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ON THE BLOW-UP OF SOLUTIONS FOR THE b-EQUATION

M. KODZHA

Abstract. We establish blow-up results for a family of equations under various classes of initial data. It turns out that it is the shape instead of the size and smoothness of the initial data which influences breakdown in finite time. Then, infinite propagation speed for the shallow water equations is proved in the following sense: the corresponding solution $u(t,x)$ with compactly supported initial datum $u_0(x)$ does not have compact x-support any longer in its lifespan.

1. Introduction

In the paper we study the following nonlinear dispersive equation

$$\begin{cases} u_t - \alpha^2 u_{txx} + c_0 u_x + (b+1)uu_x + \Gamma u_{xxx} = \alpha^2(bu_x u_{xx} + uu_{xxx}), \\ t > 0, \quad x \in \mathbb{R}, \\ \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where c_0, b, Γ, α are arbitrary real constants. This equation, model wave motion in the shallow water regime, can be derived as the family of asymptotically equivalent shallow water wave equations [**16, 17**]. Using the notation $y = u - \alpha^2 u_{xx}$, we can rewrite Eq. (1.1) as follows:

$$\begin{cases} y_t + c_0 u_x + uy_x + bu_x y + \Gamma u_{xxx} = 0, \\ \\ y(0,x) = y_0(x) = u_0(x) - \alpha u_{0xx}(x). \end{cases} \tag{1.2}$$

The b-equation (1.2) can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation, cf. [**16, 17**].

If $\alpha = 0$ and $b = 2$, then Eq. (1.2) becomes the well-known KdV equation

$$u_t + c_0 u_x + 3uu_x + \Gamma u_{xxx} = 0,$$

which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity, cf. [**15**]. In this model $u(t,x)$ represents the wave's height above a flat bottom, x is proportional to distance in the direction of propagation and t is proportional to the elapsed time.

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For $b = 2$ and $\Gamma = 0$, Eq. (1.2) becomes the Camassa-Holm equation modelling the unidirectional propagation of shallow water waves over a flat bottom. Again $u(t, x)$ stands for the fluid velocity at time t in the spatial x direction and c_0 is a nonnegative parameter related to the critical shallow water speed [**1, 15**].

The Cauchy problem for the Camassa-Holm equation has been studied extensively. It was shown that this equation is locally well-posed [**12, 22, 24**] for initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. More interestingly, it has global strong solutions [**12**] and also finite time blow-up solutions [**8, 12, 22**]. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking [**2, 8**].

If $b = 3$ and $c_0 = \Gamma = 0$ in Eq. (1.2), then we find the Degasperis-Procesi equation [**14**]. The formal integrability of the Degasperis-Procesi equation was obtained in [**13**] by constructing a Lax pair. It has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and admits exact peakon solutions which are analogous to the Camassa-Holm peakons [**13**].

The paper is organized as follows. In Section 2 we consider the b-equation. In this section, previously known results for the initial data and the bifurcation parameter in the b-equation are improved. In Section 3, the corresponding strong solution $u(t, x)$ of the b-equation in its lifespan with u_0 being compactly supported are described in detail.

2. Blow-up

In this section, we consider (1.1). The Cauchy problem for the (1.1) was studied in [**3, 18**]. For (1.1) the blow-up occurs as wave breaking, that is, the solution remains bounded, but its slope becomes infinite in finite time. Conditions on the initial data and the bifurcation parameter $b \geq 3$ for which corresponding solutions blow-up in finite time are found in [**18**]. We expand this result to $b \geq 2$ using ideas in [**26**].

Consider the differential equation

$$\begin{cases} q_t = u(t, q) - \dfrac{\Gamma}{\alpha^2}, & t \in [0, T), \\ q(0, x) = x, \quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Differentiation of Eq. (2.1) with respect to x yields to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} q_x = q_{xt} = u_x(t, q) q_x, & t \in [0, T), \\ q_x(0, x) = 1, & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

The solution of Eq. (2.2) is given by

$$q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x)) \mathrm{d}s \right), \qquad (t, x) \in [0, T) \times \mathbb{R} \tag{2.3}$$

and if $c_0 + \frac{\Gamma}{\alpha^2} = 0$, then [see [**18**]]

$$y(t, q(t, x))[q_x(t, x)]^b = y_0(x). \tag{2.4}$$

We first recall the following lemma.

Lemma 2.1 ([**26**]). *Suppose that $\Psi(t)$ is twice continuously differential satisfying*

$$\begin{cases} \Psi''(t) \geq D_0 \Psi'(t)\Psi(t), & t > 0, \ \ D_0 > 0. \\ \Psi(0) > 0, & \Psi'(0) > 0. \end{cases} \tag{2.5}$$

Then $\Psi(t)$ blows up in finite time. Moreover the blow-up time T can be estimated in terms of the initial datum as

$$T \leq \max\left\{ \frac{2}{D_0 \Psi(0)}, \frac{\Psi(0)}{\Psi'(0)} \right\}.$$

Now we will present the main result in this section. Analogously, result is obtained in [**18**] for $b \geq 3$, $c_0 = \Gamma = 0$ and some conditions on initial date $u_0(x)$. We extend this result to $b > 2$.

Theorem 2.1. *Let $b > 2$ and $c_0 = \Gamma = 0$. Suppose that $u_0 \in H^2(\mathbb{R})$ and there exists $x_0 \in \mathbb{R}$ such that $y_0(x_0) = (1 - \alpha^2 \partial_x^2) u_0(x_0) = 0$, and*

$$y_0(x) \geq 0 \ \ for \ \ x \in (-\infty, x_0) \quad and \quad y_0(x) \leq 0 \ \ for \ \ x \in (x_0, \infty). \tag{2.6}$$

Then, the corresponding solution $u(t,x)$ of (1.1) *blows up in finite time with lifespan*

$$T \leq \max\left\{ \frac{-2}{u_{0x}(x_0)}, \frac{-2\alpha^2 u_{0x}(x_0)}{\alpha^2 u_{0x}^2(x_0) - u_0^2(x_0)} \right\}.$$

Proof. Suppose that the solution exists globally. Due to equation (2.4) and the initial condition, we have $y(t, q(t, x_0)) = 0$ and

$$\begin{cases} y(t, q(t, x_0)) \geq 0 & \text{for } \ x \in (-\infty, x_0) \\ y(t, q(t, x_0)) \leq 0 & \text{for } \ x \in (x_0, \infty) \end{cases} \tag{2.7}$$

for all t.

Since $u(x,t) = G * y(t,x)$, $x \in \mathbb{R}$, $t \geq 0$ (where $G(x) := \frac{1}{2\alpha} e^{-|\frac{x}{\alpha}|}$ and $(1 - \alpha^2 \partial_x^2)^{-1} f = G * f$), one can write $u(t,x)$ and $u_x(t,x)$ as

$$u(t,x) = \frac{1}{2\alpha} e^{-\frac{x}{\alpha}} \int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} y(t,\xi) \mathrm{d}\xi + \frac{1}{2\alpha} e^{\frac{x}{\alpha}} \int_{x}^{\infty} e^{-\frac{\xi}{\alpha}} y(t,\xi) \mathrm{d}\xi,$$

$$\alpha u_x(t,x) = -\frac{1}{2\alpha} e^{-\frac{x}{\alpha}} \int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} y(t,\xi) \mathrm{d}\xi + \frac{1}{2\alpha} e^{\frac{x}{\alpha}} \int_{x}^{\infty} e^{-\frac{\xi}{\alpha}} y(t,\xi) \mathrm{d}\xi.$$

Consequently,

$$\alpha^2 u_x^2(t,x) - u^2(t,x) = -\frac{1}{\alpha^2} \int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} y(t,\xi) \mathrm{d}\xi \int_{x}^{\infty} e^{-\frac{\xi}{\alpha}} y(t,\xi) \mathrm{d}\xi.$$

From the expression of $u_x(t,x)$ in terms of $y(t,x)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}u_x(t,q(t,x_0)) &= u_{tx}(t,q(t,x_0)) + u_{xx}(t,q(t,x_0))q_t \\
&= u_{tx}(t,q(t,x_0)) + \frac{(u-y)}{\alpha^2}q_t \\
&= u_{tx}(t,q(t,x_0)) + \frac{u}{\alpha^2}(u - \frac{\Gamma}{\alpha^2}) \\
&= \frac{1}{\alpha^2}u^2(t,q(t,x_0)) - \frac{1}{2\alpha^2}\,\mathrm{e}^{-\frac{q(t,x_0)}{\alpha}}\int_{-\infty}^{q(t,x_0)}\mathrm{e}^{\frac{\xi}{\alpha}}\,y_t(t,\xi)\mathrm{d}\xi \\
&\quad + \frac{1}{2\alpha^2}\,\mathrm{e}^{\frac{q(t,x_0)}{\alpha}}\int_{q(t,x_0)}^{\infty}\mathrm{e}^{-\frac{\xi}{\alpha}}\,y_t(t,\xi)\mathrm{d}\xi.
\end{aligned}$$

Rewrite equation (1.1) as

$$y_t + uy_x + 2u_xy + \frac{b-2}{2}(u^2 - \alpha^2u_x^2)_x = 0.$$

Using the identity, we can obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}u_x(t,q(t,x_0)) = \frac{1}{\alpha^2}u^2(t,q(t,x_0)) &+ \frac{1}{2\alpha^2}\,\mathrm{e}^{-\frac{q(t,x_0)}{\alpha}}\int_{-\infty}^{q(t,x_0)}\mathrm{e}^{\frac{\xi}{\alpha}}(uy_\xi + 2u_\xi y)\mathrm{d}\xi \\
&- \frac{1}{2\alpha^2}\,\mathrm{e}^{\frac{q(t,x_0)}{\alpha}}\int_{q(t,x_0)}^{\infty}\mathrm{e}^{-\frac{\xi}{\alpha}}(uy_\xi + 2u_\xi y)\mathrm{d}\xi \\
&+ \frac{b-2}{4\alpha^2}\,\mathrm{e}^{-\frac{q(t,x_0)}{\alpha}}\int_{-\infty}^{q(t,x_0)}\mathrm{e}^{\frac{\xi}{\alpha}}(u^2 - \alpha^2u_\xi^2)_\xi\mathrm{d}\xi \\
&- \frac{b-2}{4\alpha^2}\,\mathrm{e}^{\frac{q(t,x_0)}{\alpha}}\int_{q(t,x_0)}^{\infty}\mathrm{e}^{-\frac{\xi}{\alpha}}(u^2 - \alpha^2u_\xi^2)_\xi\mathrm{d}\xi.
\end{aligned}$$

By direct calculation we have

$$\begin{aligned}
&\int_{-\infty}^{q(t,x_0)}\mathrm{e}^{\frac{\xi}{\alpha}}(uy_\xi + 2yu_\xi)(t,\xi)d\xi \\
&\quad= \int_{-\infty}^{q(t,x_0)}\mathrm{e}^{\frac{\xi}{\alpha}}(u(t,\xi)y(t,\xi))_\xi d\xi + \int_{-\infty}^{q(t,x_0)}\mathrm{e}^{\frac{\xi}{\alpha}}\,y(t,\xi)u_\xi(t,\xi)d\xi \\
&\quad= -\frac{1}{\alpha}\int_{-\infty}^{q(t,x_0)}\mathrm{e}^{\frac{\xi}{\alpha}}\,u(t,\xi)y(t,\xi)d\xi + \frac{1}{2}\int_{-\infty}^{q(t,x_0)}\mathrm{e}^{\frac{\xi}{\alpha}}(u^2(t,\xi) - \alpha^2u_\xi^2(t,\xi))_\xi d\xi \\
&\quad= -\frac{1}{\alpha}\int_{-\infty}^{q(t,x_0)}\mathrm{e}^{\frac{\xi}{\alpha}}\left[u^2(t,\xi) + \frac{1}{2}\alpha^2u_\xi^2(t,\xi)\right]d\xi \\
&\qquad+ \left[\mathrm{e}^{\frac{\xi}{\alpha}}(\alpha u(t,\xi)u_x(t,\xi) - \frac{1}{2}\alpha^2u_x^2(t,\xi))\right]_{\xi=q(t,x_0)}
\end{aligned}$$

In the above relations we used that $y = u - \alpha^2 u_{xx}$ and integration by parts. We have

$$
\begin{aligned}
&\int_{-\infty}^{x} \mathrm{e}^{\frac{\xi}{\alpha}} \left[u^2(t,\xi) + \alpha^2 u_x^2(t,\xi)\right] d\xi \\
&\qquad \geq \int_{-\infty}^{x} \mathrm{e}^{\frac{\xi}{\alpha}}\, 2\alpha u(t,\xi) u_x(t,\xi) = \alpha \int_{-\infty}^{x} \mathrm{e}^{\frac{\xi}{\alpha}} (u^2(t,\xi))_x \\
&\qquad = \alpha\, \mathrm{e}^{\frac{\xi}{\alpha}}\, u^2(t,\xi)|_{-\infty}^{x} - \frac{\alpha}{\alpha}\int_{-\infty}^{x} \mathrm{e}^{\frac{\xi}{\alpha}}\, u^2(t,\xi) = \alpha\, \mathrm{e}^{\frac{x}{\alpha}}\, u^2(t,x) - \int_{-\infty}^{x} \mathrm{e}^{\frac{\xi}{\alpha}}\, u^2(t,\xi).
\end{aligned}
$$

The above inequality yields to

$$
\int_{-\infty}^{x} \mathrm{e}^{\frac{\xi}{\alpha}} \left[u^2(t,\xi) + \frac{1}{2}\alpha^2 u_x^2(t,\xi)\right] \mathrm{d}\xi \geq \frac{\alpha}{2}\, \mathrm{e}^{\frac{x}{\alpha}}\, u^2(t,x).
$$

Hence

$$
\begin{aligned}
&\frac{1}{2\alpha^2}\, \mathrm{e}^{-\frac{q(t,x_0)}{\alpha}} \int_{-\infty}^{q(t,x_0)} \mathrm{e}^{\frac{\xi}{\alpha}} (u y_x + 2 y u_x)(t,\xi) \mathrm{d}\xi \\
&\leq -\frac{1}{4\alpha^2} u^2(t,q(t,x_0)) - \frac{1}{4} u_x^2(t,q(t,x_0)) + \frac{1}{2\alpha} u(t,q(t,x_0)) u_x(t,q(t,x_0)).
\end{aligned}
\tag{2.8}
$$

Similarly, we have

$$
\begin{aligned}
&-\frac{1}{2\alpha^2}\, \mathrm{e}^{\frac{q(t,x_0)}{\alpha}} \int_{q(t,x_0)}^{\infty} \mathrm{e}^{-\frac{\xi}{\alpha}} (u y_x + 2 y u_x)(t,\xi) \mathrm{d}\xi \\
&\leq -\frac{1}{4\alpha^2} u^2(t,q(t,x_0)) - \frac{1}{4} u_x^2(t,q(t,x_0)) - \frac{1}{2\alpha} u(t,q(t,x_0)) u_x(t,q(t,x_0)).
\end{aligned}
\tag{2.9}
$$

Now using the inequality (see [**26**]),

$$
\alpha^2 u_x^2(t,x) - u^2(t,x) \leq (\alpha^2 u_x^2 - u^2)(t,q(t,x_0))
$$

and combining (2.8) and (2.9), we obtain

$$
\frac{b-2}{4\alpha^2}\, \mathrm{e}^{-\frac{q(t,x_0)}{\alpha}} \int_{-\infty}^{q(t,x_0)} \mathrm{e}^{\frac{\xi}{\alpha}} (u^2 - \alpha^2 u_\xi^2)_\xi \mathrm{d}\xi \leq 0.
$$

Similarly, we have

$$
\frac{b-2}{4\alpha^2}\, \mathrm{e}^{\frac{q(t,x_0)}{\alpha}} \int_{q(t,x_0)}^{\infty} \mathrm{e}^{-\frac{\xi}{\alpha}} (u^2 - \alpha^2 u_\xi^2)_\xi \mathrm{d}\xi \geq 0.
$$

Combining all the above terms together, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t} u_x(t,q(t,x_0)) \leq \frac{1}{2\alpha^2} u^2(t,q(t,x_0)) - \frac{1}{2\alpha^2}\alpha^2 u_x^2(t,q(t,x_0)).
\tag{2.10}
$$

Claim. $u_x(t,q(t,x_0)) < 0$ is decreasing and $u^2(t,q(t,x_0)) < \alpha^2 u_x^2(t,q(t,x_0))$ for all $t \geq 0$.

Suppose that the claim is no true, i.e., there exists t_0 such that $u^2(t, q(t, x_0)) < \alpha^2 u_x^2(t, q(t, x_0))$ on $[0, t_0]$ and $u^2(t_0, q(t_0, x_0)) = \alpha^2 u_x^2(t_0, q(t_0, x_0))$. Now, let

$$
\begin{aligned}
I(t) &= \frac{1}{2\alpha}\,\mathrm{e}^{-\frac{q(t,x_0)}{\alpha}} \int_{-\infty}^{q(t,x_0)} \mathrm{e}^{\frac{\xi}{\alpha}}\, y(t,\xi)\mathrm{d}\xi > 0,\\
II(t) &= \frac{1}{2\alpha}\,\mathrm{e}^{\frac{q(t,x_0)}{\alpha}} \int_{q(t,x_0)}^{\infty} \mathrm{e}^{-\frac{\xi}{\alpha}}\, y(t,\xi)\mathrm{d}\xi < 0.
\end{aligned}
$$

First, by the same trick as above, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}I(t)}{\mathrm{d}t} &\geq \frac{1}{4\alpha}(\alpha^2 u_x^2 - u^2) > 0\\
\frac{\mathrm{d}II(t)}{\mathrm{d}t} &\leq -\frac{1}{4\alpha}(\alpha^2 u_x^2 - u^2) < 0\\
(\alpha^2 u_x^2 - u^2)(t, q(t, x_0)) &= -4I(t)II(t) \geq -4I(0)II(0) > 0.
\end{aligned}
$$

This implies t_0 can be extended to infinity.

Moreover, due to the above inequality we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\alpha^2 u_x^2 - u^2)(t, q(t, x_0))\\
&\quad= 4\frac{\mathrm{d}}{\mathrm{d}t}I(t)\cdot(-II(t)) + 4I(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}(-II(t))\\
&\quad\geq \frac{1}{\alpha}(\alpha^2 u_x^2 - u^2)(t, q(t, x_0))[I(t) - II(t)]\\
&\quad= -u_x(t, q(t, x_0))(\alpha^2 u_x^2 - u^2)(t, q(t, x_0)).
\end{aligned}
\tag{2.11}
$$

Now substituting (2.10) in (2.11), we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\alpha^2 u_x^2 - u^2)(t, q(t, x_0)) &\geq \frac{1}{2\alpha^2}(\alpha^2 u_x^2 - u^2)(t, q(t, x_0))\\
&\quad\times\left[\int_0^t (\alpha^2 u_x^2 - u^2)(\tau, q(\tau, x_0))d\tau - 2\alpha^2 u_{0x}(x_0)\right].
\end{aligned}
$$

Now the theorem follows from Lemma 2.1 with

$$
\Psi(t) = \int_0^t (\alpha^2 u_x^2 - u^2)(\tau, q(\tau, x_0))\mathrm{d}\tau - 2\alpha^2 u_{0x}(x_0),
$$

and $D_0 = \frac{1}{2\alpha^2}$. Then, we complete our proof. □

3. Propagation speed

The purpose of this section is to give a detailed description of the corresponding strong solution $u(t, x)$ in its lifespan with u_0 being compactly supported. We will use the same ideas as in [**26**], where this problem is considered for $\alpha = 1$, $c_0 = \Gamma = 0$. The main theorem reads as follows.

Theorem 3.1. *Let* $0 < b < 3$ *and* $c_0 + \frac{\Gamma}{\alpha^2} = 0$. *Assume that the initial datum* $u_0 \in H^3(\mathbb{R})$ *is compactly supported in* $[c, d]$. *Then, the corresponding solution of* (1.1) *has the following property. For* $0 < t < T$

$$u(t,x) = \begin{cases} L(t)\,\mathrm{e}^{-\frac{x}{\alpha}} & \text{as } x > q(t,d), \\ l(t)\,\mathrm{e}^{\frac{x}{\alpha}} & \text{as } x < q(t,c), \end{cases}$$

with $L(t) > 0$ *and* $l(t) < 0$, *respectively, where* $q(t,x)$ *is defined by* (2.1) *and* T *is its lifespan. Furthermore,* $L(t)$ *and* $l(t)$ *denote continuous non-vanishing functions with* $L(t) > 0$ *and* $l(t) < 0$ *for* $t \in (0, T]$. *And* $L(t)$ *is a strictly increasing function while* $l(t)$ *is a strictly decreasing function.*

Proof. Since $u_0(x)$ has a compact support, so does $y_0(x) = (1 - \alpha^2\partial_x^2)u_0(x)$. From the equation (2.4) follows that $y(t,x) = (1 - \alpha^2\partial_x^2)u(t,x)$ is compactly supported in $[q(t,c), q(t,d)]$ in its lifespan. Hence the following functions are well defined

$$E(t) = \int_{\mathbb{R}} \mathrm{e}^{\frac{x}{\alpha}}\, y(t,x)\mathrm{d}x \quad \text{and} \quad F(t) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{x}{\alpha}}\, y(t,x)\mathrm{d}x$$

with $E(0) = 0 = F(0)$. For $x > q(t,d)$, we have

$$u(t,x) = \frac{1}{2\alpha}\,\mathrm{e}^{-\frac{|x|}{\alpha}} * y(t,x) = \frac{1}{2\alpha}\,\mathrm{e}^{-\frac{x}{\alpha}} \int_{q(t,c)}^{q(t,d)} e^{\frac{\xi}{\alpha}} y(t,\xi)\mathrm{d}\xi = \frac{1}{2\alpha}\,\mathrm{e}^{-\frac{x}{\alpha}}\, E(t) \tag{3.1}$$

and for $x < q(t,c)$, we have

$$u(t,x) = \frac{1}{2\alpha}\,\mathrm{e}^{-\frac{|x|}{\alpha}} * y(t,x) = \frac{1}{2\alpha}\,\mathrm{e}^{\frac{x}{\alpha}} \int_{q(t,c)}^{q(t,d)} \mathrm{e}^{-\frac{\xi}{\alpha}}\, y(t,\xi)\mathrm{d}\xi = \frac{1}{2\alpha}\,\mathrm{e}^{\frac{x}{\alpha}}\, F(t). \tag{3.2}$$

Hence as consequence of (3.1) and (3.2), we have

$$u(t,x) = -\alpha u_x(t,x) = \alpha^2 u_{xx}(t,x) = \frac{1}{2\alpha}\,\mathrm{e}^{-\frac{x}{\alpha}}\, E(t), \quad \text{for } x > q(t,d) \tag{3.3}$$

and

$$u(t,x) = \alpha u_x(t,x) = \alpha^2 u_{xx}(t,x) = \frac{1}{2\alpha}\,\mathrm{e}^{\frac{x}{\alpha}}\, F(t), \qquad \text{for } x < q(t,c) \tag{3.4}$$

On the other hand,

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = \int_{\mathbb{R}} \mathrm{e}^{\frac{x}{\alpha}}\, y_t(t,x)\mathrm{d}x.$$

Differentiating equation (1.1) twice, we get

$$
\begin{aligned}
0 &= u_{xxt} + \left(\left(u - \frac{\Gamma}{\alpha^2}\right)u_x\right)_{xx} \\
&\quad + (1-\alpha^2\partial_x^2)^{-1}\partial_x^3\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2 + \left(c_0 + \frac{\Gamma}{\alpha^2}\right)u\right) \\
&= u_{xxt} + \left(\left(u - \frac{\Gamma}{\alpha^2}\right)u_x\right)_{xx} \\
&\quad + \frac{1}{\alpha^2}(1-\alpha^2\partial_x^2)^{-1}\partial_x\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2 + \left(c_0 + \frac{\Gamma}{\alpha^2}\right)u\right) \\
&\quad - \frac{1}{\alpha^2}(1-\alpha^2\partial_x^2)^{-1}(1-\alpha^2\partial_x^2)\partial_x\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2 + \left(c_0 + \frac{\Gamma}{\alpha^2}\right)u\right) \\
&= u_{xxt} + \left(\left(u - \frac{\Gamma}{\alpha^2}\right)u_x\right)_{xx} \\
&\quad + \frac{1}{\alpha^2}(1-\alpha^2\partial_x^2)^{-1}\partial_x\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2 + \left(c_0 + \frac{\Gamma}{\alpha^2}\right)u\right) \\
&\quad - \frac{1}{\alpha^2}\partial_x\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2 + \left(c_0 + \frac{\Gamma}{\alpha^2}\right)u\right).
\end{aligned}
\tag{3.5}
$$

Combining (1.1) and (3.5), we obtain

$$
\begin{aligned}
y_t = &- \left(u - \frac{\Gamma}{\alpha^2}\right)u_x + \alpha^2\left(\left(u - \frac{\Gamma}{\alpha^2}\right)u_x\right)_{xx} \\
&- \partial_x\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2 + \left(c_0 + \frac{\Gamma}{\alpha^2}\right)u\right).
\end{aligned}
\tag{3.6}
$$

Substituting the identity (3.6) into $\frac{dE(t)}{dt}$, integrating by parts and using that $c_0 + \frac{\Gamma}{\alpha^2} = 0$, (3.3) and (3.4), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}E(t)}{\mathrm{d}t} = &- \int_{\mathbb{R}} \mathrm{e}^{\frac{x}{\alpha}}\left(u - \frac{\Gamma}{\alpha^2}\right)u_x\mathrm{d}x + \alpha^2\int_{\mathbb{R}} \mathrm{e}^{\frac{x}{\alpha}}\left(\left(u - \frac{\Gamma}{\alpha^2}\right)u_x\right)_{xx}\mathrm{d}x \\
&- \int_{\mathbb{R}} \mathrm{e}^{\frac{x}{\alpha}}\,\partial_x\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2 + \left(c_0 + \frac{\Gamma}{\alpha^2}\right)u\right)\mathrm{d}x \\
= &\frac{1}{\alpha}\int_{\mathbb{R}} \mathrm{e}^{\frac{x}{\alpha}}\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2\right)\mathrm{d}x.
\end{aligned}
$$

Therefore, for $0 < b < 3$ in the lifespan of the solution, we have

$$E(t) = \int_0^t \left(\frac{1}{\alpha}\int_{\mathbb{R}} \mathrm{e}^{\frac{x}{\alpha}}\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2\right)\mathrm{d}x\right) > 0.$$

By the same argument, one can check that the following identity for $F(t)$ is true

$$F(t) = -\int_0^t \left(\frac{1}{\alpha}\int_{\mathbb{R}} \mathrm{e}^{-\frac{x}{\alpha}}\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2u_x^2\right)\mathrm{d}x\right) < 0.$$

We also have $E'(t) > 0$ and $F'(t) < 0$. In order to complete the proof, it is sufficient to let $L(t) = \frac{1}{2\alpha}E(t)$ and $l(t) = \frac{1}{2\alpha}F(t)$. $\square$

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M. Kodzha, Faculty of Mathematics and Informatics, Shumen University, 9712 Shumen, Bulgaria,
e-mail: `mehmet_85@abv.bg`