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COUNTER-EXAMPLE FOR LIOUVILLE THEOREMS FOR INDEFINITE PROBLEMS ON HALF SPACES

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Abstract. The goal of this paper is a construction of an counter example to Liouville theorems for indefinite problems on half spaces. Since Liouville theorems are closely related to the scaling method for elliptic and parabolic problems, our counter=example indicates that one has to impose additional assumptions on the nodal set of nonlinearity in order to obtain a priori estimates for indefinite elliptic problems. The counter-example is constructed by shooting method in one-dimensional case and then extended to higher dimensions.

1. Introduction

This paper is motivated by studies of the indefinite elliptic problems of the form

$$\begin{aligned} -\Delta u &= m(x)|u|^{p-1}u, && x \in \Omega\,, \\ u &= 0 && x \in \partial\Omega\,, \end{aligned} \tag{1}$$

and the parabolic counterparts. In this context the indefinite problem means that the function m changes sign in $\bar{\Omega}$. Here, and below we assume that $\Omega \subset \mathbb{R}^N$ is a smooth domain (of class $C^{2,\alpha}$ for some $\alpha > 0$) and the problem is superlinear and subcritical, that is, $1 < p < p_S$, where $p_S := \infty$ for $N = 1, 2$ and $p_S := (N+2)/(N-2)$ for $N \geq 3$. The assumptions on the function m will be specified below.

Indefinite elliptic problems attracted a lot of attention during recent decades see e.g [**1, 2, 5, 6, 7, 16**] and references therein. In order to investigate their qualitative properties it is important to obtain a priori bounds for solutions. By a priori estimates we mean estimates of the form

$$\|u\|_X \leq C(N, p, \Omega, m)\,, \tag{2}$$

where $X := L^\infty(\bar{\Omega})$. We remark that analogous estimates occur in the study of blow-up rates of solutions of parabolic problems see e.g. [**9, 14, 17**] and references therein.

A priori estimates can be obtained by various strategies (see [**15**]). In this paper we focus on the scaling method, which often yields optimal results with respect to the exponent p, if the precise asymptotics of the nonlinearity is known.

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Let us briefly explain how the scaling method connects a priori estimates and Liouville theorems. Detailed exposition for elliptic and parabolic problems can be found for example in [**7, 9, 13**]. We are not going to discuss the optimality of assumptions, especially assumptions on the exponent p. An interested reader can find a detailed analysis in [**15**], see also references therein.

In this paper the term Liouville theorem refers to the following statement. Any bounded, non-negative solution of a given problem is trivial, that is, the solution is zero everywhere. Equivalently, there is no non-trivial, non-negative, bounded solution of a given problem.

Before we proceed, we need the following notation:

$$\mathbb{R}^N_c := \{x = (x_1, x') \in \mathbb{R}^N : x_1 > c\} \qquad (c \in \mathbb{R}),$$

and

$$\Omega^+ := \{x \in \Omega : m(x) > 0\}, \qquad \Omega^- := \{x \in \Omega : m(x) < 0\},$$
$$\Omega^0 := \{x \in \Omega : m(x) = 0\}.$$

Assume that m is a continuous function and there are positive continuous functions α_1, α_2 defined on the small neighborhood of Ω_0 in Ω and $\gamma_1, \gamma_2 > 0$ such that

$$m(x) = \begin{cases} \alpha_1(x)[\mathrm{dist}(x, \Omega_0)]^{\gamma_1} & x \in \Omega^+, \\ \alpha_2(x)[\mathrm{dist}(x, \Omega_0)]^{\gamma_2} & x \in \Omega^-. \end{cases}$$

We assume that (2) fails, that is, we assume that for each $k \in \mathbb{N}$ there exist a solution u_k of the problem (1) and $x_k \in \Omega$ such that

$$u_k(x_k) \geq 2k \qquad (k \in \mathbb{N}).$$

After an application of doubling lemma (see [**13**, Lemma 5.1]), appropriate scaling, and elliptic regularity we can distinguish the following cases.

If there is a subsequence of $(x_k)_{k\in\mathbb{N}}$ (denoted again $(x_k)_{k\in\mathbb{N}}$) such that $x_k \to x_0$ with $x_0 \in \bar{\Omega}$ and $x_0 \notin \bar{\Omega}^0$, then there must exist a bounded nonnegative function v with $v(0) = 1$ that solves

$$0 = \Delta v + \kappa v^p, \qquad x \in \mathbb{R}^N, \tag{3}$$

or

$$\begin{aligned} 0 &= \Delta v + \kappa v^p, && x \in \mathbb{R}^N_{c^*}, \\ v &= 0, && x \in \partial\mathbb{R}^N_{c^*} \end{aligned} \tag{4}$$

for some $c^* < 0$, where $\kappa \in \{-1, 1\}$. However, by the results of Gidas and Spruck [**10**] if $\kappa = 1$ and by [**4, 8**] if $\kappa = -1$, the Liouville theorem holds for problem (3) and (4), provided $1 < p < p_S$. Hence, $v \equiv 0$, which contradicts $v(0) = 1$.

If $x_0 \in \bar{\Omega}^0$, then the problem is more involved and was discussed in [**7**], see also references therein, under the assumption $\bar{\Omega}^0 \subset \Omega$, that is, m does not vanish on $\partial\Omega$. Then v with $v(0) = 1$ can, in addition to (3) and (4), solve

$$0 = \Delta v + h(x_1)v^p, \qquad x \in \mathbb{R}^N, \tag{5}$$

where $h(x) = x^{\gamma_1}$ for $x > 0$, $h(x) = -|x|^{\gamma_2}$ for $x < 0$, and γ_1, γ_2 are positive constants. However, by [**7, 12**] the problem (5) satisfies the Liouville theorem for any continuous, nondecreasing function h, such that

$$h(0) = 0, \quad h \text{ is strictly increasing for } x > 0, \quad \lim_{x\to\infty} h(x) = \infty\,. \tag{6}$$

Hence $v \equiv 0$, a contradiction to $v(0) = 1$. We remark that we can allow h to depend on x_1 only, since the general problem can be transformed to (5), with h satisfying (6), by an appropriate translation and rotation.

The situation in the remaining case is more interesting. If we allow $\bar{\Omega}_0 \cap \partial\Omega \neq \emptyset$, then v with $v(0) = 1$ can, in addition to the cases above, solve

$$\begin{aligned} 0 &= \Delta v + h(x \cdot b)v^p, && x \in \mathbb{R}^N_{c^*}\,, \\ v &= 0, && x \in \partial\mathbb{R}^N_{c^*}\,, \end{aligned} \tag{7}$$

where b is a unit vector, $c^* < 0$, and $h(x) = x^{\gamma_1}$ for $x > 0$ and $h(x) = -|x|^{\gamma_2}$ for $x < 0$. Notice that we cannot guarantee $b = e_1 := (1, 0, \cdots, 0)$ by any rotation or translation, since the problem is defined on the half space. In order to obtain a contradiction as above, one has to prove Liouville theorem for (7). It follows, with additional assumptions on b and c^*, from the following result proved in [**9**].

Corollary 1. *Assume $b \neq -e_1$ and $c^* \in \mathbb{R}$, or $b = -e_1$ and $c^* \geq 0$. If $h\colon \mathbb{R} \to \mathbb{R}$ is continuous, non-decreasing function with $h(x) < 0$ for $x < 0$ such that* (6) *holds, then there is no non-negative, non-trivial, bounded solution v of* (7)*.*

We remark that the result in [**9**] treats more general nonlinearities. If $b \neq -e_1$, then the assumption $h(x) < 0$ for $x < 0$ is not needed. In the case $b = -e_1$ and $c^* \geq 0$, Liouville theorem holds under more general assumptions on h (see [**9, 17**]). One might expect that Liouville theorem will continue to be true when $b = -e_1$ and $c^* < 0$.

However, the main result of this paper (see Proposition 1 below), shows that such Liouville theorem does not hold. More precisely, if $b = -e_1$, then for each $c^* < 0$ there exists a bounded, positive solution of (7). The construction of the solution u_1 in one dimensional case ($N = 1$) is based on the shooting method in two directions. A counter-example u_N in N dimensions is obtained by the trivial extension of the one dimensional solution, that is, $u_N(x) := u_1(x_1)$ for each $x = (x_1, x') \in \mathbb{R}^N_{c^*}$. Similarly, one can obtain a counter-example to parabolic Liouville theorems.

This counter-example shows that the scaling method needs additional assumptions on m, if $m(x_0) = 0$ for some $x_0 \in \partial\Omega$. For example we need to assume, as in [**9**], that Ω_0 intersects $\partial\Omega$ transversally.

Since one might consider more general functions m, or one might be interested in the investigated ordinary differential equations without applications to Liouville theorems, we consider more general problems than required by our counter-examples.

More specifically, assume that $h \in C(\mathbb{R})$ satisfies

$$h(x) > 0 \quad \text{for} \quad x > 0, \qquad h(x) < 0 \quad \text{for} \quad x < 0, \tag{8}$$

$$\int_{-\infty}^{0} h(x)\,\mathrm{d}x = -\infty\,, \qquad \int_{0}^{\infty} h(x)\,\mathrm{d}x = \infty\,, \tag{9}$$

$$\text{there exists } \varepsilon^* > 0 \text{ such that } h \text{ is non-decreasing on } (-\varepsilon^*, 0). \tag{10}$$

The main result of the paper is the following proposition.

Proposition 1. *Let $p > 1$ and assume that a continuous function h satisfies* (8) – (10)*. Then for each $a > 0$ there exists a bounded, non-negative, nontrivial solution u of the problem*

$$\begin{aligned} &u'' = h(x)|u|^{p-1}u, && x \in (-a, \infty)\,, \\ &u(-a) = 0\,. \end{aligned} \tag{11}$$

Moreover, $u'(x) < 0$ for $x \geq 0$ and $\lim_{x\to\infty} u(x) = 0$.

Remark 1. The nonlinearity $|u|^{p-1}u$ can be replaced by a locally Lipschitz function $f : [0, \infty) \to \mathbb{R}$, such that $f(0) = 0$, $f(u) > 0$ for $u > 0$, f is non-decreasing for $u > 0$, and

$$\lim_{u\to\infty} \frac{f(u)}{u} = \infty, \qquad \lim_{u\to 0^+} \frac{f(u)}{u} = 0\,.$$

If we extend f as a locally Lipschitz function to whole $\mathbb{R}$ such that $f(u) < 0$ for $u < 0$, then the arguments are the same as for $f(u) = |u|^{p-1} = \infty$.

If the assumption

$$\int_{0}^{\infty} h(x)\,\mathrm{d}x = \infty$$

is removed, Proposition 1 still holds true without the statement $\lim_{x\to\infty} u(x) = 0$.

If the problem is scale invariant, then the proof can be simplified and we can also address the question of uniqueness.

Proposition 2. *If $h(x) = \operatorname{sign}(x)|x|^\alpha$ for some $\alpha > 0$, then the solution in Proposition 1 is unique.*

The following corollary states a counter-example to Liouville theorem for indefinite problems on half spaces. It shows that Corollary 1 cannot be improved. A counterexample is given by a function $v(x_1, \cdots, x_N) = u(x_1)$, where u is a function from Proposition 1.

Corollary 2. *If $b = -e_1$, $c^* < 0$, and h satisfies* (8) – (10)*, then the problem* (7) *possesses a bounded, nonnegative solution.*

2. Proof of Proposition 1 and Proposition 2

Let us prove Proposition 1 first. Fix $\xi \in (0, \infty)$. Let $u_k : (\tau_k, T_k) \to \mathbb{R}$ be the solution of the initial value problem

$$
\begin{aligned}
&u_k'' = h(x)|u_k|^{p-1}u_k\,, && x \in (\tau_k, T_k)\,, \\
&u_k(0) = \xi\,, && u_k'(0) = k\,,
\end{aligned}
\tag{12}
$$

where (τ_k, T_k) is the maximal existence interval of u_k. By a standard theory, $-\infty \le \tau_k < 0 < T_k \le \infty$.

Remark 2. In the first part of the proof we show that for each $\xi > 0$, there exists a unique $k(\xi)$ such that (12) has a decreasing positive solution on $(0, \infty)$, hence $T_k = \infty$. Although we use a shooting method, there are other approaches, mentioned in this remark, that yield partial results for solutions on $(0, \infty)$.

Decay at infinity. If $h(x) = |x|^\alpha$ for some $\alpha > 0$, then one can proceed as in [**11**, Theorem 2.1] and obtain that for $1 < p < p_S$, every solution u of (12) with $T_k = \infty$ satisfies $u(x) \le C|x|^{-\frac{2+\alpha}{p-1}}$ and $|u'| \le C|x|^{-\frac{p+1+\alpha}{p-1}}$ for each $x > 1$. Observe that [**11**] discusses problem with $h(x) = -|x|^\alpha$, but one can easily modify the proof of [**11**, Lemma 2.1] by replacing Liouville theorem of Gidas and Spruck [**10**] by ones in [**4, 8**].

Variational approach. Let X be the Banach space of functions with finite norm

$$
\|u\|_X := \left(\frac{1}{2}\int_0^\infty (u'(x))^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + \left(\frac{1}{p+1}\int_0^\infty h(x)|u(x)|^{p+1} \,\mathrm{d}x\right)^{\frac{1}{p+1}}.
$$

Then it is easy to check that the functional

$$
F[u] := \frac{1}{2}\int_0^\infty (u'(x))^2 \,\mathrm{d}x + \frac{1}{p+1}\int_0^\infty h(x)|u(x)|^{p+1} \,\mathrm{d}x
$$

is coercive, strictly convex, and continuous. Moreover, the set $M_\xi := \{u \in X : u(0) = \xi\}$ is convex and closed (therefore weakly closed) so there exists a unique global minimizer of F on M_ξ. The minimizer satisfies Euler-Lagrange equation (12) on $(0, \infty)$ for some $k(\xi)$. Also u is positive, as $F[u] = F[|u|]$ and every non-negative, non-trivial solution of (12) is positive. Notice that this method also implies the decay of the minimizer at infinity.

However, the variational approach guarantees the uniqueness of the solution in the space X only, but we cannot guarantee $u \in X$ a priori. Also, it gives merely existential result and it does not specify how k depends on ξ, which will be important in the second part of the proof.

Fowler transformation. If $h(x) = |x|^\alpha$, one can proceed as in [**3**] and transform the problem by Fowler transformation $X(t) := -xu'u^{-1}$, $Z(t) := x^{1+\alpha}u^p(u')^{-1}$, and $x = \mathrm{e}^t$. Then X and Z satisfy

$$
\begin{aligned}
X' &= X[X + Z + 1], \\
Z' &= Z[(1+\alpha) - pX - Z]\,.
\end{aligned}
\tag{13}
$$

The existence of solutions of (12) on $(0,\infty)$ is equivalent to the existence of heteroclinic trajectories connecting equilibria $(0,0)$, $(\frac{2+\alpha}{p-1}, -\frac{p+1+\alpha}{p-1})$ of the system (13). This approach yields very precise asymptotic behavior of u: $-XZ = x^{2+\alpha}u^{p-1} \to \frac{(2+\alpha)(p+1+\alpha)}{(p-1)^2}$ as $x \to \infty$ (and analogous expression for u').
However, since this method does not apply readily to general h and the proof of the existence of heteroclinic orbits is not elementary, we rather use other approach.

We prove the existence of solutions for (12) by shooting method. Notice that this method applies to general h and no decay of u is required. Moreover, it allows us to derive more precise information on dependence of k on ξ.

Claim 1. *If $u_k'(x_0) \geq 0$ and $u_k(x_0) > 0$ for some $x_0 > 0$, then $u_k'(x) > 0$ for each $x > x_0$ and $\lim_{x\to T_k} u_k(x) = \infty$.*

Proof. By (8), $u_k''(x_0) = h(x_0)u_k^p(x_0) > 0$, and therefore $u_k'(x) > u_k'(x_1) > u_k'(x_0) \geq 0$, for each $x > x_1 > x_0$ sufficiently close to x_0. If $u_k'(x) > u_k'(x_1) > 0$ for each $x > x_1$, Claim 1 follows.

Otherwise, there exists the smallest $x_2 > x_1$ with $u_k'(x_2) = u_k'(x_1)$. Then $u_k'(x) > 0$ on $[x_1, x_2]$, and consequently $u_k(x) > 0$ on $[x_1, x_2]$. Moreover, for each $x \in [x_1, x_2]$ one has $u_k''(x) = h(x)u_k^p(x) > 0$, that is, u_k is strictly convex on $[x_1, x_2]$, a contradiction to $u_k'(x_2) = u_k'(x_1)$. □

Claim 2. *If $u_k(x_0) \leq 0$ for some $x_0 > 0$, then $u_k(x) < 0$ for each $x > x_0$ and $\lim_{x\to T_k} u_k(x) = -\infty$.*

Proof. Let $x^* := \inf\{x > 0 : u_k(x) = 0\}$. Since $u_k(0) = \xi > 0$, x^* is well defined and $x^* > 0$. Suppose that there is $x_1 > x^*$ such that $u_k(x_1) \geq 0$. Then either $u \geq 0$ on $[x^*, x_1]$, or u has a negative minimum at $x_2 \in [x^*, x_1]$. In the first case x^* is a local minimizer of u. By the uniqueness of solutions of initial value problems one has $u \equiv 0$, a contradiction to $u(0) = \xi > 0$. In the second case $u_k''(x_2) = h(x_2)|u_k|^{p-1}u_k(x_2) < 0$, a contradiction. Hence, $x_0 = x^*$ and $u < 0$ on (x_0, ∞).

Finally, since $u_k'' = h(x)|u|^{p-1}u(x) < 0$ for each $x \in (x_0, \infty)$, u_k is concave on (x_0, ∞) and the second statement follows. □

Denote

$$\mathcal{K}_0 := \{k : u_k(x) \leq 0 \text{ for some } x \geq 0\}\,,$$
$$\mathcal{K}_2 := \{k : u_k(x) \geq 2\xi \text{ for some } x \geq 0\}\,.$$

Claim 3. *The sets $\mathcal{K}_0$ and $\mathcal{K}_2$ are non-empty, open, and disjoint. Moreover $(-\infty, -2\xi - H\xi^p) \subset \mathcal{K}_0$, where $H := \sup_{x\in[0,1]} h(x)$.*

Proof. From Claim 1 it follows that $(0,\infty) \subset \mathcal{K}_2 \neq \emptyset$. If $k \in \mathcal{K}_0$, then $\lim_{x\to T_k} u_k(x) = -\infty$ and if $k \in \mathcal{K}_2$, then $u_k'(x) > 0$ and $u(x) > 0$ for some $x > 0$, and therefore $\lim_{x\to T_k} u_k(x) = \infty$. Thus $\mathcal{K}_0 \cap \mathcal{K}_2 = \emptyset$.

If $k_0 \in \mathcal{K}_0$, then, by Claim 2, there exists $x_1 > 0$ such that $u_{k_0}(x_1) < -1$. The continuous dependence of solutions on initial data implies $u_k(x_1) < -1$ for any k sufficiently close to k_0. Thus, $\mathcal{K}_0$ is open.

Analogously if $k_0 \in \mathcal{K}_2$, then by Claim 1 there is x_0 such that $u_{k_0}(x_0) > 3\xi$. Then the continuous dependence of solutions on the initial data yields $u_k(x_0) > 3\xi$ for any k sufficiently close to k_0. Thus, $\mathcal{K}_2$ is open as well.

Finally, we show the second statement, which also implies $\mathcal{K}_0 \neq \emptyset$. Fix $k < -2\xi - H\xi^p$ and suppose that there is the smallest $x_0 \in [0,1]$ with $u_k'(x_0) = -\xi$. Without loss of generality assume $u \geq 0$ on $[0, x_0]$, otherwise $k \in \mathcal{K}_0$ and there is nothing to prove. Then $u_k(x) \leq u_k(0) = \xi$ on $(0, x_0)$. However,

$$u_k'(x_0) = u_k'(0) + \int_0^{x_0} u_k''(x)\,\mathrm{d}x = k + \int_0^{x_0} h(x)u_k^p(x)\,\mathrm{d}x \leq k + H\xi^p < -\xi\,,$$

a contradiction.

Hence, $u_k'(x) < -\xi$ on $[0,1]$, and therefore $u_k(x) \leq 0$ for some $x \in [0,1]$. $\square$

Denote

$$M := \mathbb{R} \setminus (\mathcal{K}_0 \cap \mathcal{K}_2)$$

and note that by Claim 3, $M \neq \emptyset$. Also, by Claim 1, $u_k' < 0$ in $(0,\infty)$ for each $k \in M$, and therefore

$$0 \geq u_k'(x) = u_k'(0) + \int_0^{x} u_k''(t)\,\mathrm{d}t = k + \int_0^{x} h(t)u_k^p(t)\,\mathrm{d}t \geq k + u_k^p(x)\int_0^{x} h(t)\,\mathrm{d}t\,,$$

and therefore

$$0 < u_k(x) \leq (-k)^{\frac{1}{p}} \left(\int_0^{x} h(t)\,\mathrm{d}t\right)^{-\frac{1}{p}} \tag{14}$$

and the decay of u follows from (9). Also, (14) implies $k \neq 0$ for each $\xi > 0$. Moreover, it yields a decay rate of u, which is however not optimal for $h(x) = x^\alpha$.

Claim 4. $M = \{k^*\}$.

Proof. Suppose that there are $k_1, k_2 \in M$ with $k_1 > k_2$. Then for a sufficiently small $x_0 > 0$, one has

$$u_{k_1}(x_0) - u_{k_2}(x_0) > 0 \quad \text{and} \quad (u_{k_1} - u_{k_2})'(x) > 0 \qquad (x \in [0, x_0])\,.$$

Since $\lim_{x\to\infty} u_{k_1}(x) - u_{k_2}(x) = 0$, there exists the smallest $x_1 > x_0$ with $u_{k_1}'(x_1) = u_{k_2}'(x_1)$. Then $u_{k_1}(x) > u_{k_2}(x)$ for $x \in (x_0, x_1)$; however,

$$\begin{aligned} u_{k_1}'(x_1) &= u_{k_1}'(x_0) + \int_{x_0}^{x_1} u_{k_1}''(x)\,\mathrm{d}x = u_{k_1}'(x_0) + \int_{x_0}^{x_1} h(x)u_{k_1}^p(x)\,\mathrm{d}x \\ &> u_{k_2}'(x_0) + \int_{x_0}^{x_1} h(x)u_{k_2}^p(x)\,\mathrm{d}x = u_{k_2}'(x_0) + \int_{x_0}^{x_1} u_{k_2}''(x)\,\mathrm{d}x = u_{k_2}'(x_1)\,, \end{aligned}$$

a contradiction. $\square$

Define the function $k\colon (0,\infty) \to (-\infty, 0)$ such that $k(\xi)$ is the unique k for which the problem (12) has a bounded positive solution on $(0,\infty)$. Let u_ξ be the solution of such problem:

$$\begin{aligned} u_\xi'' &= h(x)u_\xi^p\,, \qquad && x \in (\tau_\xi, \infty)\,, \\ u_\xi(0) &= \xi\,, && u_\xi'(0) = k(\xi)\,, \end{aligned} \tag{15}$$

where τ_ξ defines the existence time of u_ξ. Recall that u_ξ is decreasing and decays to 0 as $x \to \infty$. Notice that the subscript now indicates the value of $u_\xi(0)$ rather than $u_\xi'(0)$.

Claim 5. *The function $k : (0,\infty) \to (-\infty, 0)$ is a continuous, strictly decreasing with $\lim_{\xi\to\infty} k(\xi) = -\infty$, and $\lim_{\xi\to 0^+} k(\xi) = 0$.*

Proof. First, let us prove continuity. For a contradiction suppose that there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \xi_n = \xi_0 \in (0,\infty)$ such that $k(\xi_0) \neq \lim_{n\to\infty} k(\xi_n) =: M$. Let u be the solution of the problem (12) with $u'(0) = k$ replaced by $u'(0) = M$. Since $M \neq k(\xi_0)$, the solution is either not bounded above or not positive. Thus, by Claim 1 and Claim 2 there exists x_0 such that either $u(x_0) < -2$ or $u(x_0) > 3\xi_0$. The continuous dependence of solutions on initial conditions yields that $u_{\xi_n}(x_0) < -2$ or $u_{\xi_n}(x_0) > 2\xi_n$ for sufficiently large n. This contradicts the definition of $k(\xi_n)$, and proves that k is continuous.

Next, we prove monotonicity of k. Fix $\xi_1, \xi_2 \in (0,\infty)$. Subtracting equations (15) for u_{ξ_1} and u_{ξ_2}, multiplying by $u_{\xi_1} - u_{\xi_2}$ and integrating on the interval $[0,x]$, we obtain

$$\int_0^x (u_{\xi_1}'' - u_{\xi_2}'')(u_{\xi_1} - u_{\xi_2})\,\mathrm{d}t = \int_0^x h(t)[u_{\xi_1}^p - u_{\xi_2}^p](u_{\xi_1} - u_{\xi_2})\,\mathrm{d}t\,,$$

where we do not indicate the dependence of u_{ξ_i} on t. An integration by parts and positivity of h yield for any $x > 1$

$$\begin{aligned} &(u_{\xi_1}' - u_{\xi_2}')(u_{\xi_1} - u_{\xi_2})(x) - (u_{\xi_1}' - u_{\xi_2}')(u_{\xi_1} - u_{\xi_2})(0) \\ &\qquad = \int_0^x (u_{\xi_1}' - u_{\xi_2}')^2\,\mathrm{d}t + \int_0^x h(t)[u_{\xi_1}^p - u_{\xi_2}^p](u_{\xi_1} - u_{\xi_2})\,\mathrm{d}t \geq C_{\xi_1,\xi_2}\,, \end{aligned}$$

where $C_{\xi_1,\xi_2} > 0$ whenever $\xi_1 \neq \xi_2$ and C_{ξ_1,ξ_2} is independent of $x > 1$. Since u_{ξ_i} $(i = 1, 2)$ decays monotonically to 0, one has

$$\begin{aligned} 0 &= \liminf_{x\to\infty} (u_{\xi_1}' - u_{\xi_2}')(u_{\xi_1} - u_{\xi_2})(x) \geq (u_{\xi_1}' - u_{\xi_2}')(u_{\xi_1} - u_{\xi_2})(0) + C_{\xi_1,\xi_2} \\ &= (k(\xi_1) - k(\xi_2))(\xi_1 - \xi_2) + C_{\xi_1,\xi_2}\,, \end{aligned}$$

and the strict monotonicity follows.

From Claim 3 and the negativity of k it follows that $0 > k(\xi) \geq -2\xi - H\xi^p$, and the statement $\lim_{\xi\to 0^+} k(\xi) = 0$ follows.

We finish the proof by showing that $k(\xi) \leq -\frac{\xi}{2}$ for large ξ. Otherwise, there exists large ξ such that $k(\xi) > -\frac{\xi}{2}$ and the convexity of u_ξ yields that $u_\xi'(x) > -\frac{\xi}{2}$

for each $x \in [0,1]$. Hence, $u_\xi(x) > \frac{\xi}{2}$ for each $x \in [0,1]$. Since u_ξ is a nonincreasing function

$$\begin{aligned} 0 \geq u_\xi'(1) &= u_\xi'(0) + \int_0^1 u_\xi''(t)\,\mathrm{d}t \\ &= k(\xi) + \int_0^1 h(t) u_\xi^p(t)\,\mathrm{d}t \geq -\frac{\xi}{2} + \left(\frac{\xi}{2}\right)^p \int_0^1 h(t)\,\mathrm{d}t\,, \end{aligned}$$

a contradiction for sufficiently large ξ. □

Claim 6. *For each $\xi > 0$, there exists $x^* < 0$ such that $u_\xi(x^*) = 0$.*

Proof. For a contradiction assume $u_\xi(x) > 0$ for each $x \in (\tau_\xi, 0)$. Since $u_\xi''(x) = h(x)u_\xi^p < 0$, u_ξ is concave on $(\tau_\xi, 0)$. Therefore, $0 \leq u_\xi(x) \leq \xi + u_\xi'(0)x$ for each $x \in (\tau_\xi, 0)$, and in particular $\tau_\xi = -\infty$.

Next, we show that $u_\xi'(x_0) > 0$ for some $x_0 < 0$. If not, then u_ξ decreases on $(-\infty, 0)$ and $u_\xi(x) \geq u_\xi(0) = \xi$ for all $x < 0$. However,

$$\begin{aligned} 0 \geq u_\xi'(x) = u_\xi'(0) - \int_x^0 u_\xi''(s)\,\mathrm{d}s &= k(\xi) - \int_x^0 h(s) u_\xi^p(s)\,\mathrm{d}s \\ \geq k(\xi) - \xi^p \int_x^0 h(s)\,\mathrm{d}s\,, \end{aligned}$$

a contradiction to (9) for large negative x.

Thus $u_\xi'(x_0) > 0$ for some $x_0 < 0$, and since u_ξ is concave, $u_\xi'(x) \geq u_\xi'(x_0) > 0$ for each $x < x_0$. Hence, $u_\xi(x^*) = 0$ for some $x^* < 0$, a contradiction. □

Denote $a(\xi) := \sup\{x < 0 : u_\xi(x) = 0\}$. By Claim 6, a is well defined and negative for each ξ. Also, the continuous dependence of k on ξ implies the continuity of a.

Claim 7. *The range of a is $(-\infty, 0)$, that is, $\mathcal{R} := \{a(\xi) : \xi \in (0,\infty)\} = (-\infty, 0)$.*

Proof. By the continuity of a is suffices to prove $\sup \mathcal{R} = 0$ and $\inf \mathcal{R} = -\infty$.

First, for a contradiction assume $\max\{\sup \mathcal{R}, -\varepsilon^*\} =: -\varepsilon < 0$, where ε^* was defined in (10). We show that for a sufficiently large ξ, $u_\xi'(x) = 0$ for some $x \in [-\frac{\varepsilon}{4}, 0]$. For a contradiction suppose $u_\xi'(x) < 0$ for each $x \in [-\frac{\varepsilon}{4}, 0]$. Then, u_ξ decreases on $[-\frac{\varepsilon}{4}, 0]$, and by (10), $u_\xi'' = h(x)u_\xi^p$ increases on $[-\frac{\varepsilon}{4}, 0]$.

If $u_\xi'(x) \geq \frac{k(\xi)}{2}$ for some $x \in (-\frac{\varepsilon}{8}, 0)$, then the increasing second derivative of u_ξ yields $u_\xi'(x) = 0$ for some $x \in [-\frac{\varepsilon}{4}, 0]$, a contradiction. Otherwise $u_\xi'(x) < \frac{k(\xi)}{2}$ for all $x \in (-\frac{\varepsilon}{8}, 0)$, and therefore

$$u_\xi\left(-\frac{\varepsilon}{8}\right) \geq -\frac{k(\xi)}{2}\frac{\varepsilon}{8} + \xi \geq -\frac{\varepsilon}{16}k(\xi).$$

Since u_ξ decreases on $[-\frac{\varepsilon}{4}, 0]$, $u_\xi(x) \geq u_\xi(-\frac{\varepsilon}{8}) \geq -\frac{\varepsilon}{16}k(\xi)$ for each $x \in (-\frac{\varepsilon}{4}, -\frac{\varepsilon}{8})$. Moreover,

$$
\begin{aligned}
0 > u_\xi'\left(-\frac{\varepsilon}{4}\right) &= u_\xi'(0) - \int_{-\frac{\varepsilon}{4}}^{0} u_\xi''(t)\,\mathrm{d}t = k(\xi) - \int_{-\frac{\varepsilon}{4}}^{0} h(t)u_\xi^p(t)\,\mathrm{d}t \\
&\geq k(\xi) - \int_{-\frac{\varepsilon}{4}}^{-\frac{\varepsilon}{8}} h(t)\left(-\frac{\varepsilon k(\xi)}{16}\right)^p \mathrm{d}t = k(\xi) - c_\varepsilon |k(\xi)|^p\,,
\end{aligned}
$$

where $c_\varepsilon > 0$, a contradiction for a sufficiently large $k(\xi)$ (and by Claim 5, for sufficiently large ξ).

Let $b_\xi := \sup\{x < 0 : u_\xi'(x) = 0\}$. We showed that $b_\xi \geq -\frac{\varepsilon}{4}$ for any sufficiently large ξ. Let $U_\xi := u_\xi(b_\xi)$, then $U_\xi \geq \xi$ since u_ξ decreases on $(b_\xi, 0)$. Assume that there exists $x \in (-\frac{\varepsilon}{2}, b_\xi)$ such that $u_\xi(x) < U_\xi/2$. Then the concavity of u_ξ yields that $u_\xi(x) < 0$ for some $x \in (-\varepsilon, b_\xi)$, a contradiction to the definition of ε. Hence, $u_\xi(x) > U_\xi/2$ for each $x \in (-\frac{\varepsilon}{2}, b_\xi)$. However, by the Taylor's theorem

$$
\begin{aligned}
0 < u_\xi\left(-\frac{\varepsilon}{2}\right) &= u_\xi(b_\xi) + u_\xi'(b_\xi)(-\frac{\varepsilon}{2} - b_\xi) - \int_{-\frac{\varepsilon}{2}}^{b_\xi} \left(-\frac{\varepsilon}{2} - t\right) u_\xi''(t)\,\mathrm{d}t \\
&= U_\xi + \int_{-\frac{\varepsilon}{2}}^{b_\xi} \left(\frac{\varepsilon}{2} + t\right) h(t)u_\xi^p(t)\,\mathrm{d}t \leq U_\xi + \frac{U_\xi^p}{2^p} \int_{-\frac{\varepsilon}{2}}^{-\frac{\varepsilon}{4}} \left(\frac{\varepsilon}{2} + t\right) h(t)\,\mathrm{d}t \\
&= U_\xi + c_\varepsilon U_\xi^p\,,
\end{aligned}
$$

where $c_\varepsilon < 0$, a contradiction for sufficiently large U_ξ, and therefore ξ. We have showed $\sup \mathcal{R} = 0$.

Assume $M := \inf \mathcal{R} > -\infty$. First, we claim $\lim_{\xi\to 0^+} u_\xi(b_\xi) = 0$, where b_ξ was defined above. Otherwise, there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ converging to 0 such that $\lim_{n\to\infty} u_{\xi_n}(b_{\xi_n}) =: \delta > 0$. Since u_ξ is concave, $u_{\xi_n}(b_{\xi_n}) \leq \xi_n + k(\xi_n)b_{\xi_n}$, and therefore $b_{\xi_n} < (u_{\xi_n}(b_{\xi_n}) - \xi_n)/k(\xi_n)$ (recall $k(\xi) < 0$). By Claim 5, $k(\xi_n) \to 0^-$ as $n \to \infty$ and $u_{\xi_n}(b_{\xi_n}) - \xi_n \to \delta$. Thus $b_{\xi_n} \to -\infty$ as $n \to \infty$. Since u_ξ decreases on $(b_\xi, 0)$, it is positive there, and consequently $M \leq a(\xi_n) \leq b_{\xi_n} \to -\infty$, a contradiction. Therefore, $u_\xi(b_\xi) \to 0$ as $\xi \to 0^+$.

Since u_ξ is concave, u_ξ increases on $(a(\xi), b_\xi)$. Hence, $u_\xi(x) \leq u_\xi(b_\xi)$ for each $x \in (a(\xi), b_\xi)$. Then, again by the Taylor's theorem

$$
\begin{aligned}
0 = u_\xi(a(\xi)) &= u_\xi(b_\xi) + u_\xi'(b_\xi)(a(\xi) - b_\xi) - \int_{a(\xi)}^{b_\xi} (a(\xi) - t)u_\xi''(t)\,\mathrm{d}t \\
&= u_\xi(b_\xi) - \int_{a(\xi)}^{b_\xi} (a(\xi) - t)h(t)u^p(t)\,\mathrm{d}t \geq u_\xi(b_\xi) - u_\xi^p(b_\xi) \int_{a(\xi)}^{b_\xi} (a(\xi) - t)h(t)\,\mathrm{d}t \\
&\geq u_\xi(b_\xi) - u_\xi^p(b_\xi) \int_{-M}^{0} (-M - t)h(t)\,\mathrm{d}t = u_\xi(b_\xi) - c_M u_\xi^p(b_\xi)\,,
\end{aligned}
$$

where $c_M > 0$, a contradiction for small $u_\xi(b_\xi)$ (that is, small ξ). □

This finishes the proof of Proposition 1. □

Proof of Proposition 2. It is trivial to check, that assumptions (8)–(10) are satisfied for $h(x) = \operatorname{sign}(x)|x|^\alpha$, and therefore all claims in the proof of Proposition 1 holds true. In particular, for each $a < 0$ there exists a solution of (11). Fix a and two bounded, positive, nontrivial, solutions u, v of (11). Notice, by the scale invariance, that $v_\lambda(x) = \lambda^{\frac{2+\alpha}{p-1}} v(\lambda x)$ satisfies the equation in (11) and v_λ is a positive bounded function.

Without loss of generality assume $u(0) \leq v(0)$. Then there exists $\lambda \in (0, 1]$ such that $v_\lambda(0) = u(0)$. Moreover, Claim 4 yields that $v'_\lambda(0) = u'(0)$, and consequently $u = v_\lambda$ by the uniqueness of the initial value problem. If $\lambda \neq 1$, then $0 = u(-a) = v_\lambda(-a) = \lambda^{\frac{2+\alpha}{p-1}} v(-\lambda a) > 0$, a contradiction. Thus, $\lambda = 1$ and $u = v$, the uniqueness follows. □

References

1. Amann H. and López-Gómez J., *A priori bounds and multiple solutions for superlinear indefinite elliptic problems.* J. Differential Equations **146(2)**(1998), 336–374.
2. Berestycki H., Capuzzo-Dolcetta I., and Nirenberg L., *Superlinear indefinite elliptic problems and nonlinear Liouville theorems.* Topol. Methods Nonlinear Anal. **4(1)** (1994), 59–78.
3. Bidaut-Veron M. F. and Giacomini H., *A new dynamical approach of Emden-Fowler equations and systems.* Adv. Differential Equations **15(11-12)** (2010), 1033–1082.
4. Brezis H., *Semilinear equations in* $\mathbf{R}^N$ *without condition at infinity.* Appl. Math. Optim. **12(3)** (1984), 271–282.
5. Chen W. and Li C., *A priori estimates for prescribing scalar curvature equations.* Ann. of Math. (2), **145(3)** (1997), 547–564.
6. Dong W. and Mei L., *Multiple solutions for an indefinite superlinear elliptic problem on* $\mathbb{R}^N$. Nonlinear Anal. **73(7)** (2010), 2056–2070.
7. Du Y. and Li S., *Nonlinear Liouville theorems and a priori estimates for indefinite superlinear elliptic equations.* Adv. Differential Equations **10(8)** (2005), 841–860.
8. Farina A. and Serrin J., *Entire solutions of completely coercive quasilinear elliptic equations, II.* J. Differential Equations **250(12)** (2011), 4409–4436.
9. Földes J., *Liouville theorems, a priori estimates, and blow-up rates for solutions of indefinite superlinear parabolic problems.* Czechoslovak Math. J., **61(136)(1)** (2011) 169–198.
10. Gidas B. and Spruck J., *A priori bounds for positive solutions of nonlinear elliptic equations.* Comm. Partial Differential Equations **6(8)** (1981), 883–901.
11. Phan Q. H. and Souplet P., *Liouville-type theorems and bounds of solutions of Hardy-Hénon equations.* J. Differential Equations **252(3)** (2012), 2544–2562.
12. Poláčik P. and Quittner P., *Liouville type theorems and complete blow-up for indefinite superlinear parabolic equations.* In Nonlinear elliptic and parabolic problems, vol. 64 of Progr. Nonlinear Differential Equations Appl., Birkhäuser, Basel, 2005.
13. Poláčik P., Quittner P., and Souplet P., *Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems.* Duke Math. J. **139(3)** (2007), 555–579.
14. Quittner P. and F. Simondon. *A priori bounds and complete blow-up of positive solutions of indefinite superlinear parabolic problems.* J. Math. Anal. Appl. **304(2)**(2005), 614–631.
15. Quittner P. and Souplet P., *Superlinear parabolic problems.* Blow-up, global existence and steady states. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2007.

16. Ramos M., Terracini S., and Troestler C., *Superlinear indefinite elliptic problems and Pohožaev type identities.* J. Funct. Anal. **159(2)** (1998), 596–628.

17. Xing R., *The blow-up rate for positive solutions of indefinite parabolic problems and related Liouville type theorems.* Acta Math. Sin. (Engl. Ser.) **25(3)**(2009), 503–518.

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