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ON CONTRAVARIANT PRODUCT CONJUGATE CONNECTIONS

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Abstract. Invariance properties for the covariant and contravariant connections on a Riemannian manifold with respect to an almost product structure are stated. Restricting to a distribution of the contravariant connections is also discussed. The particular case of the conjugate connection is investigated and properties of the extended structural and virtual tensors for the contravariant connections are given.

1. Preliminaries

It is known that any covariant connection induces a contravariant one, but not any contravariant connection is induced by a covariant one [**5**]. In the present paper, starting with a covariant connection ∇ on a Riemannian manifold (M, g), we shall consider its extension $\widetilde{\nabla}$ to 1-forms and respectively, the contravariant connection $\overline{\nabla}$ induced by ∇ and discuss invariance properties. If besides the Riemannian structure g the manifold is endowed with an almost product structure compatible with g, we will study the product conjugate connections of $\widetilde{\nabla}$ and $\overline{\nabla}$, and determine the expressions and the properties of the structural and virtual tensors for them.

Let us point out that if M is a Riemann-Poisson manifold with the Riemannian structure g (which induces g^* on 1-forms) and the Poisson bivector field Π, it is known that the anchor map $\sharp_\Pi \colon \Gamma(T^*M) \to \Gamma(TM)$, $\beta(\sharp_\Pi\alpha) = \Pi(\alpha,\beta)$, $\alpha, \beta \in \Gamma(T^*M)$, and the Koszul bracket $[\alpha,\beta]_\Pi := L_{\sharp_\Pi\alpha}\beta - L_{\sharp_\Pi\beta}\alpha - d(\Pi(\alpha,\beta))$, α, $\beta \in \Gamma(T^*M)$ define a Lie algebroid structure associated to Π (for the definition of a Lie algebroid, see [**9**]). The contravariant connections on such manifolds proved to be important appearing in the context of noncommutative deformations [**3**], [**4**], [**10**]. Defined by I. Vaisman [**11**], the contravariant connections on Poisson manifolds were intensively studied by R. Fernandes [**2**]. If one requires for the contravariant connection to be torsion free and compatible with g^*, then we find the (unique) Levi-Civita contravariant connection associated to (Π, g^*), which is defined by the Koszul formula

$$\begin{aligned} 2g^*(\overline{\nabla}^{\alpha}\beta,\gamma) &= \sharp_\Pi\alpha(g^*(\beta,\gamma)) + \sharp_\Pi\beta(g^*(\alpha,\gamma)) - \sharp_\Pi\gamma(g^*(\alpha,\beta)) \\ &\quad + g^*([\gamma,\alpha]_\Pi,\beta) + g^*([\gamma,\beta]_\Pi,\alpha) + g^*([\alpha,\beta]_\Pi,\gamma), \quad \alpha,\beta,\gamma \in \Gamma(T^*M). \end{aligned}$$

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Let us recall the definition of the contravariant connection on the cotangent bundle of a Riemannian manifold (M,g). We say that $\overline{\nabla}\colon \Gamma(T^*M)\times\Gamma(T^*M)\to\Gamma(T^*M)$ is a *contravariant connection* on T^*M if $\overline{\nabla}$ satisfies the following properties:

1. $\overline{\nabla}$ is $\mathbb{R}$-bilinear;
2. $\overline{\nabla}^{f\alpha}\beta = f\overline{\nabla}^{\alpha}\beta$, for any $f\in C^\infty(M)$ and α, $\beta\in\Gamma(T^*M)$;
3. $\overline{\nabla}^{\alpha}(f\beta) = f\overline{\nabla}^{\alpha}\beta + \sharp_g\alpha(f)\beta$, for any $f\in C^\infty(M)$ and α, $\beta\in\Gamma(T^*M)$,

where $\sharp_g$ is the inverse of the isomorphism $\flat_g\colon\Gamma(TM)\to\Gamma(T^*M)$, $\flat_g(X):=i_Xg$.

Let E be an almost product structure on the Riemannian manifold (M,g), compatible with g, that is, $g(EX,EY)=g(X,Y)$, for any X, $Y\in\Gamma(TM)$. Consider $\nabla\colon\Gamma(TM)\times\Gamma(TM)\to\Gamma(TM)$ a covariant connection on M and define the extension of ∇ to 1-forms:

$$\begin{aligned}&\widetilde{\nabla}\colon\Gamma(TM)\times\Gamma(T^*M)\to\Gamma(T^*M),\\&(\widetilde{\nabla}_X\alpha)(Y):=X(\alpha(Y))-\alpha(\nabla_XY)\end{aligned}\tag{1.1}$$

and respectively, the contravariant connection induced by ∇:

$$\overline{\nabla}\colon\Gamma(T^*M)\times\Gamma(T^*M)\to\Gamma(T^*M),\qquad \overline{\nabla}^{\alpha}\beta:=\widetilde{\nabla}_{\sharp_g\alpha}\beta.\tag{1.2}$$

Remark that if ∇ is the Levi-Civita connection associated to g, then $\widetilde{\nabla}$ and $\overline{\nabla}$ are "natural operators", meaning that for any isometry $f\colon(M,g_M)\to(N,g_N)$, it follows that $f_*\circ\widetilde{\nabla_M}=\widetilde{\nabla_N}\circ(f_*\times f_*)$ and respectively, $(f_1^*)^{-1}\circ\overline{\nabla_M}=\overline{\nabla_N}\circ[(f_1^*)^{-1}\times(f_1^*)^{-1}]$, where $f_*:\Gamma(TM)\to\Gamma(TN)$, $f_*(X):=(f^*)^{-1}\circ X\circ f^*$, $f_1^*\colon\Gamma(T^*N)\to\Gamma(T^*M)$, $f_1^*(\alpha):=f^*\circ\alpha\circ f_*$ and ∇_M, ∇_N are the Levi-Civita connections associated to g_M, respectively, to g_N.

Let $E^*\colon\Gamma(T^*M)\to\Gamma(T^*M)$, $(E^*\alpha)(X):=\alpha(EX)$ be the dual of E and $g^*\colon\Gamma(T^*M)\times\Gamma(T^*M)\to C^\infty(M)$, $g^*(\alpha,\beta):=g(\sharp_g\alpha,\sharp_g\beta)$ the Riemannian structure induced by g. Then, for any $\alpha,\beta\in\Gamma(T^*M)$,

$$g^*(\alpha,\beta)=(i_{\sharp_g\alpha}g)(\sharp_g\beta):=\flat_g(\sharp_g\alpha)(\sharp_g\beta)=\alpha(\sharp_g\beta).$$

From the compatibility condition of g with E, it follows that for any $\alpha\in\Gamma(T^*M)$, $E(\sharp_g(E^*\alpha))=\sharp_g\alpha$. Indeed, let $E(\sharp_g(E^*\alpha))=:X$, then $\sharp_g(E^*\alpha)=EX$ and $E^*\alpha=\flat_g(EX):=i_{EX}g$. For any $Y\in\Gamma(TM)$, $(E^*\alpha)(Y)=g(EX,Y)$ is equivalent to $\alpha(EY)=g(EX,Y)=g(EX,E^2Y)=g(X,EY):=(i_Xg)(EY):=\flat_g(X)(EY)$. It follows $\alpha=\flat_g(X)$ and $\sharp_g\alpha=X$.

Note that if g is compatible with E, then g^* is compatible with E^*. Indeed,

$$\begin{aligned}g^*(E^*\alpha,E^*\beta):=&\,g(\sharp_g(E^*\alpha),\sharp_g(E^*\beta))=(E^*\alpha)(\sharp_g(E^*\beta))\\:=&\,\alpha(E(\sharp_g(E^*\beta)))=g(\sharp_g\alpha,\sharp_g\beta)\\:=&\,g^*(\alpha,\beta)\end{aligned}$$

for any α, $\beta\in\Gamma(T^*M)$.

Example 1.1. Consider the particular cases when there exists a certain relation between the connection ∇ and the almost product structure E, namely, there exists a 1-form η such that $\nabla_XE=\eta(X)E$, respectively, $\nabla_XE=\eta(X)I_{\Gamma(TM)}$

for any $X \in \Gamma(TM)$, where $I_{\Gamma(TM)}$ the identity is map on $\Gamma(TM)$. In the first case, $\widetilde{\nabla}_X E^* = \eta(X)E^*$ for any $X \in \Gamma(TM)$, $\overline{\nabla}^\alpha E^* = \eta(\sharp_g\alpha)E^*$ for any $\alpha \in \Gamma(T^*M)$ and in the second case, $\widetilde{\nabla}_X E^* = \eta(X)I_{\Gamma(T^*M)}$ for any $X \in \Gamma(TM)$, $\overline{\nabla}^\alpha E^* = \eta(\sharp_g\alpha)I_{\Gamma(T^*M)}$, for any $\alpha \in \Gamma(T^*M)$, where $I_{\Gamma(T^*M)}$ is the identity map on $\Gamma(T^*M)$.

2. Basic properties of contravariant connections

Invariance properties for $\widetilde{\nabla}$ and $\overline{\nabla}$. If we assume that E is parallel with respect to ∇ (i.e., $\nabla E = 0$) and respectively, if ∇ is a metric connection (i.e., $\nabla g = 0$), we shall establish some invariance properties for $\widetilde{\nabla}$ and $\overline{\nabla}$.

The following proposition describes the behavior of the extended connection $\widetilde{\nabla}$ and of the contravariant connection $\overline{\nabla}$ in the case when $\nabla E = 0$ and respectively, "energy-preserving" [that is, ∇ leaves invariant the "kinetic energy" $K(X) := \frac{1}{2}g(X,X)$ of the metric g]. It was proved [**7**] that a necessary and sufficient condition for a covariant connection to be energy-preserving is that its symmetric part has to vanish. In particular, it happens if $\nabla g = 0$. More exactly, we shall prove that in this case, the connections ∇ and $\widetilde{\nabla}$ commute with the isomorphism $\sharp_g$ and ∇'s extension to 1-forms, $\widetilde{\nabla}$ is energy-preserving, too (with respect to the Riemannian metric g^*). Like in the almost Hermitian case [**1**], we can state the following proposition.

Proposition 2.1. *Let E be an almost product structure on the Riemannian manifold (M, g), compatible with g and ∇ a covariant connection on M.*

1. *If E is parallel with respect to ∇, then E^* is parallel with respect to $\widetilde{\nabla}$ and $\overline{\nabla}$.*
2. *If $\nabla g = 0$, then*
   (a) *$\nabla_X \sharp_g\alpha = \sharp_g(\widetilde{\nabla}_X\alpha)$ for any $X \in \Gamma(TM)$ and $\alpha \in \Gamma(T^*M)$ and respectively, $\nabla_{\sharp_g\alpha}\sharp_g\beta = \sharp_g(\overline{\nabla}^\alpha\beta)$ for any α, $\beta \in \Gamma(T^*M)$;*
   (b) *$\widetilde{\nabla}g^* = 0$ and respectively, $\overline{\nabla}g^* = 0$.*
3. *If ∇g is symmetric, then $T^{\overline{\nabla}}(\alpha,\beta) = \flat_g(T_\nabla(\sharp_g\alpha, \sharp_g\beta))$ for any α, $\beta \in \Gamma(T^*M)$, where the $(1,2)$-tensor field $T^{\overline{\nabla}}$ is defined $T^{\overline{\nabla}}(\alpha,\beta) := \overline{\nabla}^\alpha\beta - \overline{\nabla}^\beta\alpha - [\alpha,\beta]$ for $[\alpha,\beta] := \flat_g([\sharp_g\alpha, \sharp_g\beta])$. In particular, $T^{\overline{\nabla}} = 0$ if and only if ∇ is torsion free.*

From this proposition we deduce that if ∇ is the Levi-Civita covariant connection associated to g, then $\overline{\nabla}$ is the Levi-Civita contravariant connection associated to g^*, being the unique contravariant connection satisfying

$$\begin{cases} \overline{\nabla}^\alpha\beta - \overline{\nabla}^\beta\alpha = [\alpha,\beta], \\ \sharp_g\alpha(g^*(\beta,\gamma)) = g^*(\overline{\nabla}^\alpha\beta,\gamma) + g^*(\beta,\overline{\nabla}^\alpha\gamma) \end{cases}$$

for any α, β, $\gamma \in \Gamma(T^*M)$.

$\mathcal{F}_g$**-connections.** For any $X, Y \in \operatorname{Im} \sharp_g$ [assume $X = \sharp_g\alpha$, $Y = \sharp_g\beta$, $\alpha, \beta \in \Gamma(T^*M)$], it follows that $[X, Y] = \sharp_g([\alpha, \beta])$, so $\operatorname{Im} \sharp_g$ is an integrable distribution whose associated foliation will be denoted by $\mathcal{F}_g$ and called the *canonical foliation* associated to g. If the almost product structure E is compatible with g, then the distribution $\operatorname{Im} \sharp_g$ is E-invariant. Indeed, for $X \in \operatorname{Im} \sharp_g$ [assume $X = \sharp_g\alpha$, $\alpha \in \Gamma(T^*M)$], it follows that $EX = E(\sharp_g\alpha) = \sharp_g(E^*\alpha)$. We say that an arbitrary contravariant connection $\overline{\nabla}$ is $\mathcal{F}_g$*-connection* if $\alpha \in \Gamma(\ker \sharp_g)$ implies $\overline{\nabla}^{\alpha}\beta \in \Gamma(\ker \sharp_g)$ for any $\beta \in \Gamma(T^*M)$. Following this definition, the contravariant connection $\overline{\nabla}$ induced by the covariant connection ∇ is $\mathcal{F}_g$-connection.

Proposition 2.2. *Let E be an almost product structure on the Riemannian manifold (M, g), compatible with g and ∇ a covariant connection on M.*

1. *If ∇g is symmetric, then $\alpha \in \Gamma(\ker \sharp_g)$ implies $\overline{\nabla}^{\beta}\alpha \in \Gamma(\ker \sharp_g)$ for any $\beta \in \Gamma(T^*M)$.*
2. *If $\nabla g = 0$, then $\alpha \in \Gamma((\ker \sharp_g)^{\perp})$ implies $\overline{\nabla}^{\beta}\alpha \in \Gamma((\ker \sharp_g)^{\perp})$ for any $\beta \in \Gamma(T^*M)$.*

Proof. 1. Let $\alpha \in \Gamma(\ker \sharp_g)$. Then according to Proposition 2.1,

$$\sharp_g(\overline{\nabla}^{\beta}\alpha) = -\sharp_g(T^{\overline{\nabla}}(\alpha, \beta)) - \sharp_g([\alpha, \beta]) = T_{\nabla}(\sharp_g\alpha, \sharp_g\beta) - [\sharp_g\alpha, \sharp_g\beta] = 0$$

for any $\beta \in \Gamma(T^*M)$.
2. Let $\gamma \in \Gamma(\ker \sharp_g)$. From Proposition 2.1,

$$\begin{aligned} g^*(\overline{\nabla}^{\beta}\alpha, \gamma) &= -(\overline{\nabla}g^*)(\beta, \alpha, \gamma) + \sharp_g\beta(g^*(\alpha, \gamma)) - g^*(\alpha, \overline{\nabla}^{\beta}\gamma) \\ &= -(\overline{\nabla}g^*)(\beta, \alpha, \gamma) + \sharp_g\beta(g(\sharp_g\alpha, \sharp_g\gamma)) - g(\sharp_g\alpha, \sharp_g(\overline{\nabla}^{\beta}\gamma)) = 0 \end{aligned}$$

for any α, $\beta \in \Gamma(T^*M)$. □

Restricting to a distribution. Let $\mathcal{D} \subset TM$ be an arbitrary distribution. Using the isomorphism $\flat_g$ between the tangent and cotangent bundles, we consider $\mathcal{D}^* \subset T^*M$ such that

$$\Gamma(\mathcal{D}^*) := \{\alpha \in \Gamma(T^*M) : \text{ there exists } X \in \Gamma(\mathcal{D}) \text{ such that } \alpha = i_X g\}.$$

Generalizing the definition for ∇ [**8**], we say that the extended connection $\widetilde{\nabla}$ *restricts to* $\mathcal{D}^*$ if for any $\alpha \in \Gamma(\mathcal{D}^*)$ it implies $\widetilde{\nabla}_X\alpha \in \Gamma(\mathcal{D}^*)$ for any $X \in \Gamma(TM)$ and respectively, that the contravariant connection $\overline{\nabla}$ *restricts to* $\mathcal{D}^*$ if for any $\beta \in \Gamma(\mathcal{D}^*)$ it implies $\overline{\nabla}^{\alpha}\beta \in \Gamma(\mathcal{D}^*)$ for any $\alpha \in \Gamma(T^*M)$. Then:

Proposition 2.3. *If ∇ is a metric connection with respect to g and it restricts to $\mathcal{D}$, then $\widetilde{\nabla}$ and $\overline{\nabla}$ also restrict to $\mathcal{D}^*$.*

Proof. Let $\beta \in \Gamma(\mathcal{D}^*)$. Then there exists $Y \in \Gamma(\mathcal{D})$ such that $\beta = i_Y g$ and for any $X \in \Gamma(TM)$ it follows that

$$\begin{aligned} (\widetilde{\nabla}_X\beta)(Z) = [\widetilde{\nabla}_X(i_Y g)](Z) &:= X(g(Y, Z)) - g(Y, \nabla_X Z) \\ &= g(\nabla_X Y, Z) = (i_{\nabla_X Y} g)(Z) \end{aligned}$$

for any $Z \in \Gamma(TM)$. Also,

$$\overline{\nabla}^{\alpha}\beta = \widetilde{\nabla}_{\sharp_g \alpha}\beta = i_{\nabla_{\sharp_g \alpha} Y} g$$

for any $\alpha \in \Gamma(T^*M)$. $\square$

Remark that also for any $\alpha \in \Gamma(\mathcal{D}^*)$ [assume $\alpha = i_X g$, $X \in \Gamma(\mathcal{D})$], $\overline{\nabla}^{\alpha}\alpha = \widetilde{\nabla}_{\sharp_g \alpha}\alpha = \widetilde{\nabla}_X(i_X g) = i_{\nabla_X X} g \in \Gamma(\mathcal{D}^*)$.

We can also characterize the integrability of the distribution $\mathcal{D}$ using the contravariant connection $\overline{\nabla}$ in the following way.

Proposition 2.4. *Assume that ∇g is symmetric. Then the distribution $\mathcal{D}$ is integrable if and only if $T^{\overline{\nabla}}(\alpha, \beta) \in \Gamma(\mathcal{D}^*)$ for any α, $\beta \in \Gamma(\mathcal{D}^*)$.*

Proof. Let α, $\beta \in \Gamma(\mathcal{D}^*)$. Then there exist X, $Y \in \Gamma(\mathcal{D})$ such that $\alpha = i_X g$, $\beta = i_Y g$. According to [**8**], $T_{\nabla}(X, Y) \in \Gamma(\mathcal{D})$ is equivalent to $\sharp_g(T^{\overline{\nabla}}(\alpha, \beta)) \in \Gamma(\mathcal{D})$ or to $T^{\overline{\nabla}}(\alpha, \beta) \in \Gamma(\mathcal{D}^*)$. $\square$

Concerning the invariance of the subspace $\mathcal{D}^*_x$ of $T^*_x M$, for $x \in M$, we can remark the following proposition.

Proposition 2.5. *Let $x \in M$ and u, $v \in T_x M$. Then the endomorphism $R^{\widetilde{\nabla}}_x(u, v)$ of $T^*_x M$ leaves invariant to the subspace $\mathcal{D}^*_x$.*

Proof. Let $x \in M$, u, $v \in T_x M$ and $\alpha_x \in \mathcal{D}^*_x$. Then there exists $w \in \mathcal{D}_x$ such that $\alpha_x = i_w g_x$. Then according to [**8**], $R_{\nabla x}(u, v, w) \in \mathcal{D}_x$. For any $z \in T_x M$

$$\begin{aligned}\left[R^{\widetilde{\nabla}}_x(u, v, \alpha_x)\right](z) &:= -\alpha_x(R_{\nabla x}(u, v, z)) \\ &= -i_w g_x(R_{\nabla x}(u, v, z)) = i_{R_{\nabla x}(u,v,w)} g_x(z)\end{aligned}$$

and so $R^{\widetilde{\nabla}}_x(u, v, \alpha_x) = i_{R_{\nabla x}(u,v,\sharp_{g_x}\alpha_x)} g_x \in \mathcal{D}^*_x$. $\square$

3. Contravariant product conjugate connections

We shall consider

$$\widetilde{\nabla}^{(E^*)} := \widetilde{\nabla} + E^*\widetilde{\nabla}E^*, \quad \text{and respectively,} \quad \overline{\nabla}^{(E^*)} := \overline{\nabla} + E^*\overline{\nabla}E^* \tag{3.1}$$

which have the following expressions $\widetilde{\nabla}^{(E^*)}_X \alpha = E^*(\widetilde{\nabla}_X E^*\alpha)$ and $\overline{\nabla}^{(E^*)\alpha}\beta = E^*(\overline{\nabla}^{\alpha} E^*\beta)$ for any $X \in \Gamma(TM)$ and α, $\beta \in \Gamma(T^*M)$ and whose properties are stated in the next proposition.

Proposition 3.1. *Let E be an almost product structure on the Riemannian manifold (M, g) and ∇ a covariant connection on M. Then $\widetilde{\nabla}^{(E^*)}$ and $\overline{\nabla}^{(E^*)}$ have the following properties:*

1. $\widetilde{\nabla}^{(E^*)}E^* = -\widetilde{\nabla}E^*$ *and* $\overline{\nabla}^{(E^*)}E^* = -\overline{\nabla}E^*$;

2.
$$R^{\widetilde{\nabla}^{(E^*)}}(X,Y,\alpha) = E^*(R^{\widetilde{\nabla}}(X,Y,E^*\alpha))$$
and
$$R^{\overline{\nabla}^{(E^*)}}(\alpha,\beta,\gamma) = E^*(R^{\overline{\nabla}}(\alpha,\beta,E^*\gamma))$$
for any X, $Y \in \Gamma(TM)$ and α, β, $\gamma \in \Gamma(T^*M)$, where the $(1,3)$-tensor field $R^{\overline{\nabla}}$ is defined
$$R^{\overline{\nabla}}(\alpha,\beta,\gamma) := \overline{\nabla}^{\alpha}\overline{\nabla}^{\beta}\gamma - \overline{\nabla}^{\beta}\overline{\nabla}^{\alpha}\gamma - \overline{\nabla}^{[\alpha,\beta]}\gamma,$$
for $[\alpha,\beta] := \flat_g([\sharp_g\alpha, \sharp_g\beta])$;

3. *if E is compatible with the Riemannian metric g, then*
$$(\widetilde{\nabla}^{(E^*)}_X g^*)(\alpha,\beta) = (\nabla_X g)(E(\sharp_g\alpha), E(\sharp_g\beta))$$
and
$$(\overline{\nabla}^{(E^*)\alpha} g^*)(\beta,\gamma) = (\nabla_{\sharp_g\alpha} g)(E(\sharp_g\beta), E(\sharp_g\gamma))$$
for any $X \in \Gamma(TM)$ and α, β, $\gamma \in \Gamma(T^*M)$.

Proof. 1.
$$\begin{aligned}(\widetilde{\nabla}_X E^*)\alpha &:= \widetilde{\nabla}_X E^*\alpha - E^*(\widetilde{\nabla}_X\alpha)\\ &= E^*(\widetilde{\nabla}^{(E^*)}_X\alpha) - \widetilde{\nabla}^{(E^*)}_X E^*\alpha := -(\widetilde{\nabla}^{(E^*)}_X E^*)\alpha\end{aligned}$$
for any $X \in \Gamma(TM)$ and $\alpha \in \Gamma(T^*M)$;
2.
$$\begin{aligned}R^{\widetilde{\nabla}^{(E^*)}}(X,Y,\alpha) &:= \widetilde{\nabla}^{(E^*)}_X\widetilde{\nabla}^{(E^*)}_Y\alpha - \widetilde{\nabla}^{(E^*)}_Y\widetilde{\nabla}^{(E^*)}_X\alpha - \widetilde{\nabla}^{(E^*)}_{[X,Y]}\alpha\\ &= \widetilde{\nabla}^{(E^*)}_X E^*(\widetilde{\nabla}_Y E^*\alpha) - \widetilde{\nabla}^{(E^*)}_Y E^*(\widetilde{\nabla}_X E^*\alpha) - E^*(\widetilde{\nabla}_{[X,Y]}E^*\alpha)\\ &= E^*(\widetilde{\nabla}_X\widetilde{\nabla}_Y E^*\alpha) - E^*(\widetilde{\nabla}_Y\widetilde{\nabla}_X E^*\alpha) - E^*(\widetilde{\nabla}_{[X,Y]}E^*\alpha)\\ &:= E^*(R^{\widetilde{\nabla}}(X,Y,E^*\alpha))\end{aligned}$$
for any X, $Y \in \Gamma(TM)$ and $\alpha \in \Gamma(T^*M)$;
3.
$$\begin{aligned}(\widetilde{\nabla}^{(E^*)}_X g^*)(\alpha,\beta) &:= X(g^*(\alpha,\beta)) - g^*(\widetilde{\nabla}^{(E^*)}_X\alpha,\beta) - g^*(\alpha,\widetilde{\nabla}^{(E^*)}_X\beta)\\ &= X(g^*(\alpha,\beta)) - g^*(E^*(\widetilde{\nabla}_X E^*\alpha),\beta) - g^*(\alpha,E^*(\widetilde{\nabla}_X E^*\beta))\\ &= X(g(E(\sharp_g\alpha),E(\sharp_g\beta))) - g(E(\sharp_g(E^*(\widetilde{\nabla}_X E^*\alpha))),E(\sharp_g\beta))\\ &\quad - g(E(\sharp_g\alpha),E(\sharp_g(E^*(\widetilde{\nabla}_X E^*\beta))))\\ &= X(g(E(\sharp_g\alpha),E(\sharp_g\beta))) - g(\nabla_X\sharp_g(E^*\alpha),E(\sharp_g\beta))\\ &\quad - g(E(\sharp_g\alpha),\nabla_X\sharp_g(E^*\beta))\\ &:= (\nabla_X g)(E(\sharp_g\alpha),E(\sharp_g\beta))\end{aligned}$$
for any $X \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$.

For $\overline{\nabla}$ it follows immediately from the properties of $\widetilde{\nabla}$. □

Example 3.1. Let $\eta\in\Gamma(T^*M)$ such that $\nabla_X E=\eta(X)E$ for any $X\in\Gamma(TM)$. Then $\widetilde{\nabla}^{(E^*)}_X E^* = -\eta(X)E^*$ for any $X \in \Gamma(TM)$, $\overline{\nabla}^{(E^*)\alpha} E^* = -\eta(\sharp_g\alpha)E^*$ for any $\alpha \in \Gamma(T^*M)$. If $\nabla_X E = \eta(X)I_{\Gamma(TM)}$ for any $X \in \Gamma(TM)$, then $\widetilde{\nabla}^{(E^*)}_X E^* = -\eta(X)I_{\Gamma(T^*M)}$ for any $X \in \Gamma(TM)$, $\overline{\nabla}^{(E^*)\alpha} E^* = -\eta(\sharp_g\alpha)I_{\Gamma(T^*M)}$, for any $\alpha \in \Gamma(T^*M)$.

Remark that if $\overline{\nabla}$ is $\mathcal{F}_g$-connection, the $\overline{\nabla}^{(E^*)}$ is also $\mathcal{F}_g$-connection, because for $\sharp_g\alpha = 0$, we have $\sharp_g(\overline{\nabla}^{(E^*)\alpha}\beta) = \sharp_g(E^*(\overline{\nabla}^{\alpha} E^*\beta)) = E(\sharp_g(\overline{\nabla}^{\alpha} E^*\beta)) = 0$ for any $\beta \in \Gamma(T^*M)$.

Assume now that ∇ is a metric connection and the arbitrary distribution $\mathcal{D}$ is E-invariant [that is for any $X \in \Gamma(\mathcal{D})$, it follows $EX \in \Gamma(\mathcal{D})$]. If ∇ restricts to $\mathcal{D}$, then $\widetilde{\nabla}^{(E^*)}$ and $\overline{\nabla}^{(E^*)}$ also restrict to $\mathcal{D}^*$. Indeed, let $\alpha \in \Gamma(\mathcal{D}^*)$. Then there exists $X \in \Gamma(\mathcal{D})$ such that $\alpha = i_X g$. But for any $Y \in \Gamma(TM)$, $(E^*\alpha)(Y) := \alpha(EY) = g(X, EY) = g(EX, Y) = (i_{EX} g)(Y)$, so $E^*\alpha \in \Gamma(\mathcal{D}^*)$. Therefore, from Proposition 2.3 for any $X \in \Gamma(TM)$, $\widetilde{\nabla}_X E^*\alpha \in \Gamma(\mathcal{D}^*)$ and consequently, $\widetilde{\nabla}^{(E^*)}_X \alpha = E^*(\widetilde{\nabla}_X E^*\alpha) \in \Gamma(\mathcal{D}^*)$. A similar argument holds for $\overline{\nabla}^{(E^*)}$.

4. The extended structural and virtual tensors for $\widetilde{\nabla}$ and $\overline{\nabla}$

Recall that the deformation tensor by passing from a metric g to f^*g, where f is a geodesic transformation preserving the almost product structure E [**6**], can be written

$$T(\nabla)(X, Y) = \psi(X)Y + \psi(Y)X, \qquad X, Y \in \Gamma(TM)$$

for $\psi \in \Gamma(T^*M)$ and ∇ the Levi-Civita connection is associated to g. In this case [**6**], the structural tensor is defined

$$C(X, Y) := \frac{1}{2}[(\nabla_{EX} E)Y + (\nabla_X E)EY], \qquad X, Y \in \Gamma(TM) \tag{4.1}$$

and respectively, the virtual tensor by

$$B(X, Y) := \frac{1}{2}[(\nabla_{EX} E)Y - (\nabla_X E)EY], \qquad X, Y \in \Gamma(TM). \tag{4.2}$$

Denote by ∇' the Levi-Civita connection is associated to f^*g and $\widetilde{\nabla}$, $\widetilde{\nabla}'$, $\overline{\nabla}$, $\overline{\nabla}'$ the extensions and respectively, the contravariant connections are induced by ∇ and ∇'. Similarly we can compute the deformation tensors for $\widetilde{\nabla}$ and $\overline{\nabla}$, so we get

$$\left[T(\widetilde{\nabla})(X, \beta)\right](Y) = -\beta\left[T(\nabla)(X, Y)\right] = -\beta\left[\psi(X)Y + \psi(Y)X\right],$$

$$\left[T(\overline{\nabla})(\alpha, \beta)\right](Y) = -\beta[T(\nabla)(\sharp_g\alpha, Y)] = -\beta[\psi(\sharp_g\alpha)Y + \psi(Y)\sharp_g\alpha]$$

for any $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$. Then

$$\left[(\widetilde{\nabla}'_X E^*)\beta\right](Y) = [(\widetilde{\nabla}_X E^*)\beta](Y) - \beta[\psi(Y)EX - \psi(EY)X],$$

$$\left[(\overline{\nabla}'^{\alpha} E^*)\beta\right](Y) = [(\overline{\nabla}^{\alpha} E^*)\beta](Y) - \beta[\psi(Y)E(\sharp_g\alpha) - \psi(EY)\sharp_g\alpha]$$

for any $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$.

Proposition 4.1. *The extended structural and virtual tensors (defined for* $\widetilde{\nabla}$*,* $\widetilde{\nabla}'$*,* $\overline{\nabla}$*,* $\overline{\nabla}'$*) satisfy:*

1. $[\tilde{B}'(X,\beta)](Y) = [\tilde{B}(X,\beta)](Y), \qquad [\bar{B}'(\alpha,\beta)](Y) = [\bar{B}(\alpha,\beta)](Y);$
2. $[\tilde{C}'(X,\beta)](Y) = [\tilde{C}(X,\beta)](Y) - \beta[\psi(Y)X - \psi(EY)EX],$
   $[\bar{C}'(\alpha,\beta)](Y) = [\bar{C}(\alpha,\beta)](Y) - \beta[\psi(Y)\sharp_g\alpha - \psi(EY)E(\sharp_g\alpha)]$

for any $X,\ Y \in \Gamma(TM)$ *and* $\alpha,\ \beta \in \Gamma(T^*M)$*. Moreover, the extended structural tensor satisfies*

$$E^*(\tilde{C}(X,\beta)) = -\tilde{C}(EX,\beta) = -\tilde{C}(X,E^*\beta),$$
$$E^*(\bar{C}(\alpha,\beta)) = -\bar{C}(E^*\alpha,\beta) = -\bar{C}(\alpha,E^*\beta)$$

for any $X \in \Gamma(TM)$*,* $\alpha,\ \beta \in \Gamma(T^*M)$*.*

Notice that

$$\left[\tilde{C}(EX,\beta)\right](Y) = \beta(B(EX,Y)),$$
$$\left[\tilde{C}(X,E^*\beta)\right](Y) = E^*\beta(B(X,Y))$$

for any $X, Y \in \Gamma(TM)$, $\beta \in \Gamma(T^*M)$.

Concerning $\widetilde{\nabla}^{(E^*)}$ and $\overline{\nabla}^{(E^*)}$, remark also that

$$\tilde{C}^{(E^*)}(X,\beta) = -\tilde{C}(X,\beta), \qquad \tilde{B}^{(E^*)}(X,\beta) = -\tilde{B}(X,\beta),$$
$$\bar{C}^{(E^*)}(\alpha,\beta) = -\bar{C}(\alpha,\beta), \qquad \bar{B}^{(E^*)}(\alpha,\beta) = -\bar{B}(\alpha,\beta)$$

for any $X \in \Gamma(TM)$ and $\alpha,\ \beta \in \Gamma(T^*M)$.

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References

1. Blaga A. M., *Extended connections on Hermitian manifolds*, to appear.

2. Fernandes R., *Connections in Poisson geometry, I. Holonomy anf invariants*, J. Diff. Geom., **54(2)** (2000), 303–365.

3. Hawkins E., *Noncommutative Rigidity*, Commun. Math. Phys., **246** (2004), 211–235.

4. Hawkins E., *The structure of noncommutative deformations*, arXiv:math.QA/0504232.

5. Karabegov A. V., *Fedosov's formal symplectic grupoids and contravariant connections*, J. Geom. Phys., **56** (2006), 1985–2009.

6. Kirichenko V. F., *Method of generalized Hermitian geometry in the theory of almost contact manifold*, Itogi Nauki I tech. VINITI, AN SSSR, M. **18** (1986), 25–71.

7. Lewis A. D., *Energy-preserving affine connections*, preprint, 1997.

8. Lewis A. D., *Affine connections and distributions*, Reports on Math. Phys., **42** (1/2) (1998), 135–164.

9. Pradines J., *Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux*, Comptes rendus Acad. Sci. Paris, **264 A** (1967), 245–248.

10. Reshetikhin N., Voronov A. and Weinstein A., *Semiquantum Geometry, Algebraic Geometry*, J. Math. Sci., **82(1)** (1996), 3255–3297.

11. Vaisman I., *Lecture on the geometry of Poisson manifolds*, Progr. In Math., **118** (1994), Birkhäuser.

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