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LEVY'S THEOREM AND STRONG CONVERGENCE OF MARTINGALES IN A DUAL SPACE

M. SAADOUNE

ABSTRACT. We prove Levy's Theorem for a new class of functions taking values from a dual space and we obtain almost sure strong convergence of martingales and mils satisfying various tightness conditions.

1. Introduction

This work is devoted to the study of strong convergence of martingales and mils in the space $L^1_{X^*}[X](\Omega, \mathcal{F}, P)$ of X-scalarly measurable functions f such that $\omega \to \|f(\omega)\|$ is P-integrable, where $(\Omega, \mathcal{F}, P)$ is a complete probability space, X is a separable Banach space and X^* is its topological dual without the Radon-Nikodym Property. By contrast with the well known Chatterji result dealing with strong convergence of relatively weakly compact $L^1_Y(\Omega, \mathcal{F}, P)$-bounded martingales, where Y is a Banach space, the case of the space $L^1_{X^*}[X](\Omega, \mathcal{F}, P)$ considered here is unusual because the functions are no longer strongly measurable, the dual space is not strongly separable. Our starting point of this study is to characterize functions in $L^1_{X^*}[X](\Omega, \mathcal{F}, P)$ whose associated regular martingales almost surely strong converge, by introducing the notion of σ-*measurability*. We then proceed by stating our main results, which stipule that under various tightness conditions, $L^1_{X^*}[X](\Omega, \mathcal{F}, P)$-bounded martingales and mils almost surely converge with respect to the strong topology on X^*. Further, we study the special case of martingales in the subspace of $L^1_{X^*}[X](\mathcal{F})$ of all Pettis-integrable functions that satisfy a condition formulated in the manner of Marraffa [**25**]. For the weak star convergence of martingales and mils taking values from a dual space, the reader is referred to Fitzpatrick-Lewis [**20**] and the recent paper of Castaing-Ezzaki-Lavie-Saadoune [**7**].

The paper is organized as follows. In Section 2 we set our notations and definitions, and summarize needed results. In section 3 we present a weak compactness result for uniformly integrable *weak tight* sequences in the space $L^1_{X^*}[X](\Omega, \mathcal{F}, P)$ as well as we give application to biting lemma. These results will be used in the

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next sections. In Section 4 σ-measurable functions are presented and Levy's theorem for such functions is stated. In Section 5 we give our main martingale almost surely strong convergence result (Theorem 5.1) accompanied by some important Corollaries 5.1–5.3. A version of Theorem 5.1 for mils is provided at the end of this section (Theorem 5.2). Finally, in Section 6 we discuss the special case of bounded martingales in $L^1_{X^*}[X](\Omega,\mathcal{F},P)$ whose members are also Pettis integrable. It will be shown that for such martingales it is possible to pass from convergence in a very weak sense (see [**25**], [**17**], [**4**]) to strong convergence (Proposition 6.1).

2. Notations and Preliminaries

In the sequel, X is a separable Banach space and $(x_\ell)_{\ell\geq 1}$ is a fixed dense sequence in the closed unit ball $\overline{B}_X$. We denote by X^* the topological dual of X and the dual norm by $\|.\|$. The closed unit ball of X^* is denoted by $\overline{B}_{X^*}$. If t is a topology on X^*, the space X^* endowed with t is denoted by X^*_t. Three topologies will be considered on X^*, namely the norm topology s^*, the weak topology $w=\sigma(X^*,X^{**})$ and the weak-star topology $w^*=\sigma(X^*,X)$.

Let $(C_n)_{n\geq 1}$ be a sequence of subsets of X^*. The *sequential weak upper limit* $w-ls\,C_n$ of (C_n) is defined by

$$w-ls\,C_n=\{x\in X^*:\ x=w-\lim_{j\to+\infty}x_{n_j},\quad x_{n_j}\in C_{n_j}\}$$

and the *topological weak upper limit* $w-LS\,C_n$ of (C_n) is denoted by $w-LS\,C_n$ and is defined by

$$w-LS\,C_n=\bigcap_{n\geq 1}w-\mathrm{cl}\bigcup_{k\geq n}C_n,$$

where $w-\mathrm{cl}$ denotes the closed hull operation in the weak topology. The following inclusion

$$w-ls\,C_n\subseteq w-LS\,C_n$$

is easy to check. Conversely, if the C_n are contained in a fixed weakly compact subset, then both sides coincide.

Let $(\Omega,\mathcal{F},P)$ be a complete probability space. A function $f\colon\Omega\to X^*$ is said to be X-*scalarly* $\mathcal{F}$-*measurable* (or simply scalarly $\mathcal{F}$-measurable) if the real-valued function $\omega\to<x,f(\omega)>$ is measurable with respect to (w.r.t.) the σ-field $\mathcal{F}$ for all $x\in X$. We say also that f is weak*-$\mathcal{F}$-measurable. Recall that if $f\colon\Omega\to X^*$ is a scalarly $\mathcal{F}$-measurable function such that $\langle x,f\rangle\in L^1_{\mathbb{R}}(\mathcal{F})$ for all $x\in X$, then for each $A\in\mathcal{F}$, there is $x^*\in X^*$ such that

$$\forall x\in X,\qquad \langle x,x^*\rangle=\int_A\langle x,f\rangle\,\mathrm{d}P.$$

The vector x^* is called the weak* integral (or Gelfand integral) of f over A and is denoted simply $\int_A f\,dP$. We denote by $L^0_{X^*}[X](\mathcal{F})$ (resp. $L^1_{X^*}[X](\mathcal{F})$) the space of all (classes of) scalarly $\mathcal{F}$-measurable functions (resp. scalarly $\mathcal{F}$-measurable functions f such that $\omega\to\|f(\omega)\|$ is P-integrable). By [**14**, Theorem VIII.5] (actually, a consequence of it) (see also [**3**, Proposition 2.7]), $L^1_{X^*}[X](\mathcal{F})$ endowed with the norm $\overline{N}_1$ defined by

$$\overline{N}_1(f) := \int_\Omega \|f\| \,\mathrm{d}P, \qquad f \in L^1_{X^*}[X](\mathcal{F}),$$

is a Banach space. For more properties of this space, we refer to [**3**] and [**14**].

Next, let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$. We assume without loss of generality that $\mathcal{F}$ is generated by $\cup_n \mathcal{F}_n$. A function $\tau\colon \Omega \to \mathbb{N} \cup \{+\infty\}$ is called a *stopping time* w.r.t. $(\mathcal{F}_n)$ if for each $n \geq 1$, $\{\tau = n\} \in \mathcal{F}_n$. The set of all bounded stopping times w.r.t. $(\mathcal{F}_n)$ is denoted T. Let $(f_n)_{n\geq 1}$ be a sequence in $L^1_{X^*}[X](\mathcal{F})$. If each f_n is $\mathcal{F}_n$-scalarly measurable, we say that (f_n) is adapted w.r.t. $(\mathcal{F}_n)$. For $\tau \in T$ and (f_n) an adapted sequence w.r.t. $(\mathcal{F}_n)$ recall that

$$f_\tau := \sum_{k=\min(\tau)}^{\max(\tau)} f_k 1_{\{\tau=k\}} \quad \text{and} \quad \mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau = k\} \in \mathcal{F}_k, \forall k \geq 1\}.$$

It is readily seen that f_τ is $\mathcal{F}_\tau$-scalarly measurable. Moreover, given a stopping time σ (not necessarily bounded), the following useful inclusion holds

$$(\ddagger) \qquad \{\sigma = +\infty\} \cap \mathcal{F} \subset \sigma(\cup_n \mathcal{F}_{\sigma\wedge n}),$$

which is equivalent to

$$(\ddagger)' \qquad \{\sigma = +\infty\} \cap \mathcal{F}_m \subset \sigma(\cup_n \mathcal{F}_{\sigma\wedge n}), \ \text{for all } m \geq 1,$$

where $\sigma \wedge n$ is the bounded stopping time defined by $\sigma \wedge n(\omega) := \min(\sigma(\omega), n)$ and $\sigma(\cup_n \mathcal{F}_{\sigma\wedge n})$ is the sub-σ-algebra of $\mathcal{F}$ generated by $\cup_n \mathcal{F}_{\sigma\wedge n}$. To verify $(\ddagger)'$, fix A in $\mathcal{F}_m$ and consider the sequence (f_n) defined by $f_n := 1_A$ if $n = m$, 0 otherwise. Then (f_n) is adapted w.r.t $(\mathcal{F}_n)$ and it is easy to check the following equality

$$1_{\{\sigma=+\infty\}} f_{\sigma\wedge m} = 1_{\{\sigma=+\infty\}\cap A}$$

with $1_\emptyset = 0$. As $\{\sigma = +\infty\} \in \sigma(\cup_n \mathcal{F}_{\sigma\wedge n})$ (because $\{\sigma < +\infty\} = \cup_n \{\sigma = n\}$ and $\{\sigma = n\} \in \mathcal{F}_{\sigma\wedge n}$, for all $n \geq 1$), it follows that $1_{\{\sigma=+\infty\}} f_{\sigma\wedge m}$ is measurable w.r.t. $\sigma(\cup_n \mathcal{F}_{\sigma\wedge n})$ and so is the function $1_{\{\sigma=+\infty\}\cap A}$. Equivalently $\{\sigma = +\infty\} \cap A \in \sigma(\cup_n \mathcal{F}_{\sigma\wedge n})$. Thus $\{\sigma = +\infty\} \cap \mathcal{F}_m \subset \sigma(\cup_n \mathcal{F}_{\sigma\wedge n})$. Since this holds for all $m \geq 1$, the inclusion $(\ddagger)'$ follows.

Definition 2.1. An adapted sequence $(f_n)_{n\geq 1}$ in $L^1_{X^*}[X](\mathcal{F})$ is a *martingale* if

$$\int_A f_n \,\mathrm{d}P = \int_A f_{n+1} \,\mathrm{d}P$$

for each $A \in \mathcal{F}_n$ and each $n \geq 1$. Equivalently $E^{\mathcal{F}_n}(f_{n+1}) = f_n$ for each $n \geq 1$.

$E^{\mathcal{F}_n}$ denotes the (Gelfand) conditional expectation w.r.t. $\mathcal{F}_n$. It must be noted that the conditional expectation of a Gelfand function in $L^1_{X^*}[X](\mathcal{F})$ always exists, (see [**32**, Proposition 7, p. 366] and [**35**, Theorem 3]).

Definition 2.2. An adapted sequence $(f_n)_{n\geq 1}$ in $L^1_{X^*}[X](\mathcal{F})$ is a mil if for every $\varepsilon > 0$, there exists p such that for each $n \geq p$, we have

$$P(\sup_{n\geq q\geq p} \|f_q - E^{\mathcal{F}_q} f_n\| > \varepsilon) < \varepsilon.$$

It is obvious that if $(f_n)_{n\geq 1}$ is a mil in $L^1_{X^*}[X](\mathcal{F})$, then for every x in $\overline{B}_X$, the sequence $(\langle x, f_n\rangle)_{n\geq 1}$ is a mil in $L^1_{\mathbb{R}}(\mathcal{F})$.

We end this section by recalling two concepts of tightness which permit us to pass from weak star to strong convergence. For this purpose, let $\mathcal{C} = cwk(X^*_w)$ or $\mathcal{R}(X^*_w)$, where $cwk(X^*_w)$ (resp. $\mathcal{R}(X^*_w)$) denotes the space of all nonempty $\sigma(X^*, X^{**})$-compact convex subsets of X^*_w (resp. closed convex subsets of X^*_w such that their intersections with any closed ball are weakly compact). A $\mathcal{C}$-valued multifunction $\Gamma : \Omega \Rightarrow X^*$ is $\mathcal{F}$-measurable if its graph $Gr(\Gamma)$ defined by

$$Gr(\Gamma) := \{(\omega, x^*) \in \Omega \times X^* : x^* \in \Gamma(\omega)\}$$

belongs to $\mathcal{F} \otimes \mathcal{B}(X^*_{w^*})$.

Definition 2.3. A sequence (f_n) in $L^0_{X^*}[X](\mathcal{F})$ is $\mathcal{C}$-*tight* if for every $\varepsilon > 0$, there is a $\mathcal{C}$-valued $\mathcal{F}$-measurable multifunction $\Gamma_\varepsilon : \Omega \Rightarrow X^*$ such that

$$\inf_n P(\{\omega \in \Omega : f_n(\omega) \in \Gamma_\varepsilon(\omega)\}) \geq 1 - \varepsilon.$$

In view of the completeness hypothesis on the probability space $(\Omega, \mathcal{F}, P)$, the measurability of the set $\{\omega \in \Omega : f_n(\omega) \in \Gamma_\varepsilon(\omega)\}$ is a consequence of the classical Projection Theorem [**14**, Theorem III.23] since $X^*_{w^*}$ is a Suslin space and Γ_ε has its graph in $\mathcal{F} \otimes \mathcal{B}(X^*_{w^*})$ (see [**8**, p. 171–172] and also [**6, 11**]).

Now let us introduce a weaker notion of tightness, namely $\mathcal{S}(\mathcal{C})$-*tightness*. It is a dual version of a similar notion in [**6**] dealing with primal space X.

Definition 2.4. A sequence (f_n) in $L^0_{X^*}[X](\mathcal{F})$ is $\mathcal{S}(\mathcal{C})$-*tight* if there exists a $\mathcal{C}$-valued $\mathcal{F}$-measurable multifunction $\Gamma\colon \Omega \Rightarrow X^*$ such that for almost all $\omega \in \Omega$, one has

$$(*) \qquad f_n(\omega) \in \Gamma(\omega) \text{ for infinitely many indices } n.$$

The following two results reformulate [**6**, Proposition 3.3] for sequences of measurable functions with values in a dual space.

Proposition 2.1. *Let (f_n) be an $\mathcal{R}(X^*_w)$-tight sequence. If it is bounded in $L^1_{X^*}[X](\mathcal{F})$, then it is also $cwk(X^*_w)$-tight.*

Proof. Let $\varepsilon > 0$. By the $\mathcal{R}(X^*_w)$-tightness assumption, there exists a $\mathcal{F}$-measurable $\mathcal{R}(X^*_w)$-valued multifunction $\Gamma_\varepsilon\colon \Omega \Rightarrow X^*$ such that

$$\inf_n P(\{\omega \in \Omega : f_n(\omega) \in \Gamma_\varepsilon(\omega)\}) \geq 1 - \varepsilon. \tag{2.1}$$

On the other hand, since $(\|f_n\|)$ is bounded in $L^1_{\mathbb{R}^+}(\mathcal{F})$, one can find $r_\varepsilon > 0$ such that

$$\sup_n P(\{\|f_n\| > r_\varepsilon\}) \leq \varepsilon. \tag{2.2}$$

For each $n \geq 1$, put

$$A_{n,\varepsilon} := \{\omega \in \Omega : f_n(\omega) \in \Gamma_\varepsilon(\omega) \cap \overline{B}(0, r_\varepsilon)\}$$

and let us consider the multifunction Δ_ε defined on Ω by

$$\Delta_\varepsilon := s^*\text{-cl}\,\text{co}\big[\bigcup_{n\geq 1} \{1_{A_{n,\varepsilon}} f_n\}\big].$$

The values of multifunction Δ_ε are $cwk(X^*_w)$-valued, because $\Delta_\varepsilon(\omega) \subset s^*\text{-cl}\,\text{co}(\{0\} \cup [\Gamma_\varepsilon(\omega) \cap \overline{B}(0, r_\varepsilon)])$ and $\Gamma_\varepsilon(\omega) \in \mathcal{R}(X^*_w)$, for all ω. Therefore, Δ_ε is $\mathcal{F}$-measurable (see [**6**], [**10**]). Finally, using (2.1), (2.2) and the following inclusions

$$A_{n,\varepsilon} \subseteq \{\omega \in \Omega : f_n(\omega) \in \Delta_\varepsilon(\omega)\}, \quad n \geq 1,$$

we get

$$P(\{\omega \in \Omega : f_n(\omega) \in \Delta_\varepsilon(\omega)\}) > 1 - 2\varepsilon \quad \text{for all } n.$$

□

Proposition 2.2. *Every $\mathcal{C}$-tight sequence is $\mathcal{S}(\mathcal{C})$-tight.*

Proof. Let (f_n) be a $\mathcal{C}$-tight sequence in $L^0_{X^*}[X](\mathcal{F})$ and consider $\varepsilon_q := \frac{1}{q}$, $q \geq 1$. By the $\mathcal{C}$-tightness assumption, there is a $\mathcal{F}$-measurable $\mathcal{C}$-valued multifunction $\Gamma_{\varepsilon_q} : \Omega \rightrightarrows X^*$ denoted simply Γ_q such that

$$\inf_n P(A_{n,q}) \geq 1 - \varepsilon_q, \tag{2.3}$$

where

$$A_{n,q} := \{\omega \in \Omega : f_n(\omega) \in \Gamma_q(\omega)\}.$$

Now, we define the sequence $(\Omega_q)_{q\geq 1}$ by

$$\Omega_q = \limsup_{n\to+\infty} A_{n,q}$$

and the multifunction Γ on Ω by

$$\Gamma = 1_{\Omega'_1}\,\Gamma_1 + \sum_{q\geq 2} 1_{\Omega'_q}\,\Gamma_q,$$

where $\Omega'_1 = \Omega_1$ and $\Omega'_q = \Omega_q \setminus \cup_{i<q} \Omega_i$ for all $q > 1$. Then inequality (2.3) implies

$$P(\Omega_q) = \lim_{n\to\infty} P\big(\bigcup_{m\geq n} A_{m,q}\big) \geq 1 - \varepsilon_q \to 1.$$

Further, for each $\omega \in \Omega_q$, one has

$$\omega \in A_{n,q} = \{\omega \in \Omega : f_n(\omega) \in \Gamma(\omega)\} \quad \text{for infinitely many indices } n.$$

This proves the $\mathcal{S}(\mathcal{C})$-tightness. □

Remark 2.5. By the Eberlein-Smulian theorem, the following implication

$$(f_n)\ \mathcal{S}(cwk(X^*_w))\text{-tight} \Rightarrow w - ls\, f_n \neq \emptyset \text{ a.s.}$$

holds true. Conversely, if $w - ls\, f_n \neq \emptyset$ a.s. then the condition (*) in Definition 2.4 is satisfied, but the multifunction $\mathcal{C}$ may fail to be $\mathcal{F}$-measurable.

Actually, in all results involving the $\mathcal{S}(\mathcal{C})$-tightness condition, the measurability of the multifunction Γ is not essential.

3. WEAK COMPACTNESS IN THE SPACE $L^1_{X^*}[X](\mathcal{F})$

We recall first the following weak compactness result in the space $L^1_{X^*}[X](\mathcal{F})$ due to Benabdellah and Castaing [**3**].

Proposition 3.1. ([**3**, Proposition 4.1]) *Suppose that $(f_n)_{n\geq 1}$ is a uniformly integrable sequence in $L^1_{X^*}[X](\mathcal{F})$ and Γ is a $cw(X^*_w)$-valued multifunction such that*

$$f_n(\omega) \in \Gamma(\omega) \ \text{a.s. for all } n \geq 1,$$

then (f_n) is relatively weakly compact in $L^1_{X^*}[X](\mathcal{F})$.

Proceeding as in the primal case (see [**5**], [**1**], [**30**]), it is possible to extend this result to uniformly integrable $\mathcal{R}(X^*_w)$-tight sequences in $L^1_{X^*}[X](\mathcal{F})$

Proposition 3.2. *Suppose that $(f_n)_{n\geq 1}$ is a uniformly integrable $\mathcal{R}(X^*_w)$-tight sequence in $L^1_{X^*}[X](\mathcal{F})$. Then (f_n) is relatively weakly compact in $L^1_{X^*}[X](\mathcal{F})$.*

Proof. By Proposition 2.1, (f_n) is $cwk(X^*_w)$-tight since it is bounded and $\mathcal{R}(X^*_w)$-tight. Consequently, for every $q \geq 1$, there is a $\mathcal{F}$-measurable $cwk(X^*_w)$-valued multifunction $\Gamma_{\frac{1}{q}} : \Omega \rightrightarrows X^*$, denoted simply Γ_q, such that

$$\inf_n P(A_{n,q}) \geq 1 - \frac{1}{q},$$

where

$$A_{n,q} := \{\omega \in \Omega : f_n(\omega) \in \Gamma_q(\omega)\}.$$

Now, for each $q \geq 1$, we consider the sequence $(f_{n,q})$ defined by

$$f_{n,q} = 1_{A_{n,q}} f_n \quad n \geq 1.$$

By Proposition 3.1, the sequence $(f_{n,q})$ is relatively weakly compact in $L^1_{X^*}[X](\mathcal{F})$ since it is $L^1_{X^*}[X](\mathcal{F})$-bounded and $f_{n,q}(\omega)$ belongs to the w-compact set $\Gamma(\omega)$ for all $\omega \in \Omega$ and all n, $q \geq 1$. Furthermore, we have the following estimation

$$\sup_n \int_\Omega \|f_n - f_{n,q}\| \, dP \leq \sup_n \int_{\Omega \setminus A_{n,q}} \|f_n\| \, \mathrm{d}P$$

for all $q \geq 1$. As (f_n) is uniformly integrable and $\inf_n P(A_{n,q}) \geq 1 - \frac{1}{q}$, we get

$$\lim_{q\to\infty} \sup_n \int_{\Omega \setminus A_{n,q}} \|f_n\| \, \mathrm{d}P = 0.$$

Hence

$$\lim_{q\to\infty} \sup_n \int_\Omega \|f_n - f_{n,q}\| \, \mathrm{d}P = 0.$$

Consequently, by Grothendieck's weak relative compactness lemma ([**22**, Chap. 5, 4, n°1]), the sequence (f_n) is relatively weakly compact in $L^1_{X^*}[X](\mathcal{F})$. □

Now, we provide the following version of the biting lemma in the space $L^1_{X^*}[X](\mathcal{F})$. See [**13**] for other related results involving a weaker mode of convergence; see also [**9**] dealing with the primal case.

Proposition 3.3. *Let (f_n) be a bounded $\mathcal{R}(X^*_w)$-tight sequence in $L^1_{X^*}[X](\mathcal{F})$. Then there exist a subsequence (f'_n) of (f_n), a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ and an increasing sequence (B_p) of measurable sets with $\lim_{p\to\infty} P(B_p) = 1$ such that $(1_{B_p} f'_n)$ converges to $1_{B_p} f_\infty$ in the weak topology of $L^1_{X^*}[X](\mathcal{F})$ for all $p \geq 1$.*

Proof. In view of the biting lemma (see [**21**], [**33**] [**31**]), there exist an increasing sequence (B_p) of measurable sets with $\lim_{p\to\infty} P(B_p) = 1$ and a subsequence (f'_n) of (f_n) such that for all $p \geq 1$, the sequence $(1_{B_p} f'_n)$ is uniformly integrable. It is also $\mathcal{R}(X^*_w)$-tight. Consequently, by Proposition 3.2, for each $p \geq 1$, $(1_{B_p} f'_n)$ is relatively weakly compact in $L^1_{X^*}[X](\mathcal{F})$. By applying the Eberlein-Smulian theorem via a standard diagonal procedure, we provide a subsequence of (f'_n), not relabeled, such that for each $p \geq 1$, $(1_{B_p} f'_n)$ converges to a function $f_{\infty,p} \in L^1_{X^*}[X](\mathcal{F})$ in the weak topology of $L^1_{X^*}[X](\mathcal{F})$, also denoted $\sigma(L^1_{X^*}[X](\mathcal{F}), (L^1_{X^*}[X](\mathcal{F}))')$. Finally, define

$$f_\infty := \sum_{p=1}^{p=\infty} 1_{C_p} f_{\infty,p},$$

where

$$C_1 := B_1 \text{ and } C_p := B_p \setminus \cup_{i<p} B_i \quad \text{for} \quad p > 1.$$

It is not difficult to verify that $(1_{B_p} f'_n)$ converges to $1_{B_p} f_\infty$ in the weak topology of $L^1_{X^*}[X](\mathcal{F})$. Since the norm $\overline{N}_1(.)$ of $L^1_{X^*}[X](\mathcal{F})$ is $\sigma(L^1_{X^*}[X](\mathcal{F}), (L^1_{X^*}[X](\mathcal{F}))')$-lower semi-continuous, we have

$$\int_{B_p} \|f_\infty\| \, \mathrm{d}P \leq \liminf_{n\to\infty} \int_{B_p} \|f'_n\| \, \mathrm{d}P \leq \sup_n \int_\Omega \|f_n\| \, \mathrm{d}P < \infty \text{ for all } p \geq 1.$$

As $\lim_{p\to\infty} P(B_p) = 1$, we deduce that $\|f_\infty\| \in L^1_{\mathbb{R}}(\mathcal{F})$. This completes the proof of Proposition 3.3. □

As a consequence of Proposition 3.3 and Mazur theorem we get the following corollary.

Corollary 3.1. *Let (f_n) be a bounded $\mathcal{R}(X^*_w)$-tight sequence in $L^1_{X^*}[X](\mathcal{F})$. Then there exist a sequence (g_n) with $g_n \in co\{f_i : i \geq n\}$ and a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that*

$$(g_n) \ s^*\text{-converges to } f_\infty \ a.s.$$

Proof. By the assumptions and Proposition 3.3, there exist a subsequence (f'_n) of (f_n), a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ and increasing sequence (B_p) of measurable sets with $\lim_{p\to\infty} P(B_p) = 1$ such that for all $p \geq 1$, $(1_{B_p} f'_n)$ converges to $1_{B_p} f_\infty$ in the weak topology of $L^1_{X^*}[X](\mathcal{F})$. So, appealing to a diagonal procedure based on successively applying Mazur's theorem (see [**10**, Lemma 3.1]), one can show the existence of a sequence (g_n) of convex combinations of (f'_n), such that for all $p \geq 1$, $(1_{B_p} g_n)$ s^*-converges almost surely to $1_{B_p} f_\infty$ and also strongly in $L^1_{X^*}[X](\mathcal{F})$. Since $\lim_{p\to\infty} P(B_p) = 1$, (g_n) s^*-converges almost surely to f_∞. □

4. Levy's theorem in $L^1_{X^*}[X](\mathcal{F})$

In this section, we present a new class of functions in $L^1_{X^*}[X](\mathcal{F})$ whose associated regular martingales almost surely converge with respect to the strong topology of X^*.

Definition 4.1. A function f in $L^0_{X^*}[X](\mathcal{F})$ is said to be σ-measurable, if there exists an adapted sequence $(\Gamma_n)_{n\geq 1}$ (that is, for each integer $n \geq 1$, Γ_n is $\mathcal{F}_n$-measurable) of $\mathcal{R}(X^*_w)$-valued multifunctions such that $f(\omega) \in s^*\text{-cl}\operatorname{co}(\cup_n \Gamma_n)$ a.s.

Remark 4.2. The sequence (Γ_n) given in this definition can be assumed to be adapted w.r.t. a subsequence of $(\mathcal{F}_n)$.

Remark 4.3. As a special case note that *every strongly measurable function* $f\colon \Omega \to X^*$ is σ-measurable. Indeed, if $(\xi_n)_{n\geq 1}$ is a sequence of measurable functions assuming a finite number of values and which norm converges a.s. to f, then $f(\omega) \in s^*\text{-cl}(\cup_{n\geq 1}\xi_n(\Omega))$ a.s., $(\Gamma_n := \xi_n(\Omega))$.

Proposition 4.1. *Let $f \in L^0_{X^*}[X](\mathcal{F})$ and suppose there exists a sequence $(\Gamma_n)_{n\geq 1}$ of $\mathcal{R}(X^*_w)$-valued multifunctions which is adapted w.r.t. a subsequence of $(\mathcal{F}_n)$ such that $f(\omega) \in s^*\text{-}cl\, co\, w\text{-}LS\,\Gamma_n$ a.s., then f is σ-measurable.*

Proof. Indeed, since

$$w\text{-}LS\,\Gamma_n := \bigcap_{k\geq 1} w\text{-cl}\big(\bigcup_{n\geq k}\Gamma_n\big) \subset \bigcap_{k\geq 1} s^*\text{-cl}\operatorname{co}\big(\bigcup_{n\geq k}\Gamma_n\big) \subset s^*\text{-cl}\operatorname{co}\big(\bigcup_{n\geq 1}\Gamma_n\big),$$

we have

$$s^*\text{-cl}\operatorname{co} w\text{-}LS\,\Gamma_n \subset s^*\text{-cl}\operatorname{co}\big(\bigcup_{n\geq 1}\Gamma_n\big).$$

□

In particular, we have the following result.

Corollary 4.1. *Let $f \in L^0_{X^*}[X](\mathcal{F})$. If there exists a sequence (f_n) in $L^0_{X^*}[X](\mathcal{F})$, adapted w.r.t. a subsequence of $(\mathcal{F}_n)$ which weak converges a.s. to f, then f is σ-measurable.*

The following proposition will be useful in this work.

Proposition 4.2. *Let $(f_n)_{n\geq 1}$ be an adapted $\mathcal{S}(cwk(X^*_w))$-tight sequence in $L^0_{X^*}[X](\mathcal{F})$ and f_∞ a function in $L^0_{X^*}[X](\mathcal{F})$ such that*

$$\lim_{n\to\infty} \langle x_\ell, f_n\rangle = \langle x_\ell, f_\infty\rangle \text{ a.s. for all } \ell.$$

Then f_∞ is σ-measurable.

Proof. $\mathcal{S}(cwk(X^*_w))$-tightness and Remark 2.5 imply

$$w\text{-}ls\, f_n \neq \emptyset \quad \text{a.s.}$$

Since $\lim_{n\to\infty}\langle x_\ell, f_n\rangle = \langle x_\ell, f_\infty\rangle$, it is easy to prove that

$$w\text{-}ls\, f_n = \{f_\infty\} \text{ a.s.}$$

Thus f_∞ is σ-measurable, in view of Proposition 4.1 $\square$

There are two significant variants of Proposition 4.2. involving the $\mathcal{R}(X^*_w)$-tightness condition. The first one is essentially based on Proposition 3.2.

Proposition 4.3. *Let $(f_n)_{n\geq 1}$ be a uniformly integrable $\mathcal{R}(X^*_w)$-tight adapted sequence in $L^1_{X^*}[X](\mathcal{F})$ and f_∞ a function in $L^1_{X^*}[X](\mathcal{F})$. Suppose there exists a sequence (g_n) in $L^1_{X^*}[X](\mathcal{F})$ with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ such that*

$$\lim_{n\to\infty} \langle x_\ell, g_n\rangle = \langle x_\ell, f_\infty\rangle \text{ a.s. for all } \ell.$$

Then f_∞ is σ-measurable.

Proof. Let (g_n) be given as in the proposition. By Proposition 3.2 and Krein-Smulian theorem, the convex hull of the set $\{f_n : n \geq 1\}$ is relatively weakly compact in $L^1_{X^*}[X](\mathcal{F})$; hence (g_n) is relatively weakly compact in $L^1_{X^*}[X](\mathcal{F})$. Consequently, by the Eberlein Smulian theorem, there exists a subsequence of (g_n), not relabeled, such that for each $p \geq 1$, (g_n) converges to a function $f'_\infty \in L^1_{X^*}[X](\mathcal{F})$ in the weak topology of $L^1_{X^*}[X](\mathcal{F})$. So, invoking Mazur's theorem it can be shown the existence of a sequence of convex combinations of (g_n), still denoted in the same manner such that (g_n) s^*-converges almost surely to f'_∞. As $\lim_{n\to\infty}\langle x_\ell, g_n\rangle = \langle x_\ell, f_\infty\rangle$ a.s. for all ℓ, we get $f_\infty = f'_\infty$ a.s. Therefore, since (g_n) is adapted w.r.t. a subsequence of $(\mathcal{F}_n)$, it follows that f_∞ is σ-measurable. $\square$

The second variant is a consequence of the proof of Corollary 3.1.

Proposition 4.4. *Let $(f_n)_{n\geq 1}$ be a bounded $\mathcal{R}(X^*_w)$-tight adapted sequence in $L^1_{X^*}[X](\mathcal{F})$ and f_∞ a function in $L^1_{X^*}[X](\mathcal{F})$ such that the following condition holds.*

For any subsequence (f'_n) of (f_n), there is a sequence (g_n) in $L^1_{X^*}[X](\mathcal{F})$ with $g_n \in \mathrm{co}\{f'_i : i \geq n\}$ such that

$$\lim_{n\to\infty} \langle x_\ell, g_n\rangle = \langle x_\ell, f_\infty\rangle \text{ a.s. for all } \ell.$$

Then f_∞ is σ-measurable.

Now our main result comes and shows that a regular martingale associated to a σ-measurable function in $L^1_{X^*}[X](\mathcal{F})$ norm converges a.s.

Proposition 4.5. *Let f be a function in $L^1_{X^*}[X](\mathcal{F})$. Then the following two statements are equivalent:*

(a) $(E^{\mathcal{F}_n}(f))$ s^*-converges a.s.to f;
(b) f is σ-measurable.

Proof. Step 1. The implication (a) $\Rightarrow$ (b) is trivial. Conversely, suppose that f is σ-measurable. Then there exists an adapted sequence (Γ_n) of $\mathcal{R}(X^*_w)$-valued multifunctions such that

$$f(\omega) \in s^*\text{-cl}\,\mathrm{co}\Big(\bigcup_n \Gamma_n(\omega)\Big) \text{ a.s.} \tag{4.1}$$

Without loss of generality, we may suppose that $0 \in \Gamma_n(\omega)$, for all $\omega \in \Omega$ and all $n \geq 1$. For each $n, p \geq 1$, define the multifunction Γ_n^p by

$$\Gamma_n^p := \Gamma_n \cap \overline{B}_{X^*}(0, p).$$

Since this multifunction is $\mathcal{F}_n$-measurable, namely $Gr(\Gamma_n^p) \in \mathcal{F}_n \otimes \mathcal{B}(X^*_{w^*})$ and $X^*_{w^*}$ is a Suslin space, invoking [**14**, Theorem III.22], one can find a sequence $(\sigma_{n,i}^p)_{i\geq 1}$ of scalarly $\mathcal{F}_n$-measurable selectors of Γ_n^p that are also $L^1_{X^*}[X](\mathcal{F})$-integrable (because the multifunctions Γ_n^p are integrably bounded) such that for every $\omega \in \Omega$,

$$w^* - \mathrm{cl}(\Gamma_n^p(\omega)) = w^* - \mathrm{cl}(\{\sigma_{n,i}^p(\omega)\}_{i\geq 1}).$$

Equivalently

$$\Gamma_n^p(\omega) = w - \mathrm{cl}(\{\sigma_{n,i}^p(\omega)\}_{i\geq 1}),$$

since Γ_n^p is w-compact valued. So

$$\Gamma_n^p(\omega) \subset w - \mathrm{cl}\,\mathrm{co}(\{\sigma_{n,i}^p(\omega)\}_{i\geq 1}) = s^*\text{-}\mathrm{cl}\,\mathrm{co}(\{\sigma_{n,i}^p(\omega)\}_{i\geq 1}). \tag{4.2}$$

Let $(s_m)_{m\geq 1}$ be the sequence of all linear combinations with rational coefficients of $\sigma_{n,i}^p$, $(n, p, i \geq 1)$. It is easy to check that

$$s^*\text{-}\mathrm{cl}\,\mathrm{co}(\{\sigma_{n,i}^p(\omega)\}_{n,i,p\geq 1}) \subset s^* - \mathrm{cl}(\{s_m(\omega)\}_{m\geq 1}).$$

Combining this with (4.2) we get

$$s^*\text{-}\mathrm{cl}\,\mathrm{co}\big(\bigcup_n \Gamma_n(\omega)\big) = s^*\text{-}\mathrm{cl}\,\mathrm{co}\big(\bigcup_n \bigcup_p \Gamma_n^p(\omega)\big) \subset s^* - \mathrm{cl}(\{s_m(\omega)\}_{m\geq 1}),$$

whence, by (4.1)

$$f(\omega) \in s^* - \mathrm{cl}(\{s_m(\omega)\}_{m\geq 1}) \text{ a.s.} \tag{4.3}$$

Now, for each $q \geq 1$, let us define the sets

$$B_m^q := \left\{\omega \in \Omega : \ \|f(\omega) - s_m(\omega)\| < \frac{1}{q}\right\} \qquad (m \geq 1),$$

$$\Omega_1^q := B_1^q, \qquad \Omega_m^q := B_m^q \setminus \bigcup_{i<m} B_i^q \qquad \text{for } m > 1$$

and the function

$$f_q := \sum_{m=1}^{+\infty} 1_{\Omega_m^q} s_m.$$

Since the functions $\omega \to \|f(\omega) - s_m(\omega)\|$ are $\mathcal{F}$-measurable, $B_m^q \in \mathcal{F}$, for all $m \geq 1$, and then each f_q is scalarly $\mathcal{F}$-measurable. Further, from (4.3) it follows that $\cup_m B_m^q = \Omega$ a.s., so that $(\Omega_m^q)_m$ constitutes a sequence of pairwise disjoint members of $\mathcal{F}$ which satisfies $\cup_m \Omega_m^q = \Omega$ a.s., and so we have

$$\|f(\omega) - f_q(\omega)\| \leq \frac{1}{q} \text{ for almost all } \omega \in \Omega. \tag{4.4}$$

Next, we claim that

$$\lim_{n\to\infty} \|E^{\mathcal{F}_n}(f) - f\| = 0 \text{ a.s.}$$

First, observe that by construction of the s_m's, we can find a strictly increasing sequence (p_m) of positive integers such that (s_m) is adapted w.r.t. $(\mathcal{F}_{p_m})$. Now, let $k \geq 1$ be a fixed integer. For each $n \geq p_k$, one has

$$E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} B_m^q} f_q) = E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} \Omega_m^q} f_q) = E^{\mathcal{F}_n} \sum_{m=1}^{m=k} 1_{\Omega_m^q} s_m = \sum_{m=1}^{m=k} (E^{\mathcal{F}_n} 1_{\Omega_m^q}) s_m,$$

whence by the classical Levy theorem

$$\lim_{n\to\infty} E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} B_m^q} f_q) = \sum_{m=1}^{m=k} 1_{\Omega_m^q} s_m = 1_{\cup_{m=1}^{m=k} B_m^q} f_q \text{ a.s.} \tag{4.5}$$

w.r.t. the norm topology of X^*. On the other hand, from (4.4) we deduce the following estimation

$$\begin{aligned}
\|E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} B_m^q} f) - 1_{\cup_{m=1}^{m=k} B_m^q} f\| &\leq \|E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} B_m^q} f) - E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} B_m^q} f_q)\| \\
&\quad + \|E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} B_m^q} f_q) - 1_{\cup_{m=1}^{m=k} B_m^q} f_q\| \\
&\quad + \|1_{\cup_{m=1}^{m=k} B_m^q} f(\omega) - 1_{\cup_{m=1}^{m=k} B_m^q} f_q(\omega)\| \\
&\leq \|E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} B_m^q} f_q) - 1_{\cup_{m=1}^{m=k} B_m^q} f_q\| + \frac{2}{q},
\end{aligned}$$

which leads to

$$\begin{aligned}
\|E^{\mathcal{F}_n}(f) - f\| &\leq \|E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} B_m^q} f) - 1_{\cup_{m=1}^{m=k} B_m^q} f\| \\
&\quad + \|E^{\mathcal{F}_n}(1_{\Omega\setminus\cup_{m=1}^{i=k} B_m^q} f) - 1_{\Omega\setminus\cup_{m=1}^{m=k} B_m^q} f\| \\
&\leq \|E^{\mathcal{F}_n}(1_{\cup_{m=1}^{m=k} B_m^q} f_q) - 1_{\cup_{m=1}^{m=k} B_m^q} f_q\| + \frac{2}{q} \\
&\quad + E^{\mathcal{F}_n}(1_{\Omega\setminus\cup_{m=1}^{m=k} B_m^q} \|f\|) + 1_{\Omega\setminus\cup_{m=1}^{m=k} B_m^q} \|f\|.
\end{aligned}$$

Consequently, from (4.5) and the classical Levy Theorem ($\|f\|$ being in $L^1_{\mathbb{R}}(\mathcal{F})$), it follows that

$$\limsup_{n\to\infty} \|E^{\mathcal{F}_n}(f) - f\| \leq 2\left(1_{\Omega\setminus\cup_{m=1}^{m=k} B_m^q} \|f\| + \frac{1}{q}\right),$$

a.s. for all $k \geq 1$ and all $q \geq 1$. Since $P(\cup_m B_m^q) = 1$, by passing to the limit when $k \to \infty$ and $q \to \infty$, respectively, we get the desired conclusion, and the proof is finished. $\square$

5. STRONG CONVERGENCE OF MARTINGALES IN $L^1_{X^*}[X](\mathcal{F})$

The main result of this section asserts that under the $\mathcal{S}(\mathcal{R}(X_w^*))$-tightness condition every bounded martingale in $L^1_{X^*}[X](\mathcal{F})$ norm converges a.s. We begin with the following decomposition result for martingales which is borrowed from [**7**]. For the convenience of the reader we give a detailed proof.

Proposition 5.1. *Let $(f_n)_{n\geq 1}$ be a bounded martingale in $L^1_{X^*}[X](\mathcal{F})$. Then there exists $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that*

$$\lim_{n\to\infty} \|f_n - E^{\mathcal{F}_n} f_\infty\| = 0 \text{ a.s and,}$$

$$(f_n)\ w^*\text{-converges to } f_\infty \text{ a.s.}$$

Proof. As (f_n) is a bounded martingale in $L^1_{X^*}[X](\mathcal{F})$ for each $x \in X$, $(\langle x, f_n\rangle)$ is a bounded real martingale in $L^1_{\mathbb{R}}(\mathcal{F})$, hence it converges a.s. to a function $r_x \in L^1_{\mathbb{R}}(\mathcal{F})$ for every $x \in X$. By using [**11**, Theorem 6.1(4)], we provide an increasing sequence $(A_p)_{p\geq 1}$ in $\mathcal{F}$ with $\lim_{p\to\infty} P(A_p) = 1$, a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ and a subsequence $(f'_n)_{n\geq 1}$ of (f_n) such that

$$\lim_{n\to\infty} \int_{A_p} \langle h, f'_n\rangle \mathrm{d}P = \int_{A_p} \langle h, f_\infty\rangle \mathrm{d}P$$

for all $p \geq 1$ and all $h \in L^\infty_X(\mathcal{F})$. So by identifying the limit, we get $r_x = \langle x, f_\infty\rangle$ a.s. Hence

$$\lim_{n\to\infty} \langle x, f_n\rangle = \langle x, f_\infty\rangle, \text{ a.s. for all } x \in X \tag{5.1}$$

and then in view of the classical Levy's theorem

$$\lim_{n\to\infty} [\langle x, f_n\rangle - \langle x, E^{\mathcal{F}_n}(f_\infty)\rangle] = 0 \text{ a.s. for all } x \in X.$$

Furthermore, $\{(\langle x_\ell, f_n\rangle - \langle x_\ell, E^{\mathcal{F}_n}(f_\infty)\rangle)_{n\geq 1} : \ell \geq 1\}$ is a countable family of real-valued $L^1_{\mathbb{R}}(\mathcal{F})$-bounded martingales, thus invoking [**28**, Lemma V.2.9], we see that

$$\begin{aligned}\lim_{n\to\infty} \|f_n - E^{\mathcal{F}_n} f_\infty\| &= \lim_{n\to\infty} \sup_{\ell\geq 1} [\langle x_\ell, f_n\rangle - \langle x_\ell, E^{\mathcal{F}_n}(f_\infty)\rangle)] \\ &= \sup_{\ell \geq 1} \lim_{n\to\infty} [\langle x_\ell, f_n\rangle - \langle x_\ell, E^{\mathcal{F}_n}(f_\infty)\rangle] = 0.\end{aligned} \tag{5.2}$$

Since

$$\sup_n \|E^{\mathcal{F}_n}(f_\infty)\| \leq \sup_n E^{\mathcal{F}_n} \|f_\infty\| < \infty,$$

equation (5.2) entails

$$\sup_n \|f_n\| < \infty \text{ a.s.}$$

Invoking the separability of X and (5.1), we get

$$(f_n)\ w^*\text{-converges to } f_\infty \text{ a.s.},$$

by a routine argument. This completes the proof. □

Propositions 4.5 and 5.1 together allow us to pass from weak star convergence to strong convergence of martingales.

Theorem 5.1. *Let $(f_n)_{n\geq 1}$ be a bounded martingale in $L^1_{X^*}[X](\mathcal{F})$ satisfying the following condition.*

($\mathcal{T}$) *There exists a $\mathcal{S}(\mathcal{R}(X^*_w))$-tight sequence (g_n) in $L^1_{X^*}[X](\mathcal{F})$ with $g_n \in \mathrm{co}\{f_i : i \geq n\}$.*

Then there exists $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that

$$(f_n)\ s^*\text{-converges to } f_\infty \text{ a.s.}$$

Proof. Let (g_n) be as in condition $(\mathcal{T})$. By Proposition 5.1, there exists $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that

(a) $$\|f_n - E^{\mathcal{F}_n}(f_\infty)\| \to 0 \text{ a.s.}$$

(b) $$(f_n)\ w^*\text{-converges to } f_\infty \text{ a.s.}$$

By (b), (f_n) is pointwise bounded a.s., and so is the sequence (g_n). Consequently, (g_n) is $\mathcal{S}(cwk(X^*_w))$-tight, since it is $\mathcal{S}(\mathcal{R}(X^*_w))$-tight (by $(\mathcal{T})$). Furthermore, we have

$$(g_n)\ w^*\text{-converges to } f_\infty \text{ a.s.}$$

Therefore, noting that (g_n) is adapted w.r.t. a subsequence of $\mathcal{F}_n$, we conclude that f_∞ is σ-measurable in view of Proposition 4.2. In turn, by Proposition 4.5, this ensures the a.s. s^*-convergence of $E^{\mathcal{F}_n}(f_\infty)$ to f_∞. Coming back to (a), we get the desired conclusion. □

An alternative proof of Theorem 5.1 via a standard stopping time argument is also available. We want to emphasize that some of the arguments used in this proof will be helpful in the next section.

Second proof. Reasoning as at the beginning of the proof of Proposition 5.1 we find a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that

(5.3) $$\lim_{n\to\infty} \langle x, f_n(\omega)\rangle = \langle x, f_\infty(\omega)\rangle \text{ a.s. for all } x \in X.$$

1) Suppose that $\sup_n \|f_n\| \in L^1_{\mathbb{R}}(\mathcal{F})$. Then equation (5.3) implies

$$\lim_{n\to\infty} \int_A \langle x, f_n\rangle\, \mathrm{d}P = \int_A \langle x, f_\infty\rangle\, \mathrm{d}P$$

for all $x \in X$ and for all $A \in \mathcal{F}$. Since (f_n) is a martingale, it follows that

$$\begin{aligned}\int_A \langle x, f_m\rangle\, \mathrm{d}P &= \lim_{n\to\infty, n\geq m} \int_A \langle x, f_n\rangle\, \mathrm{d}P \\ &= \int_A \langle x, f_\infty\rangle\, \mathrm{d}P = \int_A \langle x, E^{\mathcal{F}_m}(f_\infty)\rangle\, \mathrm{d}P\end{aligned}$$

for all $x \in X$, $m \geq 1$ and $A \in \mathcal{F}_m$. Hence

$$f_m = E^{\mathcal{F}_m}(f_\infty) \text{ a.s. for all } m \geq 1,$$

by the separability of X. On the other hand, the sequence (g_n) appearing in the condition $(\mathcal{T})$ above is $\mathcal{S}(cwk(X^*_w))$-tight, since it is $\mathcal{S}(\mathcal{R}(X^*_w))$-tight and pointwise-bounded almost surely in view of the inequality

$$\sup_{n\geq 1} \|g_n(\omega)\| \leq \sup_{n\geq 1} \|f_n(\omega)\| < \infty \quad \text{a.s.}$$

Further, from (5.3) it follows

$$\lim_{n\to\infty} \langle x, g_n\rangle = \langle x, f_\infty\rangle \text{ a.s.},$$

for every $x \in X$. Taking into account Proposition 4.2, it follows that f_∞ is σ-measurable.Therefore, by Proposition 4.5, (f_n) s^*-converges a.s. to f_∞.

2) The case $\sup_n \int_\Omega \|f_n\| \, \mathrm{d}P < \infty$. For each $t > 0$, define the following well known stopping time

$$\sigma_t(\omega) = \begin{cases} n & \text{if } \|f_i(\omega)\| \leq t, \quad \text{for } i = 1, \dots, n-1 \text{ and } \|f_n(\omega)\| \geq t, \\ +\infty & \text{if } \|f_i(\omega)\| \leq t, \quad \text{for all } i. \end{cases}$$

Then, following the same lines as those of the $L^1_E(\mathcal{F})$ case ([**15**], [**19**]) we show that:

(i) $(f_{\sigma_t \wedge n}, \mathcal{F}_{\sigma_t \wedge n})$ is a $L^1_{X^*}[X](\mathcal{F})$-bounded martingale.
(ii) The function $\omega \to \sup_n \|f_{\sigma_t \wedge n}(\omega)\|$ is integrable.
(iii) $P(A_t := \{\omega : \sigma_t(\omega) = \infty\}) \to 1$ as $t \to \infty$.

Moreover, using (5.3) it is not difficult to check that

$$\lim_{n \to \infty} \langle x, f_{\sigma_t \wedge n}(\omega) \rangle = \langle x, f^t_\infty(\omega) \rangle, \text{ a.s.} \tag{5.4}$$

for every $x \in X$, where

$$f^t_\infty(\omega) := \begin{cases} f_\infty(\omega) & \text{if } \omega \in A_t, \\ f_{\sigma_t(\omega)}(\omega) & \text{otherwise.} \end{cases}$$

By (5.4), it is clear that f^t_∞ is scalarly $\mathcal{F}$-measurable. Furthermore, one has

$$\|f^t_\infty\| \leq \liminf_{n \to +\infty} \|f_{\sigma_t \wedge n}\| \text{ a.s.}$$

which in view of (i) and Fatou's lemma (or (ii)) shows that $\|f^t_\infty\|$ is integrable. Thus $f^t_\infty \in L^1_{X^*}[X](\mathcal{F})$.

Now, writing each g_n in the form

$$g_n = \sum_{i=n}^{k_n} \mu^i_n f_i \text{ with } 0 \leq \mu^i_n \leq 1 \quad \text{and} \quad \sum_{i=n}^{k_n} \mu^i_n = 1,$$

we define

$$g^t_n(\omega) := \sum_{i=n}^{k_n} \mu^i_n f_{\sigma_t \wedge n}(\omega), \quad (t > 0).$$

Observing that

$$g^t_n(\omega) = \begin{cases} g_n(\omega) & \text{if } \omega \in A_t, \\ f_{\sigma_t(\omega)}(\omega) & \text{otherwise} \ \text{ for all } n \geq \sigma_t(\omega), \end{cases}$$

we conclude that $(g^t_n(\omega))$ is $\mathcal{S}(\mathcal{R}(X^*_w))$-tight and equation (5.4) entails the following convergence

$$\lim_{n \to \infty} \langle x, g^t_n(\omega) \rangle = \langle x, f^t_\infty(\omega) \rangle, \text{ a.s.}$$

for every $x \in X$. Consequently, by (i), (ii), (5.4) and the first part of the proof, it follows that $(f_{\sigma_t \wedge n})$ s^*-converges a.s. to f^t_∞. Since $(f_{\sigma_t \wedge n})$ and f^t_∞ respectively, coincide with (f_n) and f_∞ on A_t and $P(A_t) \to 1$ when $t \to \infty$ (in view of (iii)), we deduce that (f_n) s^*-converges a.s. to f_∞. $\square$

Now here are some important corollaries.

Corollary 5.1. *Let $(f_n)_{n\geq 1}$ be a bounded martingale in $L^1_{X^*}[X](\mathcal{F})$ satisfying the following condition*

$(\mathcal{T}^+)$ *There exists a $\mathcal{R}(X^*_w)$-tight sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$.*

Then there exists $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that

$$(f_n) \ s^*\text{-converges a.s. to } f_\infty.$$

Proof. In view of Proposition 2.2, $(\mathcal{T}^+)$ implies $(\mathcal{T})$. This implication is also a consequence of Corollary 3.1. □

As a special case of this corollary we obtain the following extension of Chatterji result [**16**] (see also [**19**, Corollary II.3.1.7]) to the space $L^1_{X^*}[X](\mathcal{F})$.

Corollary 5.2. *Let $(f_n)_{n\geq 1}$ be a bounded martingale in $L^1_{X^*}[X](\mathcal{F})$. Suppose there exists a $cwk(X^*_w)$-valued multifunction K such that*

$$f_n(\omega) \in K(\omega) \quad \text{for all } n \geq 1.$$

Then there exists $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that (f_n) s^*-converges a.s. to f_∞.

Corollary 5.3. *Let $(f_n)_{n\geq 1}$ be a bounded martingale in $L^1_{X^*}[X](\mathcal{F})$ and let $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ be such that*

$$(\star) \qquad \lim_{n\to\infty} \langle x_\ell, f_n(\omega)\rangle = \langle x_\ell, f_\infty(\omega)\rangle \ \text{a.s. for all } \ell \geq 1.$$

Then the following statements are equivalent

(1) (f_n) s^*-converges to f_∞ a.s.
(2) There exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ which a.s. w-converges to f_∞.
(3) f_∞ is σ-measurable.

Proof. The implication (1) ⇒ (2) is obvious, whereas (2) ⇒ (3) follows from Corollary 4.1.
(3) ⇒ (1): A close look at the first proof of Theorem 5.1 reveals that the condition $(\mathcal{T})$ may be replaced with $(\star)$ and (3). □

It is worth to give the following variant of Proposition 5.1–Theorem 5.1.

Proposition 5.2. *Let $(f_n)_{n\geq 1}$ be a martingale in $L^1_{X^*}[X](\mathcal{F})$ satisfying the following two conditions:*

(C_1) For each $\ell \geq 1$, there exists a sequence (h_n) with $h_n \in \mathrm{co}\{f_i : i \geq n\}$ such that $(\langle x_\ell, h_n\rangle)$ is uniformly integrable.
(C_2) $\liminf_{n\to\infty} \|f_n\| \in L^1_{\mathbb{R}}(\mathcal{F})$

Then there exists $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that

$$f_n = E^{\mathcal{F}_n}(f_\infty) \text{ for all } n \geq 1 \text{ a.s. and}$$

$$(f_n) \ w^*\text{-converges to } f_\infty \text{ a.s.}$$

Furthermore, if the condition $(\mathcal{T})$ is satisfied, then

$$(f_n) \ s^*\text{-converges to } f_\infty \text{ a.s.}$$

Proof. Let $\ell \geq 1$ be fixed and let (h_n) be the sequence associated to ℓ according with (C_1). As the sequence $(\langle x_\ell, h_n\rangle)$ is uniformly integrable, there exist a subsequence (h_{n_k}) of (h_n) (possibly depending upon ℓ) and a function $\varphi_\ell \in L^1_{\mathbb{R}}(\mathcal{F})$ such that

$$\lim_{k\to\infty} \int_A \langle x_\ell, h_{n_k}\rangle \,\mathrm{d}P = \int_A \varphi_\ell \,\mathrm{d}P$$

for every $A \in \mathcal{F}$. Since $h_n \in \mathrm{co}\{f_i : i \geq n\}$ and $(\langle x_\ell, f_n\rangle)_n$ is a martingale, it is easy to check that

$$\int_A \langle x_\ell, h_{n_k}\rangle \,\mathrm{d}P = \int_A \langle x_\ell, f_m\rangle \,\mathrm{d}P$$

for all $k \geq m$ and $A \in \mathcal{F}_m$. Therefore

$$\int_A \langle x_\ell, f_m\rangle \,\mathrm{d}P = \int_A \varphi_\ell \,\mathrm{d}P \quad \text{for all } A \in \mathcal{F}_m$$

which is equivalent to

$$\langle x_\ell, f_m\rangle = E^{\mathcal{F}_m}(\varphi_\ell) \text{ a.s.} \tag{5.5}$$

This holds for all $\ell \geq 1$ and $m \geq 1$. Using the classical Levy's theorem, we get

$$\lim_{n\to+\infty} \langle x_\ell, f_n\rangle = \varphi_\ell \text{ a.s. for all } \ell \geq 1. \tag{5.6}$$

On the other hand, by (C_2) and the cluster point approximation theorem [**2**, Theorem 1]), (see also [**18**]), there exists an increasing sequence (τ_n) in T with $\tau_n \geq n$ for all n, such that

$$\lim_{n\to\infty} \|f_{\tau_n}\| = \liminf_{n\to\infty} \|f_n\| \text{ a.s.}$$

Then, for each ω outside a negligible set N, the sequence $(f_{\tau_n}(\omega))$ is bounded in X^*; hence it is relatively w^*-sequentially compact (the weak star topology being metrizable on bounded sets). Therefore, there exists a subsequence of (f_{τ_n}) (possibly depending upon ω) not relabeled and an element $x^*_\omega \in X^*$ such that

$$(f_{\tau_n}(\omega)) \ w^*\text{-converges to } x^*_\omega.$$

Define $f_\infty(\omega) := x^*_\omega$ for $\omega \in \Omega \setminus N$ and $f_\infty(\omega) := 0$ for $\omega \in N$. Then, taking into account (5.6), we get

$$\lim_{n\to+\infty} \langle x_\ell, f_n\rangle = \langle x_\ell, f_\infty\rangle = \varphi_\ell \text{ a.s. for all } \ell \geq 1. \tag{5.7}$$

This implies the scalar $\mathcal{F}$-measurability of f_∞. Furthermore, one has

$$\|f_\infty\| \leq \liminf_{n\to+\infty} \|f_n\| \text{ a.s.}$$

which in view of (C_2) shows that $\|f_\infty\|$ is integrable. Thus $f_\infty \in L^1_{X^*}[X](\mathcal{F})$. Next, replacing φ_ℓ in (5.5) with $\langle x_\ell, f_\infty\rangle$ (because of the second equality of (5.7)), we get

$$f_n = E^{\mathcal{F}_n}(f_\infty) \text{ a.s. for all } n \geq 1.$$

In particular, this yields

$$\sup_n \|f_n\| \leq \sup_n E^{\mathcal{F}_n}\|f_\infty\| < \infty \text{ a.s.} \tag{5.8}$$

Using the separability of X, (5.7) and (5.8), we get

$$(f_n)\ w^*\text{-converges to } f_\infty \text{ a.s.}$$

Finally, if the condition $(\mathcal{T})$ is satisfied, then, reasoning as in the first proof (or the first part of the second proof) of Theorem 5.1, we deduce that (f_n) s^*-converges a.s. to f_∞. $\square$

We finish this section by extending Theorem 5.1 to mils. For this purpose the following decomposition result is needed [**7**, Corollary 3.1].

Proposition 5.3. *Let $(f_n)_{n\geq 1}$ be a bounded mil in $L^1_{X^*}[X](\mathcal{F})$. Then there exists $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that*

$$\|f_n - E^{\mathcal{F}_n}(f_\infty)\| \to 0 \textit{ a.s. and}$$

$$(f_n)\ w^*\textit{-converges to } f_\infty \textit{ a.s.}$$

Proof. As (f_n) is a bounded mil in $L^1_{X^*}[X](\mathcal{F})$ for each $x \in X$, $(\langle x, f_n\rangle)$ is a bounded real mil in $L^1_{\mathbb{R}}(\mathcal{F})$, hence it converges a.s. to a function $r_x \in L^1_{\mathbb{R}}(\mathcal{F})$. On the other hand, using [**11**, Theorem 6.1(4)], we provide an increasing sequence $(A_p)_{p\geq 1}$ in $\mathcal{F}$ with $\lim_{p\to\infty} P(A_p) = 1$, a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ and a subsequence $(f'_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} \int_{A_p} \langle h, f'_n\rangle \mathrm{d}P = \int_{A_p} \langle h, f_\infty\rangle \mathrm{d}P$$

for all $p \geq 1$ and $h \in L^\infty_X(\mathcal{F})$. By identifying the limit, we get $r_x = \langle x, f_\infty\rangle$ a.s. Thus

$$\lim_{n\to\infty} \langle x, f_n(\omega)\rangle = \langle x, f_\infty(\omega)\rangle \text{ a.s.,} \tag{5.9}$$

for every $x \in X$. So the real mil $(\langle x, f_n - E^{\mathcal{F}_n}(f_\infty)\rangle)$ converges to 0 a.s. Consequently, it is possible to invoke an important result of Talagrand, ([**34**, Theorem 6]) which gives

$$\|f_n - E^{\mathcal{F}_n}(f_\infty)\| \to 0 \text{ a.s.}$$

As

$$\sup_{n\geq 1} \|E^{\mathcal{F}_n}(f_\infty)\| \leq \sup_{n\geq 1} E^{\mathcal{F}_n}(\|f_\infty\|) < \infty \text{ a.s.,}$$

we deduce that

$$\sup_{n\geq 1} \|f_n\| < \infty \text{ a.s.}$$

Then, using (5.9), the separability of X and the point-wise boundedness of (f_n), we obtain the a.s. w^*-convergence of (f_n) to f_∞. $\square$

Theorem 5.2. *Let $(f_n)_{n\geq 1}$ be a bounded mil in $L^1_{X^*}[X](\mathcal{F})$ satisfying the condition $(\mathcal{T})$. Then there exists $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that*

$$(f_n)\ s^*\textit{-converges a.s. to } f_\infty.$$

Proof. The proof is similar to the one given in Theorem 5.1 by using Proposition 5.3 instead of Proposition 5.1. $\square$

6. The special case of martingales in the subspace of $L^1_{X^*}[X](\mathcal{F})$ of all Pettis-integrable functions

In this section we provide a version of Theorem 5.1 in the special case of martingales in $L^1_{X^*}[X](\mathcal{F})$ whose members are also Pettis-integrable and satisfy a condition formulated in the manner of Marraffa [**25**]. For this purpose we need to recall a few extra definitions.

A function $f : \Omega \to X^*$ is said to be X^{**}*-scalarly* $\mathcal{F}$*-measurable* if for each $x^{**} \in X^{**}$, the real-valued function $\langle x^{**}, f\rangle$ is $\mathcal{F}$-measurable. We say also that f is *weakly* $\mathcal{F}$*-measurable*. If $f : \Omega \to X^*$ is a weakly $\mathcal{F}$-measurable function such that $\langle x^{**}, f\rangle \in L^1_{\mathbb{R}}(\mathcal{F})$ for all $x^{**} \in X^{**}$, then for each $A \in \mathcal{F}$, there is $x^{***}_A \in X^{***}$ such that

$$\langle x^{**}, x^{***}_A\rangle = \int_A \langle x^{**}, f\rangle \, \mathrm{d}P.$$

The vector x^{***}_A is called the *Dunford integral* of f over A. In the case that $x^{***}_A \in X^*$ for all $A \in \mathcal{F}$, then f is called *Pettis-integrable* and we write $P - \int_A f \, \mathrm{d}P$ instead of x^{***}_A to denote the *Pettis integral* of f over A. We denote by $P^1_{X^*}(\mathcal{F})$ the space of (equivalence class of) Pettis-integrable X^*-valued functions defined on $(\Omega, \mathcal{F}, P)$. Clearly, we have

$$P - \int_A f \, \mathrm{d}P = \int_A f \, \mathrm{d}P$$

for all $A \in \mathcal{F}$, whenever f is Pettis-integrable.

Before going further, we reformulate Corollary 3.1 under condition $(\mathcal{T}^+)$ for functions in the subspace $L^1_{X^*}[X](\mathcal{F}) \cap P^1_{X^*}(\mathcal{F})$.

Proposition 6.1. *Let $(f_n)_{n\geq 1}$ be a bounded sequence in $L^1_{X^*}[X](\mathcal{F})$ whose members are also Pettis-integrable. If (f_n) satisfies the condition $(\mathcal{T}^+)$, then there exist a function $f_\infty \in L^1_{X^*}[X](\mathcal{F}) \cap P^1_{X^*}(\mathcal{F})$ and a sequence (g_n) in $L^1_{X^*}[X](\mathcal{F})$ with $g_n \in co\{f_i : i \geq n\}$ such that*

$$(g_n) \ s^*\text{-converges to } f_\infty \ a.s.$$

Proof. Let (g_n) be as mentioned in $(\mathcal{T}^+)$. Since (f_n) is bounded in $L^1_{X^*}[X](\mathcal{F})$, so is the sequence (g_n) which is also $\mathcal{R}(X^*_w)$-tight (by $(\mathcal{T}^+)$). Consequently, using Corollary 3.1 and its proof, we provide a sequence of convex combinations of the (g_n) not relabeled, a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ and an increasing sequence (B_p) of measurable sets with $\lim_{p\to\infty} P(B_p) = 1$ such that

(i) (g_n) s^*-converges almost surely to f_∞.
(ii) For each $p \geq 1$, $(1_{B_p} g_n)$ converges strongly in $L^1_{X^*}[X](\mathcal{F})$ to $1_{B_p} f_\infty$.

Next, we will prove that f_∞ is Pettis integrable. Noting that the functions g_n are Pettis integrable, conclusion (i) shows that f_∞ is X^{**}-scalarly $\mathcal{F}$-measurable, and hence X^{**}-scalarly integrable (that is, for every $x^{**} \in X^{**}$, the scalar function $\omega \to \langle x^{**}, f_\infty(\omega)\rangle$ is integrable) because $\|f_\infty\| \in L^1_{\mathbb{R}}(\mathcal{F})$. Consequently, (ii) implies

$$\lim_{n\to\infty} \int_A \langle x^{**}, 1_{B_p} g_n\rangle \, \mathrm{d}P = \int_\Omega \langle x^{**}, 1_{B_p} f_\infty\rangle \, \mathrm{d}P$$

for all $p \geq 1$, $x^{**} \in X^{**}$ and $A \in \mathcal{F}$. This equation together with (ii) entails

$$\begin{aligned}
&|\langle x^{**}, \int_A 1_{B_p} f_\infty \,\mathrm{d}P\rangle - \int_A \langle x^{**}, 1_{B_p} f_\infty\rangle \,\mathrm{d}P| \\
&\qquad = \lim_{n\to\infty} |\langle x^{**}, \int_A 1_{B_p} f_\infty \,\mathrm{d}P\rangle - \int_A \langle x^{**}, 1_{B_p} g_n\rangle \,\mathrm{d}P| \\
&\qquad = \lim_{n\to\infty} |\langle x^{**}, \int_A 1_{B_p} f_\infty \,\mathrm{d}P\rangle - \langle x^{**}, \int_A 1_{B_p} g_n \,\mathrm{d}P\rangle| \\
&\qquad \leq \lim_{n\to\infty} \| \int_A 1_{B_p} f_\infty \,\mathrm{d}P - \int_A 1_{B_p} g_n \,\mathrm{d}P\| \\
&\qquad \leq \lim_{n\to\infty} \int_A \|1_{B_p} f_\infty \,\mathrm{d}P - 1_{B_p} g_n\| \,\mathrm{d}P = 0
\end{aligned}$$

for all $x^{**} \in X^{**}$ and $A \in \mathcal{F}$. Hence $1_{B_p} f_\infty$ is Pettis integrable for all $p \geq 1$. This yields to

$$\begin{aligned}
&\left|\langle x^{**}, \int_A f_\infty \,\mathrm{d}P\rangle - \int_A \langle x^{**}, f_\infty\rangle \,\mathrm{d}P\right| \\
&\leq \left|\langle x^{**}, \int_A 1_{B_p} f_\infty \,\mathrm{d}P\rangle - \int_\Omega \langle x^{**}, 1_{B_p} f_\infty\rangle \,\mathrm{d}P\right| \\
&\quad + \left|\langle x^{**}, \int_A 1_{\Omega\setminus B_p} f_\infty \,\mathrm{d}P\rangle - \int_\Omega \langle x^{**}, 1_{\Omega\setminus B_p} f_\infty\rangle \,\mathrm{d}P\right| \\
&\leq 2\int_A 1_{\Omega\setminus B_p} \|f_\infty\| \,\mathrm{d}P
\end{aligned}$$

for all $p \geq 1$. Letting $p \to +\infty$, we get

$$\langle x^{**}, \int_A f_\infty \,\mathrm{d}P\rangle = \int_A \langle x^{**}, f_\infty\rangle \,\mathrm{d}P,$$

for all $x^{**} \in X^{**}$ and $A \in \mathcal{F}$, thus f_∞ is Pettis integrable. □

Remark 6.1. If X does not contain any isomorphic copy of ℓ_1 (or equivalently X^* has the weak Radon-Nikodym property (WRNP)), then, according to Theorem 3 of Musial [**26**], every X-scalarly integrable function $f\colon \Omega \to X^*$ (that is, for every $x \in X$, $\langle x, f\rangle \in L^1_{\mathbb{R}}(\mathcal{F})$) is Pettis integrable, and hence each member of $L^1_{X^*}[X](\mathcal{F})$ is Pettis-integrable, i.e. $L^1_{X^*}[X](\mathcal{F}) \subset P^1_{X^*}(\mathcal{F})$.

Now we are ready to state the main result of this section which can be seen as a dual variant of [**17**, Theorem 3]. For this purpose, recall that a subset $H \subset X^{**}$ is *total* if it separates the points of X^*, that is, H is total if $\langle x^{**}, x^*\rangle = 0$ for all $x^{**} \in H$ implies $x^* = 0$.

Proposition 6.2. *Let $(f_n)_{n\geq 1}$ be a bounded martingale in $L^1_{X^*}[X](\mathcal{F})$ whose members are also Pettis-integrable satisfying the following condition.*

($\mathcal{T}^-$) There exist a total subset H of X^{**}, a σ-measurable function f in $L^0_{X^*}[X](\mathcal{F})$ and an increasing sequence (B_p) of measurable sets with $\lim_{p\to\infty} P(B_p) = 1$ such that for every $p \geq 1$, $1_{B_p} f \in P^1_{X^*}(\mathcal{F})$ and to each corresponds a sequence (g_n) in $L^1_{X^*}[X](\mathcal{F})$ with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ satisfying

$$\lim_{n\to\infty}\langle x^{**}, g_n\rangle = \langle x^{**}, f\rangle \text{ a.s. for each } x^{**} \in H.$$

Then $f \in L^1_{X^*}[X](\mathcal{F}) \cap P^1_{X^*}(\mathcal{F})$ *and we have*

$$(f_n) \ s^*\text{-converges to } f \text{ a.s.}$$

Note that the function f in condition $(\mathcal{T}^-)$ is necessarily weakly measurable, since the functions $1_{B_p}f$ $(p \geq 1)$ are weakly measurable and $\lim_{p\to\infty} P(B_p) = 1$.

Remark 6.2. In view of Proposition 6.1, condition $(\mathcal{T}^-)$ is weaker than $(\mathcal{T}^+)$ when dealing with bounded sequences in $L^1_{X^*}[X](\mathcal{F})$ whose members are also Pettis-integrable.

Taking account of Proposition 5.2, we deduce the following variant of Proposition 6.2.

Corollary 6.1. *Let* $(f_n)_{n\geq 1}$ *be a martingale in* $L^1_{X^*}[X](\mathcal{F}) \cap P^1_{X^*}(\mathcal{F})$ *satisfying the conditions* (C_1), (C_2) *and* $(\mathcal{T}^-)$. *Then the limit function* f *in* $(\mathcal{T}^-)$ *belongs to* $L^1_{X^*}[X](\mathcal{F}) \cap P^1_{X^*}(\mathcal{F})$ *and we have*

$$(f_n) \ s^*\text{-converges to } f \text{ a.s.}$$

Remark 6.3. In Proposition 6.2 as well as in Corollary 6.1, condition $(\mathcal{T})^-$ may be replaced with $\mathcal{T}^{--}$.

$(\mathcal{T}^{--})$ There exist a total subspace H of X^{**}, a σ-measurable function f in $L^0_{X^*}[X](\mathcal{F})$ and an increasing sequence (B_p) of measurable sets with $\lim_{p\to\infty} P(B_p) = 1$ such that, for every $p \geq 1$, $1_{B_p}f \in P^1_{X^*}(\mathcal{F})$, and $\langle x^{**}, f\rangle$ is a cluster point of $(\langle x^{**}, f_n\rangle)$ a.s. for each $x^{**} \in H$.

Indeed, let H and f be as mentioned in $(\mathcal{T}^{--})$. Let $x^{**} \in H$ be arbitrary fixed. Then $\langle x^{**}, f\rangle$ is a cluster point of $(\langle x^{**}, f_n\rangle)$ a.s. Further, f is weakly measurable. Consequently, by the cluster point approximation theorem (Theorem 1, [**2**]), there exists an increasing sequence (τ_n) in T (which may depend on x^{**}) with $\tau_n \geq n$ such that

$$\lim_{n\to\infty}\langle x^{**}, f_{\tau_n}\rangle = \langle x^{**}, f\rangle \text{ a.s.}$$

On the other hand, as (f_n) is a bounded martingale in $L^1_{X^*}[X](\mathcal{F})$ and each f_n is Pettis integrable, it is easy to see that the sequence $(\langle x^{**}, f_n\rangle)$ is a bounded real-valued martingale in $L^1_{\mathbb{R}}(\mathcal{F})$. So it converges a.s. to an integrable function in $L^1_{\mathbb{R}}(\mathcal{F})$ which is necessarily a.s. equal to $\langle x^{**}, f\rangle$. Thus condition $(\mathcal{T}^-)$ is satisfied.

Proof of Proposition 6.2. As (f_n) is a bounded martingale in $L^1_{X^*}[X](\mathcal{F})$, by Proposition 5.1, there exists $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that

(a) $$\lim_{n\to\infty}\|f_n - E^{\mathcal{F}_n}f_\infty\| \text{ a.s.}$$

(b) $$(f_n) \ w^*\text{-converges to } f_\infty \text{ a.s.}$$

Next, let H, f and (B_p) be as mentioned in $(\mathcal{T}^-)$. We will show that $f_\infty = f$ a.s.; once this is done we can invoke (a), the σ-measurability of f and Proposition 4.5 to conclude that (f_n) s^*-converges to f a.s.

We will use again a standard stopping time argument. Let σ_t, A_t and f^t_∞ be defined exactly as in the second proof of Theorem 5.1. First, by the expression $f_{\sigma_t\wedge n} = \sum_{k=\min\sigma_t\wedge n}^{\max\sigma_t\wedge n} f_k 1_{\{\sigma_t\wedge n=k\}}$ and since the functions f_n are Pettis-integrable, it is clear that the functions $f_{\sigma_t\wedge n}$ are also Pettis-integrable. Further, from (b) it follows

$$\lim_{n\to\infty}\langle x, f_{\sigma_t\wedge n}\rangle = \langle x, f^t_\infty\rangle. \tag{6.1}$$

Since $(f_{\sigma_t\wedge n})$ is a uniformly integrable martingale, from (6.1) it follows

$$\begin{aligned}\int_A \langle x, f_{\sigma_t\wedge m}\rangle \, \mathrm{d}P &= \lim_{n\to\infty, n\geq m}\int_A \langle x, f_{\sigma_t\wedge n}\rangle \, \mathrm{d}P\\ &= \int_A \langle x, f^t_\infty\rangle \, \mathrm{d}P = \int_A \langle x, E^{\mathcal{F}_{\sigma_t\wedge m}}(f^t_\infty)\rangle \, \mathrm{d}P\end{aligned}$$

for all $x\in X$, $m\geq 1$ and $A\in\mathcal{F}_{\sigma_t\wedge m}$. Hence

$$f_{\sigma_t\wedge m} = E^{\mathcal{F}_{\sigma_t\wedge m}}(f^t_\infty) \text{ a.s. for all } m\geq 1,$$

by the separability of X. Next, let x^{**} be an arbitrary fixed element in H. Then, by $(\mathcal{T}^-)$, there exists a sequence (g_n) of the form

$$g_n = \sum_{i=n}^{k_n}\mu^i_n f_i \quad\text{with}\quad 0\leq \mu^i_n\leq 1 \quad\text{and}\quad \sum_{i=n}^{k_n}\mu^i_n = 1$$

such that

$$\lim_{n\to\infty}\langle x^{**}, g_n\rangle = \langle x^{**}, f\rangle$$

which implies

$$\lim_{n\to\infty}\langle x^{**}, g^t_n\rangle = \langle x^{**}, f^t\rangle, \tag{6.2}$$

where

$$g^t_n := \sum_{i=n}^{k_n}\mu^i_n f_{\sigma_t\wedge i}$$

and

$$f^t(\omega) := \begin{cases} f(\omega) & \text{if } \omega\in A_t,\\ f_{\sigma_t(\omega)}(\omega) & \text{otherwise}\end{cases}$$

(see the second proof of Theorem 5.1). As the function $\sup_n \|g^t_n\|$ is also integrable, (by (ii) and the inequality $\sup_n\|g^t_n\|\leq \sup_n\|f_{\sigma_t\wedge n}\|$) equation (6.2) entails

$$\lim_{n\to\infty}\int_A\langle x^{**}, g^t_n\rangle \, \mathrm{d}P = \int_A \langle x^{**}, f^t\rangle \, \mathrm{d}P \text{ for all } A\in\mathcal{F}. \tag{6.3}$$

On the other hand, recalling that for each $m\geq 1$, $f_{\sigma_t\wedge m} = E^{\mathcal{F}_{\sigma_t\wedge m}}(f^t_\infty)$ a.s. and $f_{\sigma_t\wedge m}\in P^1_{X^*}(\mathcal{F})$, we obtain

$$\begin{aligned}\int_A\langle x^{**}, g^t_n\rangle \, \mathrm{d}P &= \sum_{i=n}^{k_n}\mu^i_n\int_A\langle x^{**}, f_{\sigma_t\wedge i}\rangle \, \mathrm{d}P = \int_A\langle x^{**}, f_{\sigma_t\wedge m}\rangle \, \mathrm{d}P\\ &= \langle x^{**}, \int_A f_{\sigma_t\wedge m}\, \mathrm{d}P\rangle = \langle x^{**}, \int_A E^{\mathcal{F}_{\sigma_t\wedge m}}(f^t_\infty)\, \mathrm{d}P\rangle = \langle x^{**}, \int_A f^t_\infty \, \mathrm{d}P\rangle\end{aligned}$$

for all $m \geq 1$, $n \geq m$ and $A \in \mathcal{F}_{\sigma_t \wedge m}$. Together with (6.3), we get

$$\int_A \langle x^{**}, f^t \rangle \, \mathrm{d}P = \langle x^{**}, \int_A f^t_\infty \, \mathrm{d}P \rangle$$

for all $m \geq 1$ and $A \in \mathcal{F}_{\sigma_t \wedge m}$. Since the functions $\langle x^{**}, f^t \rangle$ and $\|f^t_\infty\|$ are integrable, this equality extends easily to all $A \in \sigma(\cup_n \mathcal{F}_{\sigma_t \wedge n})$. In particular, for all $p \geq 1$ and for all $A \in \mathcal{F}$, we have

$$\int_{A \cap A^t \cap B_p} \langle x^{**}, f^t \rangle \, \mathrm{d}P = \langle x^{**}, \int_{A \cap A^t \cap B_p} f^t_\infty \, \mathrm{d}P \rangle$$

because $A_t \cap \mathcal{F} \subset \sigma(\cup_n \mathcal{F}_{\sigma_t \wedge n})$ in view of the inclusion (‡) presented in Section 2. As $1_{A^t} f^t_\infty = 1_{A^t} f_\infty$, $1_{A^t} f^t = 1_{A^t} f$ and $1_{B_p} f \in P^1_{X^*}(\mathcal{F})$ for all $p \geq 1$, it follows that

$$\langle x^{**}, \int_{A \cap A^t \cap B_p} f \, \mathrm{d}P \rangle = \langle x^{**}, \int_{A \cap A^t \cap B_p} f_\infty \, \mathrm{d}P \rangle$$

for all $p \geq 1$ and $A \in \mathcal{F}$. Since this holds for all $x^{**} \in H$ and H is total, we get

$$\int_A 1_{A^t \cap B_p} f \, \mathrm{d}P = \int_A 1_{A^t \cap B_p} f_\infty \, \mathrm{d}P$$

for every $p \geq 1$ and $A \in \mathcal{F}$. Equivalently

$$1_{A^t \cap B_p} \langle x, f \rangle = 1_{A^t \cap B_p} \langle x, f_\infty \rangle \text{ a.s.}$$

for every $p \geq 1$ and $x \in X$. Since X is separable, $P(B_p) \to 1$ and $P(A_t) \to 1$ when $p, t \to \infty$, it follows that

$$f = f_\infty \text{ a.s.}$$

Finally, to prove that $f \in P^1_{X^*}(\mathcal{F})$, it suffices to repeat the arguments used at the end of the proof of Proposition 6.1. □

We finish this work by providing the following result which extends Proposition 6.2 to mils. It can be seen as a mil version of the Ito-Nisio theorem (the main implication of it) (see [**23**]) in the framework of a dual space. For various martingale generalizations dealing with primal space, one can look at the contributions of Marraffa [**25**], Bouzar [**4**] and Luu [**24**].

Proposition 6.3. *Let $(f_n)_{n \geq 1}$ be a bounded mil in $L^1_{X^*}[X](\mathcal{F})$ whose members are also Pettis-integrable satisfying the condition ($\mathcal{T}^{--}$). Then f is a member of $L^1_{X^*}[X](\mathcal{F}) \cap P^1_{X^*}(\mathcal{F})$ and we have*

$$(f_n) \ s^*\text{-converges to } f \ \text{a.s.},$$

where f is the limit function in ($\mathcal{T}^{--}$).

For the sake of shortness, we refrain from giving the details of proofs and refer to our forthcoming work [**29**].

References

1. Amrani A., Castaing C. and Valadier M., *Méthodes de troncatures appliquées à des problèmes de convergences faible ou forte dand* L^1, Arch. Rational Mech. Anal. **117** (1992), 167–191.

2. Austin D. G., Edgar G. A. and Ionescu Tulcea A., *Pointwise Convergence in Terms of Expectations*, Wahrscheinlichkeitstheorie verw. Geb. **30**(1974), 17–26.

3. Benabdellah H. and Castaing C., *Weak compactness and convergences in* $L^1_{E'}[E]$, Adv. Math. Econ. **3** (2001), 1–44.

4. Bouzar N., *On almost sure convergence without the Radon-Nikodym property*, Acta Math. Univ. Comenianae **LXX(2)** (2001), 167–175.

5. Castaing C., *Quelques résultats de convergence des suites adaptées*, Sém. Anal. Conv. Montpellier, Exposé **2** (1987), 2.1–2.24.

6. Castaing C., Hess Ch. and Saadoune M., *Tightness conditions and integrability of the sequential weak upper limit of a sequence of multifunctions*, Adv. Math. Econ. **11** (2008), 11–44.

7. Castaing C., Ezzaki, Lavie M. and Saadoune M., *Weak star convergence of martingales in a dual space*, Banach center publications **92**, Institute of Mathematics, Polish Academy of Sciences, Warszawa 2011.

8. Castaing C., Raynaud de Fitte P. and Valadier M., *Young measures on topological spaces. With applications in control theory and probability theory*. Kluwer Academic Publishers, Dordrecht 2004.

9. Castaing C. and Guessous M., *Convergences in* $L^1_X(\mu)$, Adv. Math. Econ. **1** (1999), 17–37.

10. Castaing C. and Saadoune M., *Dunford-Pettis-types theorem and convergences in set-valued integration*, Journal of Nonlinear and Convex Analysis **1(1)** (2000), 37–71.

11. ______, *Komlos type convergence for random variables and random sets with applications to minimizations problems*, Adv. Math. Econ. **10** (2007), 1–29.

12. ______, *Various versions of Fatou type lemma*, Working paper, March 2006, Unviversite Montpellier II. 23 pages.

13. ______, *Convergences in a dual space with applications to Fatou Lemma*, Adv. Math. Econ. **12** (2009), 23–69.

14. Castaing C. and Valadier M., *Convex Analysis and Measurable Multifunctions*, Lectures Notes in Mathematics, Springer-Verlag, Berlin **580**, 1977.

15. Chacon R. V. and Sucheston L., *On convergence of vector-valued asymptotic martingales*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **33** (1975), 55–59.

16. Chatterji S. D., *Vector-valued martingales and their applications*, Proc. Conf. measure theory Oberwolfach (1975). Lect. Notes in Math. 526, 33–51, Springer Verlag Berlin.

17. Davis W. J., Ghoussoub N., Johnson W. B., Kwapien S. and Maurey B., *Weak convergence of vector-valued martingales*, Probability in Banach spaces 6 (Sandbjerg 1986), 41–40, Birkhauser Boston, Boston, Ma, 1990.

18. Edgar G. A. and Sucheston L., *Amarts: a class of asymptotic martingales, A. discrete parameter*, J. Multivariate Anal. **6** (1976), 193–221.

19. Egghe L., *Convergences of adapted sequences of Pettis-integrable functions*, Pacific Journal of Mathematics 114(2), (1984), 345–366.

20. Fitzpatrick S. and Lewis A. S., *Weak-Star Convergence of Convex Sets*, Journal of Convex Analysis **13(3-4)** (2006), 711–719.

21. Gaposhkin V. F., *Convergence and limit theorems for sequences of random variables*, Theory Prob. and Appl. **17(3)** (1972), 379–400.

22. Grothendieck A., *Espaces vectoriels topologiques*, Pub. de la Soc. Math. de Sao-Paulo, 1954.

23. Ito K. and Nisio M, *On the convergence of sums of independent Banach space valued random variables*, Osaka Math. J. **5** (1968), 25–48.

24. Luu D. Q., *Convergence of adapted sequences in Banach spaces without the Radon-Nikodym property*, Acta Math. Vietnam. **30(3)** (2005), 289–297.

25. Marraffa V., *On almost sure convergence of amarts and martingales without the Radon-Nikodym property*, Journal of Theoret. Probab **1** (1988), 255–261.

26. Musial K., *Topics in the theory of Pettis integration*, In: *School of Measure Theory and Real Analysis* Italy, May 1992.

27. ———, *The weak Radon-Nikodym property in Banach spaces*, Studia Mathematica, **LXIV** (1979) 151–173.

28. Neveu J., *Martingales à temps discret*, Masson, 1972.

29. Saadoune M., *Strong convergence of martingales in the limit in a dual space*, Working paper May 2010.

30. ———, *A new extension of Komlós theorem in infinite dimensions. Application: Weak compactness in* L^1_X, Portugaliae Mathematica **55** (1998), 113–128.

31. Saadoune M. and Valadier M., *Extraction of a "good" subsequence from a bounded sequence of integrable functions*, Journal of Convex Analysis **2** (1995), 345–357.

32. Sainte-Beuve M. F., *Some topological properties of vector measures with bounded variations and its applications*, Annali di Mat. Pura ed Appl. **116** (1978), 317–379.

33. Slaby M., *Strong convergence of vector-valued pramarts and subpramarts*, Probab. and Math. Statist. **5** (1985), 187–196.

34. Talagrand M., *Some structure results for martingales in the limit and pramarts*, The Annals of Probability **13(40)** (1985), 1192–1203.

35. Valadier M., *On conditional Expectation of random sets*, Annali di Mathematica pura ed applicada **CXXVI(iv)** (1980), 81–91.

M. Saadoune, Mathematics Department, Ibn Zohr University, Addakhla, B.P. 8106, Agadir, Maroc, *e-mail*: `mohammed.saadoune@gmail.com`