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ISOMETRIES AND ISOMORPHISMS IN QUASI-BANACH ALGEBRAS

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Abstract. In this paper, we prove the Hyers-Ulam-Rassias stability of isometries and of homomorphisms for additive functional equations in quasi-Banach algebras. This is applied to investigate isomorphisms between quasi-Banach algebras.

1. Introduction and preliminaries

Stability is investigated when one concerns whether a small error of parameters causes a large deviation of the solution. Generally speaking, given a function which satisfies a functional equation approximately called an *approximate solution*, we ask: Is there a solution of this equation which is close to the approximate solution in some accuracy? An ealier work was done by Hyers [**11**] in order to answer Ulam's question ([**20**]) on approximately additive mappings. Later there have been given lots of results on stability in the Hyers-Ulam sense or some generalized sense (see books and papers [**1, 3, 8, 9, 12, 17, 18**] and references therein).

G. Z. Eskandani [**7**] established the general solution and investigated the Hyers-Ulam-Rassias stability of the following functional equation

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right) \tag{1.1}$$

in quasi-Banach spaces, where $m \in \mathbb{N}$ and $m \geq 2$. The stability of isometries in norms spaces and Banach spaces was investigated in several papers [**4, 6, 10, 13**]. However, C. Park and Th. M. Rassias [**15**] proved the Hyers-Ulam stability of isometric additive functional equations in quasi-Banach spaces. C. Park [**16**] studied the Hyers-Ulam stability of homomorphisms in quasi-Banach algebras. Recently, M. S. Moslehian and Gh. Sadeghi [**14**] have proved the Hyers-Ulam-Rassias stability of linear mappings in quasi-Banach modules associated to the Cauchy functional equation and a generalized Jensen functional equation.

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The main purpose of this paper is to study the Hyers-Ulam-Rassias stability of equation (1.1). More precisely, we prove the Hyers-Ulam-Rassias stability of isometric additive functional equations (1.1) in quasi-Banach algebras. Furthermore, we investigate the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to additive functional equations (1.1). This is applied to investigate isomorphisms between quasi-Banach algebras.

We now give some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 (cf. [**5, 19**])**.** Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and for all $x \in X$.
(3) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\|+\|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a *quasi-norm* on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete *quasi-normed space*.

A *quasi-norm* $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a *quasi-Banach space* is called a *p-Banach space*.

Given a p-norm, the formula $d(x, y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [**19**] (see also [**5**]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Definition 1.2 (cf. [**2**])**.** Let $(X, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(X, \|\cdot\|)$ is called a *quasi-normed algebra* if X is an algebra and there is a constant $C > 0$ such that $\|xy\| \leq C\|x\|\|y\|$ for all $x, y \in X$.

A *quasi-Banach algebra* is a complete *quasi-normed algebra*. If the quasi-norm $\|\cdot\|$ is a p-norm, then the *quasi-Banach algebra* is called a *p-Banach algebra*.

Definition 1.3 (cf. [**15**])**.** Let X and Y be quasi-Banach algebras with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. An additive mapping $A\colon X \to Y$ is called an isometric additive mapping if the additive mapping $A\colon X \to Y$ satisfies

$$\|A(x) - A(y)\|_Y = \|x-y\|_X$$

for all $x, y \in X$.

2. Stability of isometric additive mappings in quasi-Banach algebras

Throughout this section and Section 3, assume that X is a quasi-normed algebra with quasi-norm $\|\cdot\|_X$ and that Y is a p-Banach algebra with p-norm $\|\cdot\|_Y$. Let

K be the modulus of concavity of $\|\cdot\|_Y$. For convenience, we use the following abbreviation for a given mapping $f\colon X \to Y$:

$$Df(x_1, \cdots, x_m) = \sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) - 2f\left(\sum_{i=1}^{m} mx_i\right)$$

for all $x_j \in X$ $(1 \le j \le m)$. We prove the Hyers-Ulam-Rassias stability of the isometric additive functional equation (1.1) in quasi-Banach algebras.

Theorem 2.1. *Let $\varphi\colon X^m \to [0,\infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{m^n}\varphi(m^n x_1, \cdots, m^n x_m) = 0 \tag{2.1}$$

$$\tilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{m^{ip}} (\varphi(m^i x, 0, \cdots, 0))^p < \infty \tag{2.2}$$

for all $x, x_j \in X$ $(1 \le j \le m)$. Suppose that a mapping $f\colon X \to Y$ satisfies

$$\|Df(x_1, \cdots, x_m)\|_Y \le \varphi(x_1, \cdots, x_m) \tag{2.3}$$

$$|\ \|f(x)\|_Y - \|x\|_X\ | \le \varphi(\underbrace{x, \cdots, x}_{m-times}) \tag{2.4}$$

for all $x, x_j \in X$ $(1 \le j \le m)$. Then there exists a unique isometric additive mapping $A\colon X \to Y$ such that

$$\|f(x) - A(x)\|_Y \le \frac{1}{m}[\tilde{\varphi}(x)]^{\frac{1}{p}} \tag{2.5}$$

for all $x \in X$.

Proof. By the Eskandani's theorem [**7**, Theorem 2.2], it follows from (2.1), (2.2) and (2.3) that there exists a unique additive mapping $A\colon X \to Y$ satisfying (2.5). The additive mapping $A\colon X \to Y$ is given by

$$A(x) := \lim_{n\to\infty} \frac{1}{m^n} f(m^n x) \tag{2.6}$$

for all $x \in X$.

It follows from (2.4) that

$$\begin{aligned} |\ \|\frac{1}{m^n} f(m^n x)\|_Y - \|x\|_X\ | &\le \frac{1}{m^n}\ |\ \|f(m^n x)\|_Y - \|m^n x\|_X\ | \\ &\le \frac{1}{m^n}\varphi(\underbrace{m^n x, \cdots, m^n x}_{m-times}) \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So

$$\|A(x)\|_Y = \lim_{n\to\infty} \|\frac{1}{m^n} f(m^n x)\|_Y = \|x\|_X$$

for all $x \in X$. Since $A\colon X \to Y$ is additive,

$$\|A(x) - A(y)\|_Y = \|A(x-y)\|_Y = \|x - y\|_X$$

for all $x \in X$. So the mapping $A\colon X \to Y$ is an isometry. Thus the mapping $A\colon X \to Y$ is a unique isometric additive mapping satisfying (2.5). This completes the proof of the theorem. $\square$

Theorem 2.2. *Let $\phi\colon X^m \to [0,\infty)$ be a mapping such that*

$$\lim_{n\to\infty} m^n \phi(\frac{x_1}{m^n}, \cdots, \frac{x_m}{m^n}) = 0 \tag{2.7}$$

$$\tilde{\phi}(x) := \sum_{i=1}^{\infty} m^{ip} (\phi(\frac{x}{m^i}, 0, \cdots, 0))^p < \infty \tag{2.8}$$

for all $x, x_j \in X$ $(1 \le j \le m)$. Suppose that a mapping $f\colon X \to Y$ satisfies

$$\|Df(x_1, \cdots, x_m)\|_Y \le \phi(x_1, \cdots, x_m) \tag{2.9}$$

$$| \, \|f(x)\|_Y - \|x\|_X \, | \le \phi(\underbrace{x, \cdots, x}_{m-times}) \tag{2.10}$$

for all $x, x_j \in X$ $(1 \le j \le m)$. Then there exists a unique isometric additive mapping $A\colon X \to Y$ such that

$$\|f(x) - A(x)\|_Y \le \frac{1}{m}[\tilde{\phi}(x)]^{\frac{1}{p}} \tag{2.11}$$

for all $x \in X$.

Proof. By the Eskandani's theorem [**7**, Theorem 2.3], it follows from (2.7), (2.8) and (2.9) that there exists a unique additive mapping $A\colon X \to Y$ satisfying (2.11). The additive mapping $A\colon X \to Y$ is given by

$$A(x) := \lim_{n\to\infty} m^n f(\frac{x}{m^n}) \tag{2.12}$$

for all $x \in X$.

By (2.10), we have

$$\begin{aligned} | \, \|m^n f(\frac{x}{m^n})\|_Y - \|x\|_X \, | &\le m^n \, | \, \|f(\frac{x}{m^n})\|_Y - \|\frac{x}{m^n}\|_X \, | \\ &\le m^n \varphi(\underbrace{\frac{x}{m^n}, \cdots, \frac{x}{m^n}}_{m-times}) \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. By (2.12), we obtain

$$\|A(x)\|_Y = \lim_{n\to\infty} \|m^n f(\frac{x}{m^n})\|_Y = \|x\|_X$$

for all $x \in X$. Hence

$$\|A(x) - A(y)\|_Y = \|A(x-y)\|_Y = \|x-y\|_X$$

for all $x \in X$. So the additive mapping $A\colon X \to Y$ is an isometry. This completes the proof of the theorem. $\square$

Corollary 2.1. *Let θ, r_j $(1 \leq j \leq m)$ be non-negative real numbers such that $r_j > 1$ or $0 < r_j < 1$. Suppose that a mapping $f: X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \leq \theta \sum_{i=1}^{m} \|x_i\|_X^{r_i}$$

$$| \, \|f(x)\|_Y - \|x\|_X \, | \leq \theta \sum_{i=1}^{m} \|x\|_X^{r_i}$$

for all $x, x_j \in X$ $(1 \leq j \leq m)$. Then there exists a unique isometric additive mapping $A: X \to Y$ such that

$$\|f(x) - A(x)\|_Y \leq \frac{\theta}{|m^p - m^{pr_1}|^{\frac{1}{p}}} \|x\|_X^{r_1}$$

for all $x \in X$.

Proof. The result follows from the proofs of Theorems 2.1 and 2.2. □

3. Stability of homomorphisms in quasi-Banach algebras

We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras, associated to the additive functional equation (1.1).

Theorem 3.1. *Suppose that a mapping $f: X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \leq \varphi(x_1, \cdots, x_m) \tag{3.1}$$

$$\|f(xy) - f(x)f(y)\|_Y \leq \psi(x, y) \tag{3.2}$$

for all $x, y, x_j \in X$ $(1 \leq j \leq m)$, where $\varphi: X^m \to [0, \infty)$ satisfies (2.1) and (2.2), and $\psi: X \times X \to [0, \infty)$ satisfies the following

$$\lim_{n \to \infty} \frac{1}{m^n} \psi(m^n x, m^n y) = 0 \tag{3.3}$$

for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H: X \to Y$ such that

$$\|f(x) - H(x)\|_Y \leq \frac{1}{m} [\tilde{\varphi}(x)]^{\frac{1}{p}} \tag{3.4}$$

for all $x \in X$.

Proof. By Theorem 2.1, there exists a unique additive mapping $H: X \to Y$ satisfying (3.4). The additive mapping $H: X \to Y$ is given by

$$H(x) := \lim_{n \to \infty} \frac{1}{m^n} f(m^n x) \tag{3.5}$$

for all $x \in X$. By the same reasoning as in the proof of Theorem of [**17**], the mapping $H: X \to Y$ is $\mathbb{R}$-linear.

It follows from (3.2) that

$$\begin{aligned}\|H(xy) - H(x)H(y)\|_Y &= \lim_{n\to\infty} \frac{1}{m^{2n}} \|f(m^{2n}xy) - f(m^n x)f(m^n y)\|_Y \\ &\leq \lim_{n\to\infty} \frac{1}{m^{2n}} \psi(m^n x, m^n y) = 0\end{aligned}$$

for all $x, y \in X$. Hence, we get

$$H(xy) = H(x)H(y)$$

for all $x, y \in X$. Thus the mapping $H\colon X \to Y$ is a unique homomorphism satisfying (3.4). This completes the proof of the theorem. □

Theorem 3.2. *Suppose that a mapping $f\colon X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \leq \phi(x_1, \cdots, x_m) \tag{3.6}$$

$$\|f(xy) - f(x)f(y)\|_Y \leq \Psi(x, y) \tag{3.7}$$

for all $x, y, x_j \in X$ $(1 \leq j \leq m)$, where $\phi\colon X^m \to [0, \infty)$ satisfies (2.7) *and* (2.8), *and $\Psi\colon X \times X \to [0, \infty)$ satisfies the following*

$$\lim_{n\to\infty} m^n \Psi(\frac{x}{m^n}, \frac{y}{m^n}) = 0 \tag{3.8}$$

for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H\colon X \to Y$ such that

$$\|f(x) - H(x)\|_Y \leq \frac{1}{m}[\tilde{\phi}(x)]^{\frac{1}{p}} \tag{3.9}$$

for all $x \in X$.

Proof. By Theorem 2.2, there exists a unique additive mapping $H\colon X \to Y$ satisfying (3.9). The additive mapping $H\colon X \to Y$ is given by

$$H(x) := \lim_{n\to\infty} m^n f(\frac{x}{m^n}) \tag{3.10}$$

for all $x \in X$. By the same reasoning as in the proof of Theorem of [**17**], the mapping $H\colon X \to Y$ is $\mathbb{R}$-linear.

It follows from (3.8) that

$$\begin{aligned}\|H(xy) - H(x)H(y)\|_Y &= \lim_{n\to\infty} m^{2n} \|f(\frac{xy}{m^n \cdot m^n}) - f(\frac{x}{m^n})f(\frac{y}{m^n})\|_Y \\ &\leq \lim_{n\to\infty} m^{2n} \Psi(\frac{x}{m^n}, \frac{y}{m^n}) = 0\end{aligned}$$

for all $x, y \in X$. Hence, we get

$$H(xy) = H(x)H(y)$$

for all $x, y \in X$. Thus the mapping $H\colon X \to Y$ is a unique homomorphism satisfying (3.9). This completes the proof of the theorem. □

Corollary 3.1. *Let θ, δ be non-negative real numbers and let r_j $(1 \le j \le m)$, s_1, s_2 be non-negative real numbers such that $r_j > 1$, $s_1, s_2 > 2$ or $0 < r_j < 1$, $s_1, s_2 < 2$. Suppose that a mapping $f\colon X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \le \theta \sum_{i=1}^{m} \|x_i\|_X^{r_i} \tag{3.11}$$

$$\|f(xy) - f(x)f(y)\|_Y \le \delta(\|x\|_X^{s_1} + \|y\|_X^{s_2}) \tag{3.12}$$

for all $x, y, x_j \in X$ $(1 \le j \le m)$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H\colon X \to Y$ such that

$$\|f(x) - H(x)\|_Y \le \frac{\theta}{|m^p - m^{pr_1}|^{\frac{1}{p}}} \|x\|_X^{r_1}$$

for all $x \in X$.

Proof. The result follows from the proofs of Theorems 3.1 and 3.2. □

Corollary 3.2. *Let θ, δ be non-negative real numbers and let r_j $(1 \le j \le m)$, s_1, s_2 be non-negative real numbers such that $\sum_{i=1}^{m} r_i > 1$, $s_1 + s_2 > 2$ or $\sum_{i=1}^{m} r_i < 1$, $s_1 + s_2 < 2$ and $r_j \ne 0$ for some j $(2 \le j \le m)$. Suppose that a mapping $f\colon X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \le \theta \prod_{i=1}^{m} \|x_i\|_X^{r_i} \tag{3.13}$$

$$\|f(xy) - f(x)f(y)\|_Y \le \delta \|x\|_X^{s_1} \|y\|_X^{s_2} \tag{3.14}$$

for all $x, y, x_j \in X$ $(1 \le j \le m)$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping $f\colon X \to Y$ is a homomorphism.

Proof. The result follows from the proofs of Theorems 3.1 and 3.2. □

4. Isomorphisms between quasi-Banach algebras

Throughout this section, assume that X is a quasi-Banach algebra with quasi-norm $\|\cdot\|_X$ and unit e and that Y is a p-Banach algebra with p-norm $\|\cdot\|_Y$ and unit e'. Let K be the modulus of concavity of $\|\cdot\|_Y$.

We investigate isomorphisms between quasi-Banach algebras associated to the additive functional equation (1.1).

Theorem 4.1. *Suppose that $f\colon X \to Y$ is a bijective mapping satisfying* (3.1) *such that*

$$f(xy) = f(x)f(y) \tag{4.1}$$

for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} \frac{1}{m^n} f(m^n e) = e'$, then the mapping $f\colon X \to Y$ is an isomorphism.

Proof. By Theorem 3.1, there exists a homomorphism $H\colon X\to Y$ satisfying (3.4). The mapping $H\colon X\to Y$ is given by

$$H(x):=\lim_{n\to\infty}\frac{1}{m^n}f(m^nx) \tag{4.2}$$

for all $x\in X$.

By (4.1), we have

$$\begin{aligned}H(x)=H(ex)&=\lim_{n\to\infty}\frac{1}{m^n}f(m^nex)=\lim_{n\to\infty}\frac{1}{m^n}f(m^ne\cdot x)\\&=\lim_{n\to\infty}\frac{1}{m^n}f(m^ne)f(x)=e'f(x)=f(x)\end{aligned}$$

for all $x\in X$. So the bijective mapping $f\colon X\to Y$ is an isomorphism. This completes the proof of the theorem. □

Theorem 4.2. *Suppose that $f\colon X\to Y$ is a bijective mapping satisfying (3.6) and (4.1). If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in X$ and $\lim\limits_{n\to\infty}m^nf(\frac{e}{m^n})=e'$, then the mapping $f\colon X\to Y$ is an isomorphism.*

Proof. By Theorem 3.2, there exists a homomorphism $H\colon X\to Y$ satisfying (3.9). The mapping $H\colon X\to Y$ is given by

$$H(x):=\lim_{n\to\infty}m^nf(\frac{x}{m^n}) \tag{4.3}$$

for all $x\in X$.

The rest of the proof is similar to the proof of Theorem 4.1. This completes the proof of the theorem. □

Corollary 4.1. *Let θ, r_j $(1\le j\le m)$ be non-negative real numbers such that $r_j>1$ or $0<r_j<1$. Suppose that a bijective mapping $f\colon X\to Y$ satisfies (3.11) and (4.1). If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in X$ and $\lim\limits_{n\to\infty}m^nf(\frac{e}{m^n})=e'$ or $\lim\limits_{n\to\infty}\frac{1}{m^n}f(m^ne)=e'$, then the mapping $f\colon X\to Y$ is an isomorphism.*

Proof. The result follows from the proofs of Theorems 4.1 and 4.2. □

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