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REMARKS ON ŠEDA THEOREM

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Abstract. We found sufficient conditions on a sequences (λ_n) and (b_n) when the equation $f'' + a_0 f = 0$ has an entire solution f such that $f(\lambda_n) = b_n$.

In [**10**] V. Šeda proved that for any sequence (λ_n) of distinct complex numbers with no finite limit points there exists an entire function A_0 such that the equation

$$f'' + A_0 f = 0 \tag{1}$$

has an entire solution f with zeros only at points λ_n. On the other hand ([**3**, p. 201], [**7**, p. 300–301]), for every sequence (λ_n) of distinct complex numbers with no finite limit points and for every sequence (b_n) of complex numbers there exists an entire function f such that

$$f(\lambda_n) = b_n. \tag{2}$$

This result was extended to the case of functions holomorphic in open subsets of the complex plane $\mathbb{C}$ by C. Berenstein and B. Taylor [**2**]. In particular, we generalize the above-mentioned results from [**10**] and [**3**].

Theorem 1. *For any sequence (λ_n) of distinct complex numbers in the domain $D \subset \mathbb{C}$ with no limit points in D and every sequence (b_n) of complex numbers there exists a holomorphic in D function A_0 such that the equation* (1) *has a holomorphic solution f satisfying* (2).

Šeda result was developed in papers [**1, 4, 5, 8, 9**]. For meromorphic function A_0 it was extended in [**11**]. Bank [**1**] obtained a necessary condition for a sequence with a finite exponent of convergence to be the zero-sequence of a solution of the equation (1). In [**1**] there is also proved the following proposition.

Theorem A ([**1**, p. 3])**.** *Let $K > 1$ be a real number and let (λ_n) be any sequence of non-zero complex points satisfying $|\lambda_{n+1}| \geq K|\lambda_n|$ for $n \in \mathbb{N}$. Then there exists an entire transcendental function A(z) of order zero such that the equation* (1) *possesses a solution whose zero-sequence is (λ_n).*

In [**8**] Sauer obtain a more general sufficient condition.

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Theorem B ([**8**, p. 1144])**.** *Let (λ_n) be a sequence with finite exponent of convergence, p be its genus and*

$$\mu_k := \prod_{m\neq k} \left(1 - \frac{\lambda_k}{\lambda_m}\right)^{-1} e_p\left(\frac{\lambda_k}{\lambda_m}\right)^{-1},$$

where $e_p(z)$ denotes the Weierstrass convergence factor. If there exists a real number $b > 0$ and a positive integer k_0 such that

$$|\mu_k| \leq \exp\left(|\lambda_k|^b\right)$$

for all $k \geq k_0$, then (λ_n) is the zero-sequence of a solution of an equation (1) *with entire transcendental function $A_0(z)$ of finite order.*

In [**4**] J. Heittokangas and I. Laine improved the above results and, in particular, proved the following statement.

Theorem C ([**4**, p. 300])**.** *Let (λ_n) be an infinite sequence of non-zero complex points having a finite exponent of convergence λ, a finite genus p and no finite limit points. Let L be the canonical product associated with (λ_n),*

$$\inf_k \left\{ |\lambda_k| \, e^{|\lambda_k|^q} \left|L'\left(\lambda_k\right)\right| \right\} > 0$$

for some $q \geq 0$ and arbitrary $\varepsilon > 0$. Then (λ_n) is the zero-sequence of a solution of an equation (1) *with entire transcendental function A_0 such that*

$$\rho_{A_0} \leq \max\{\lambda + \varepsilon; q\}.$$

From estimates in [**4**] it is possible to get the following result.

Corollary 1. *Let $\rho \in (0; +\infty)$, L be the canonical product associated with the sequence (λ_n) of distinct complex numbers and the conditions*

$$\lambda := \overline{\lim_{j\to\infty}} \frac{\log j}{\log|\lambda_j|} \leq \rho, \tag{3}$$

$$\overline{\lim_{j\to\infty}} \frac{\log^+ \log^+ |1/L'(\lambda_j)|}{\log|\lambda_j|} \leq \rho \tag{4}$$

be satisfied. Then there exists an entire function A_0 of order $\rho_{A_0} \leq \rho$ such that the equation (1) *has an entire solution f for which (λ_n) is the zero-sequence.*

This corollary also follows from the following theorem. The Theorem 2 is our second main result.

Theorem 2. *Let $\rho \in (0; +\infty)$, (b_n) be an arbitrary sequence of complex numbers and L be the canonical product associated with the sequence (λ_n) of distinct complex numbers. If the conditions* (3), (4) *and*

$$\overline{\lim_{j\to\infty}} \frac{\log^+\log^+\log^+|b_j|}{\log|\lambda_j|} \leq \rho \tag{5}$$

hold, then there exists an entire function A_0 of order $\rho_{A_0} \leq \rho$ such that the equation (1) *has an entire solution f satisfying* (2).

To prove Theorem 1 we need the following lemma.

Lemma 1 ([**2**, p. 118]). *Let $(a_{j,1})$ and $(a_{j,2})$ be sequences of complex numbers, (λ_j) be a sequence of distinct complex numbers in domain $D \subset \mathbb{C}$ with no limit points in D. Then there exists a holomorphic in D function g such that*

$$g(\lambda_j) = a_{j,1}, \qquad g'(\lambda_j) = a_{j,2} \tag{6}$$

for all $j \in \mathbb{N}$.

Proof of Theorem 1. Let

$$\{n_k : k \in \mathbb{N}\} = \{n \in \mathbb{N} : b_n = 0\} \qquad \text{and} \qquad \{m_k : k \in \mathbb{N}\} = \mathbb{N}\backslash\{n_k : k \in \mathbb{N}\}.$$

Then $\{\lambda_{n_k}\} \cup \{\lambda_{m_k}\} = \{\lambda_n\}$. Let $\log u = \log|u| + \mathrm{i}\,\varphi$, $\varphi = \arg u \in [-\pi; \pi)$, and Q be a holomorphic function in D with simple zeros at the points λ_{n_k} and $Q(\lambda_{m_k}) \neq 0$ for all k. Denote

$$a_{j,1} = \begin{cases} \log \dfrac{b_j}{Q(\lambda_j)}, & j \in \{m_k\}, \\ 0, & j \notin \{m_k\}, \end{cases} \qquad a_{j,2} = \begin{cases} 0, & j \notin \{n_k\}, \\ -\dfrac{Q''(\lambda_j)}{2Q'(\lambda_j)}, & j \in \{n_k\}. \end{cases}$$

By Lemma 1 it follows that there exists a holomorphic function g in D such that (6) is valid. Hence the function

$$A_0 = -\frac{Q'' + 2Q'g'}{Q} - g'' - {g'}^2$$

is holomorphic in D and the function $f = Qe^g$ is a solution of the equation (1) and satisfies the condition (2). □

To prove Theorem 2 we need the following statement.

Lemma 2 ([**6**, p. 146–147]). *Let $\rho \in (0; +\infty)$ and (λ_n) be a sequence of distinct complex numbers. For any sequences $(a_{j,1})$ and $(a_{j,2})$ of complex numbers such that*

$$\varlimsup_{j\to\infty} \frac{\log^+\log^+|a_{j,s}|}{\log|\lambda_j|} \le \rho, \qquad s \in \{1; 2\}, \tag{7}$$

there exists at least one entire function g of order $\rho_g \le \rho$ satisfying (6) *if and only if the condition* (3) *and*

$$\varlimsup_{j\to\infty} \frac{\log^+\log^+|\gamma_{j,s}|}{\log|\lambda_j|} \le \rho, \qquad s \in \{1; 2\}, \tag{8}$$

hold, where $F = L^2$,

$$\gamma_{j,1} = \left(\frac{(z-\lambda_j)^2}{F(z)}\right)\bigg|_{z=\lambda_j}, \qquad \gamma_{j,2} = \left(\frac{(z-\lambda_j)^2}{F(z)}\right)'\bigg|_{z=\lambda_j},$$

$$L(z) = \prod_{j=1}^{\infty} (1 - z/\lambda_j) \exp\left(\sum_i^p \frac{1}{i}\left(\frac{z}{\lambda_j}\right)^i\right)$$

and p is the smallest integer for which the series

$$\sum_j \frac{1}{|\lambda_j|^{p+1}}$$

converges.

Proof of Theorem 2. Let $\{n_k : k \in \mathbb{N}\} = \{n \in \mathbb{N} : b_n = 0\}$ and $\{m_k : k \in \mathbb{N}\} = \mathbb{N} \backslash \{n_k : k \in \mathbb{N}\}$. Then $\{\lambda_{n_k}\} \cup \{\lambda_{m_k}\} = \{\lambda_n\}$. Denote

$$Q(z) = \prod_{j=1, j\in\{n_k\}}^{\infty} (1 - z/\lambda_j) \exp\left(\sum_i^p \frac{1}{i}\left(\frac{z}{\lambda_j}\right)^i\right),$$
$$G(z) = \prod_{j=1, j\in\{m_k\}}^{\infty} (1 - z/\lambda_j) \exp\left(\sum_i^p \frac{1}{i}\left(\frac{z}{\lambda_j}\right)^i\right)$$

and

$$a_{j,1} = \begin{cases} \log \dfrac{b_j}{Q(\lambda_j)}, & j \in \{m_k\}, \\ 0, & j \notin \{m_k\}, \end{cases} \qquad a_{j,2} = \begin{cases} 0, & j \notin \{n_k\}, \\ -\dfrac{Q''(\lambda_j)}{2Q'(\lambda_j)}, & j \in \{n_k\}. \end{cases}$$

Since $L(z) = Q(z)G(z)$, $L'(z) = Q'(z)G(z) + Q(z)G'(z)$, we see that $1/Q(\lambda_{m_k}) = G'(\lambda_{m_k})/L'(\lambda_{m_k})$ and $1/Q'(\lambda_{n_k}) = G(\lambda_{n_k})/L'(\lambda_{n_k})$. Using (3)–(5), we get that the sequences $(a_{j,1})$ and $(a_{j,2})$ satisfy the condition (7). Since

$$F(z) = \sum_{j=0}^{m} \frac{F^{(j)}(\lambda_j)}{j!}(z-\lambda_j)^j + o(z-\lambda_j)^m, z \to \lambda_j$$

for each $m \in \mathbb{Z}_+$, we have

$$\gamma_{j,1} = \frac{2}{F''(\lambda_j)}, \qquad \gamma_{j,2} = -\frac{2}{3}\frac{F'''(\lambda_j)}{(F''(\lambda_j))^2}.$$

Since

$$F''(\lambda_j) = 2(L'(\lambda_j))^2, \qquad F'''(\lambda_j) = -2L''(\lambda_j)/L'(\lambda_j),$$

then

$$\gamma_{j,1} = \frac{1}{(L'(\lambda_j))^2}, \qquad \gamma_{j,2} = \frac{L''(\lambda_j)}{3(L'(\lambda_j))^5}.$$

Taking into account (3) and (4), we obtain (8). From Lemma2 it follows that there exists an entire function g such that the condition (6) holds. Moreover $\rho_g \leq \rho$. Then $f = Qe^g$ is a solution of the equation (1), where

$$A_0 = -\frac{Q'' + 2Q'g'}{Q} - g'' - {g'}^2.$$

By standard metods we obtain $\rho_{A_0} \leq \rho$. □

A question of sharpness of the condition (7) remains open.

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