^E)(0)}{\alpha_1(0)} \\ \frac{\delta^D(0)}{\alpha_1(0)} & 0 & 0 & 0 \\ 0 & \frac{f_6^D(0)}{\alpha_1(0)} & 0 & \frac{(f_6^F - f_7^F)(0)}{\alpha_1(0)} \end{pmatrix}, \tag{46}$$

where we put

$$\delta^P(0) \quad = \quad (f_4^P + f_5^P + f_6^P)(0), \qquad P = C, D. \tag{47}$$

Hence if (TM, G) is pointwise Riemannian Osserman manifold, according to Proposition 4.1 the quotient $\frac{k(x)}{(\alpha_1+\alpha_3)(0)}$ is necessarily an eigenvalue of the matrix (46) that does not depend on x. So the Gaussian curvature k is constant. This completes the proof. □

Let us consider the orthonormal frame bundle $\mathcal{O}(M)$ over (M, g). It is a subbundle of the tangent bundle TM and a g-natural metric on $\mathcal{O}(M)$ is the restriction of a g-natural metric on TM. In [**10**] the authors proved that if (M, g) has constant sectional curvature, then an orthonormal frame bundle equipped with a g-natural metric is always locally homogeneous (cf. [**10**, Corollary 4.5]). From this observation and Proposition 4.1, we get the following corollary.

Corollary 4.2. *Let (M, g) a connected Riemannian surface, and $\widetilde{G}$ a g-natural metric on its orthonormal frame bundle $\mathcal{O}(M)$. Then $(\mathcal{O}(M), \widetilde{G})$ is globally Osserman if and only if it is pointwise Osserman.*

Proof. If $(\mathcal{O}(M), \widetilde{G})$ is pointwise Osserman, then by Corollary 4.1, (M, g) is of constant Gaussian curvature and by [**10**, Corollary 4.5], $(\mathcal{O}(M), \widetilde{G})$ is locally homogenous. Hence the spectrum of its Jacobi operators is the same for all points and then $(\mathcal{O}(M), \widetilde{G})$ is globally Osserman. □

In the sequel we assume that (M, g) is of constant Gaussian curvature k. Then the following proposition holds.

Proposition 4.2. *Let (M, g) be a connected Riemannian surface with constant Gaussian curvature and $(x, u) \in TM$ with $u \neq 0_x$. Put $t = g(u, u)$. Then the family $(u^h, (\mathbf{i}u)^h, u^v, (\mathbf{i}u)^v)$ is a basis of $T_{(x,u)}TM$ and the non-vanishing entries of the matrix $(J_{ij})_{1\le i,j\le 4}$ of the Jacobi operator $\bar{J}_{u^h_{(x,u)}}$ with respect to this basis are:*

$$\begin{aligned} J_{22} = t^2\{&(f_5^A - kf_1^A)[(f_4^A - kf_2^A) - (f_4^A + f_5^A + f_6^A + tf_7^A)] \\ &+ (f_4^C - kf_2^C)(f_5^B + k(1 - f_1^B)) \\ &- (f_5^C - kf_1^C)(f_4^B + f_5^B + f_6^B + tf_7^B)\} + kt \end{aligned} \tag{48}$$

$$\begin{aligned} J_{42} = t^2\{&(f_5^A - kf_1^A)(f_4^B - kf_2^B) - (f_4^A + f_5^A + f_6^A + tf_7^A)(f_5^B - kf_1^B) \\ &+ (f_4^D - kf_2^D)(f_5^B + k(1 - f_1^B)) - (f_4^B + f_5^B + f_6^B + tf_7^B)(f_5^D - kf_1^D)\} \end{aligned}$$

$$
\begin{aligned}
J_{13} = {} & t^2[(f_4^C + f_5^C + f_6^C + tf_7^C)(f_4^D + f_5^D + f_6^D + tf_7^D) \\
& - (f_4^B + f_5^B + f_6^B + tf_7^B)(f_4^E + f_5^E + f_6^E + tf_7^E)] \\
& - t[(f_4^A + f_5^A + f_6^A + 3tf_7^A) + 2t(f_4^{A'} + f_5^{A'} + f_6^{A'} + tf_7^{A'})], \\
J_{33} = {} & t^2\{(f_4^B + f_7^B + f_6^B + tf_7^B) \\
& \cdot [(f_4^C + f_5^C + f_6^C + tf_7^C) - (f_4^F + f_5^F + f_6^F + tf_7^F)] \\
& + (f_4^D + f_5^D + f_6^D + tf_7^D) \\
& \cdot [(f_4^D + f_5^D + f_6^D + tf_7^D) - (f_4^A + f_5^A + f_6^A + tf_7^A)]\} \\
& - t[(f_4^B + f_5^B + f_6^B + 3tf_7^B) + 2t(f_4^{B'} + f_5^{B'} + f_6^{B'} + tf_7^{B'})],
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
J_{24} = {} & t^2\{(f_4^C - kf_2^C)[(f_4^A - kf_2^A) + (f_4^D - kf_2^D)] \\
& - (f_4^C - kf_2^C)(f_4^A + f_5^A + f_6^A + tf_7^A) \\
& - (f_5^E - kf_1^E)(f_4^B + f_5^B + f_6^B + tf_7^B)\} \\
& - t[(f_6^A + tf_7^A) + k(f_1^A + f_2^A)], \\
J_{44} = {} & t^2\{(f_4^D - kf_2^D)^2 + (f_4^B - kf_2^B)(f_4^C - kf_2^C) \\
& - (f_4^D - kf_2^D)(f_4^A + f_5^A + f_6^A + tf_7^A) \\
& - (f_5^F - kf_1^F)(f_4^B + f_5^B + f_6^B + tf_7^B)\} \\
& - t[(f_6^B + tf_7^B) + k(f_1^B + f_2^B)].
\end{aligned}
\tag{50}
$$

Remark 4.1. 1. It is easy to check that

$$(\phi_1 + \phi_3)J_{13} + \phi_2 J_{33} = 0, \tag{51}$$

$$\alpha_2(J_{44} - J_{22}) + (\alpha_1 + \alpha_3)J_{24} = \alpha_1 J_{42}. \tag{52}$$

2. The following vectors

$$v_1 = \frac{1}{\sqrt{t(\phi_1 + \phi_3)(t)}}\, u^h, \tag{53}$$

$$v_2 = \sqrt{\frac{(\phi_1 + \phi_3)(t)}{t\phi(t)}}\, u^v - \frac{\phi_2(t)}{\sqrt{t\phi(t)(\phi_1 + \phi_3)(t)}}\, u^h, \tag{54}$$

$$v_3 = \frac{1}{\sqrt{t(\alpha_1 + \alpha_3)(t)}}\, (\mathbf{i}u)^h, \tag{55}$$

$$v_4 = \sqrt{\frac{(\alpha_1 + \alpha_3)(t)}{t\alpha(t)}}\, (\mathbf{i}u)^v - \frac{\alpha_2(t)}{\sqrt{t\alpha(t)(\alpha_1 + \alpha_3)(t)}}\, (\mathbf{i}u)^h, \tag{56}$$

where the lifts are taken at (x, u), determine an orthonormal basis of $(T_{(x,u)}TM, G_{(x,u)})$.

Proposition 4.3. *Let $(x,u) \in TM$ such that $u \neq 0_x$ and $t = g(u,u)$. Then the spectrum of Jacobi operator $\bar{J}_{u^h_{(x,u)}}$ is given by the set*

$$\left\{0,\ J_{33},\ \frac{(J_{22}+J_{44})+\sqrt{\Delta}}{2},\ \frac{(J_{22}+J_{44})-\sqrt{\Delta}}{2}\right\}, \tag{57}$$

where $\Delta = \left(J_{22} - J_{44} + 2\frac{\alpha_2}{\alpha_1+\alpha_3}J_{42}\right)^2 + 4\frac{\alpha}{(\alpha_1+\alpha_3)^2}J_{42}^2$.

Proof. According to Remark 4.1 and Proposition 4.2, the matrix of $\bar{J}_{u^h_{(x,u)}}$ in the orthonormal basis $(v_1,\ v_2,\ v_3,\ v_4)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & J_{33} & 0 & 0 \\ 0 & 0 & (J_{22} + \frac{\alpha_2}{\alpha_1+\alpha_3}J_{42}) & \frac{\sqrt{\alpha}}{\alpha_1+\alpha_3}J_{42} \\ 0 & 0 & \frac{\sqrt{\alpha}}{\alpha_1+\alpha_3}J_{42} & (J_{44} - \frac{\alpha_2}{\alpha_1+\alpha_3}J_{42}) \end{pmatrix}. \tag{58}$$

So by computing the eigenvalues of this matrix, we obtain the proof. □

Using Proposition 4.3 and by notifying that $G(u^h,u^h) = t(\phi_1+\phi_3)(t)$ with $t = g(u,u)$, we obtain the following results:

Theorem 4.1. *(TM,G) is a pointwise Osserman manifold if and only if*

1. *(M,g) has constant Gauss curvature k.*
2. *The eigenvalues of its Jacobi operators on the unit tangent bundle $S(TTM)$ are the functions $(\lambda_i)_{i=1,2,3}$ defined on TM by*

$$\begin{aligned} \lambda_0(x,u) &= 0, \\ \lambda_1(x,u) &= \frac{J_{33}}{t(\phi_1+\phi_3)}, \\ \lambda_2(x,u) &= \frac{(J_{22}+J_{44})+\sqrt{\Delta}}{2t(\phi_1+\phi_3)}, \\ \lambda_3(x,u) &= \frac{(J_{22}+J_{44})-\sqrt{\Delta}}{2t(\phi_1+\phi_3)}, \end{aligned} \tag{59}$$

if $u \neq 0_x$
and

$$\begin{aligned} \lambda_0(x,0_x) &= 0, \\ \lambda_1(x,0_x) &= -\frac{(f_4^B+f_5^B+f_6^B)(0)}{(\alpha_1+\alpha_3)(0)}, \\ \lambda_2(x,0_x) &= \frac{k}{(\alpha_1+\alpha_3)(0)}, \\ \lambda_3(x,0_x) &= -\frac{f_6^B + k(f_1^B+f_2^B)(0)}{(\alpha_1+\alpha_3)(0)}. \end{aligned} \tag{60}$$

Theorem 4.2. *(TM, G) is a globally Osserman manifold if and only if*

1. *(M, g) has constant Gauss curvature k.*
2. *The eigenvalues of its Jacobi operators on the unit tangent bundle $S(TTM)$ are the real numbers $(\tilde{\lambda}_i)_{i=1,2,3}$ given by*

$$\begin{aligned}
\tilde{\lambda}_0 &= 0, \\
\tilde{\lambda}_1 &= -\frac{(f_4^B + f_5^B + f_6^B)(0)}{(\alpha_1 + \alpha_3)(0)}, \\
\tilde{\lambda}_2 &= \frac{k}{(\alpha_1 + \alpha_3)(0)}, \\
\tilde{\lambda}_3 &= -\frac{f_6^B + k(f_1^B + f_2^B)(0)}{(\alpha_1 + \alpha_3)(0)}.
\end{aligned} \tag{61}$$

In the following we apply the result in Theorem 4.2 to the Sasaki metric and to the Cheeger-Gromoll metric on the tangent bundle.

Applications:

1. Let G be the Sasaki metric on the tangent bundle TM. In this case the functions α_i and β_i of Proposition 2.1 are given by

$$\begin{aligned}
&\alpha_1 = 1; \quad \alpha_2 = \alpha_3 = 0 \quad \text{and} \\
&\beta_1 = \beta_2 = \beta_3 = 0.
\end{aligned}$$

The eigenvalues $\tilde{\lambda}_0$, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$ of Theorem 4.2 are

$$\tilde{\lambda}_0 = \tilde{\lambda}_1 = 0; \quad \tilde{\lambda}_2 = k; \quad \tilde{\lambda}_3 = 0.$$

2. Let G be the Cheeger-Gromoll metric on the tangent bundle TM. Then the functions α_i and β_i of Proposition 2.1 are given by

$$\begin{aligned}
&\alpha_1 = \beta_1 = \frac{1}{1+2t}; \quad \alpha_2 = \beta_2 = 0; \quad \text{and} \\
&\alpha_3 = \frac{2t}{1+2t}; \quad \beta_3 = -\frac{1}{1+2t}.
\end{aligned}$$

The eigenvalues $\tilde{\lambda}_0$, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$ of Theorem 4.2 are in this case

$$\tilde{\lambda}_0 = \tilde{\lambda}_1 = 0; \quad \tilde{\lambda}_2 = k; \quad \tilde{\lambda}_3 = 0.$$

We can conclude that the tangent bundle TM with the Sasaki metric or the Cheeger-Gromoll metric is globally Ossermann if and only if (M, g) is of constant Gaussian curvature k and the eigenvalues of its Jacobi operators are 0 (with multiplicity three) and k.

The following consequence for the sectional curvature of g-natural metrics can be derived from Theorem 4.2.

Corollary 4.3. *Only flat g-natural metrics on the tangent bundle of a Riemannian surface (M, g) are of constant sectional curvature.*

Proof. Let G be a g-natural metric on TM of constant sectional curvature. Then (TM, G) is globally Osserman. But then also (M, g) is flat (cf. [**7**]) and according to (61), the eigenvalue $\bar{\lambda}_2 = \frac{k}{(\alpha_1+\alpha_3)(0)}$ of the Jacobi operators of (TM, G) is like the eigenvalue $\bar{\lambda}_0$ equal to zero. Thus 0 is an eigenvalue of the Jacobi operators of (TM, G) with multiplicity at least two. Hence (TM, G) is flat. □

Remark 4.2. This corollary extends Proposition 4.3 in [**7**] to the case where $\dim M = 2$.

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S. Degla, Institut de Mathématiques et de Sciences Physiques, 01 BP 613 Porto-Novo, Benin, *e-mail*: `sdegla@imsp-uac.org`

L. Todjihounde, Institut de Mathématiques et de Sciences Physiques, 01 BP 613 Porto-Novo, Benin, *e-mail*: `leonardt@imsp-uac.org`