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THE CESÀRO χ^2 SEQUENCE SPACES DEFINED BY A MODULUS

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Abstract. In this paper we define the Cesàro χ^2 sequence space $\mathrm{Ces}_p^q\left(\chi_f^2\right)$ defined by a modulus and exhibit some general properties of the space.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

An initial work on double sequence spaces is found in Bromwich [**4**]. Later on, they were investigated by Hardy [**5**], Moricz [**9**], Moricz and Rhoades [**10**], Basarir and Solankan [**2**], Tripathy [**17**], Turkmenoglu [**19**] and many others.

Let us define the following sets of double sequences:

$$\begin{aligned}
\mathcal{M}_u(t) &:= \left\{(x_{mn}) \in w^2 : \sup_{m,n\in\mathbb{N}} |x_{mn}|^{t_{mn}} < \infty\right\},\\
\mathcal{C}_p(t) &:= \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C}\right\},\\
\mathcal{C}_{0p}(t) &:= \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},\\
\mathcal{L}_u(t) &:= \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},\\
\mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \quad \text{and} \quad \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t)
\end{aligned}$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$, $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ are reduced to the sets

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$\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some documents related to the double sequence spaces. Gökhan and Colak [**21, 22**] proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α-, β-, γ- duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, Zelter [**23**] in her PhD thesis, essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [**24**] recently introduced the statistical convergence and Cauchy for double sequences and gave the relation between statistical convergent and strongly Cesàro summable double sequences. Next, Mursaleen [**25**] and Mursaleen and Edely [**26**] defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [**27**] defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Further Basar and Sever [**28**] introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [**29**] studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

Spaces that are strongly summable sequences were discussed by Kuttner [**31**], Maddox [**32**], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [**8**] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections among strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. In [**34**] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [**35**]–[**38**] and [**39**] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. This will be accomplished by presenting the following sequence spaces:

$$\mathrm{Ces}_p^q\left(\chi_f^2\right) = d(x,0)$$
$$= \left\{x\in\chi^2 := \lim_{m,n\to\infty}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)\left|x_{mn}\right|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}} = 0\right\}$$

and

$$\mathrm{Ces}_p^q\left(\Lambda_f^2\right) = d(x,0)$$
$$= \left\{x\in\chi^2 := \sup\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left(\left|x_{mn}\right|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}} < \infty\right\}$$

where f is a modulus function. Other implications, general properties and variations will also be presented.

In the sequel of the paper we need the following inequality

$$(a+b)^p \leq a^p + b^p \tag{1.1}$$

for $a, b, \geq 0$ and $0 < p < 1$. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [**1**]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{$all finite sequences$\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})$ $(m, n \in \mathbb{N})$ are also continuous.

Orlicz [**13**] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [**7**] investigated Orlicz sequence spaces in more detail and proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p \, (1 \leq p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [**14**], Mursaleen et al. [**11**], Bektas and Altin [**3**], Tripathy et al. [**18**], Rao and Subramanian [**15**] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [**6**].

Recalling [**13**] and [**6**], an Orlicz function is a function $M \colon [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [**12**] and further discussed by Ruckle [**16**] and Maddox [**8**], and many others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u) \, (u \geq 0)$. The Δ_2-condition is equivalent to $M(\ell u) \leq K\ell M(u)$ for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [**7**] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some} \quad \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p$, $(1 \le p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions:

(i) X' is the continuous dual of X;

(ii) $X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \quad \text{for each } \; x \in X\right\}$;

(iii) $X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \; \text{ is convergent, for each } \; x \in X\right\}$;

(iv) $X^\gamma = \left\{a = (a_{mn}) : \sup_{mn} \ge 1, \quad \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{ for each } \; x \in X\right\}$;

(v) let X be an FK-space $\supset \phi$; then $X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi) $X^\delta = \left\{a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \quad \text{for each } x \in X\right\}$;

X^α, X^β, X^γ are called α- (or Köthe-Toeplitz)-dual of X, β- (or generalized- Köthe-Toeplitz)-dual of X, γ-dual of X, δ-dual of X, respectively. X^α was defined by Gupta and Kamptan [**20**]. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [**30**] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences, respectively. The difference space bv_p of the classical space ℓ_p was introduced and studied in the case $1 \le p \le \infty$ by Basar and Altay in [**42**] and in the case $0 < p < 1$ by Altay and Basar in [**43**]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \ge 1} |\Delta x_k| \quad \text{and} \quad \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, \quad (1 \le p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2$, χ^2 and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Definitions and Preliminaries

$\mathrm{Ces}_p^q\left(\chi_f^2\right)$ and $\mathrm{Ces}_p^q\left(\Lambda_f^2\right)$ denote the Pringscheims sense of Cesàro double gai sequence space of modulus and Pringscheims sense of Cesàro double analytic sequence space of modulus, respecctively.

Definition 2.1. A modulus function was introduced by Nakano [**12**]. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x+y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from here that f is continuous on $[0,\infty)$.

Definition 2.2. Let $A = \left(a_{k,\ell}^{mn}\right)$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the k,ℓ-th term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}.$$

Such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [**40**] and Toeplitz [**41**]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded.

Definition 2.3. Let $p \in [1,\infty)$ and q be a double gai sequence of positive real numbers such that

$$Q_{ij} = \sum_{m=0}^{i}\sum_{n=0}^{j} q_{mn}, \qquad i,j \in \mathbb{N},$$

$$\mathrm{Ces}_p^q\left(\chi_f^2\right) = d(x,0)$$
$$= \left\{ x \in \chi^2 := \lim_{m,n\to\infty} \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left(\frac{1}{Q_{ij}} \sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)\,|x_{mn}| \right)^{\frac{1}{m+n}} \right) \right)^{p_{mn}} \right)^{\frac{1}{p_{mn}}} = 0 \right\}$$

If $q_{mn} = 1$ for all $m,n \in \mathbb{N}$, then $\mathrm{Ces}_p^q\left(\chi_f^2\right)$ reduces to $\mathrm{Ces}_p\left(\chi_f^2\right)$, and if $f(x) = x$, then $\mathrm{Ces}_p^q\left(\chi_f^2\right)$ reduces to $\mathrm{Ces}_p^q\left(\chi^2\right)$.

Definition 2.4. Let $p \in [1,\infty)$ and q be a double analytic sequence of positive real numbers such that

$$Q_{ij} = \sum_{m=0}^{i}\sum_{n=0}^{j} q_{mn}, \qquad i,j \in \mathbb{N},$$

$$\operatorname{Ces}_p^q\left(\Lambda_f^2\right)=d\left(x,0\right)$$
$$=\left\{x\in\chi^2:=\sup\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(\left(\left|x_{mn}\right|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}<\infty\right\}.$$

If $q_{mn}=1$ for all $m,n\in\mathbb{N}$, then $\operatorname{Ces}_p^q\left(\Lambda_f^2\right)$ reduces to $\operatorname{Ces}_p\left(\Lambda_f^2\right)$, and if $f\left(x\right)=x$, then $\operatorname{Ces}_p^q\left(\Lambda_f^2\right)$ reduces to $\operatorname{Ces}_p^q\left(\Lambda^2\right)$.

The space $Ces_p^q\left(\chi_f^2\right)$ is a metric space with the metric

$$d\left(x,y\right)$$
$$=\inf\left\{\sup\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(\left((m+n)!\left|x_{mn}-y_{mn}\right|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\leq 1\right\}$$

The space $Ces_p^q\left(\Lambda_f^2\right)$ is a metric space with the metric

$$d\left(x,y\right)$$
$$=\inf\left\{\sup\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(\left(\left|x_{mn}-y_{mn}\right|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\leq 1\right\}.$$

3. Main Results

Proposition 3.1. *Let $x,y\in Ces_p^q\left(\chi_f^2\right)$. Then for any $\varepsilon>0$ and $L>0$, there exists $\delta>0$ such that $\left(d\left(x+y,0\right),0\right)^{p_{mn}}=d\left(x,0\right)^{p_{mn}}+\varepsilon$, whenever $d\left(x,0\right)^{p}\leq L$ and $d\left(y,0\right)^{p_{mn}}\leq\delta$.*

Proof. For any fix $\varepsilon>0$,

$$\begin{aligned}d\left(x+y,0\right)^{p_{mn}}&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(\left((m+n)!\left|x_{mn}+y_{mn}\right|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\\&\leq\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(\left((m+n)!\left|x_{mn}\right|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\\&\quad+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(\left((m+n)!\left|y_{mn}\right|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\end{aligned}$$

$$\le (1-\beta)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}$$
$$+(\beta)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}$$
$$\cdot(\beta)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}\frac{q_{mn}f\left(((m+n)!\,|y_{mn}|)^{\frac{1}{m+n}}\right)}{\beta}\right)^{p_{mn}}$$
$$\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}$$
$$+\frac{\beta}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}2q_{mn}f\left(((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}$$
$$\cdot\frac{\beta}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}\frac{2q_{mn}f\left(((m+n)!\,|y_{mn}|)^{\frac{1}{m+n}}\right)}{\beta}\right)^{p_{mn}}$$
$$\le d(x,0)^{p_{mn}}+\frac{\varepsilon}{2}$$
$$+\left(\frac{2}{\beta}\right)^{p_{mn}-1}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(((m+n)!\,|y_{mn}|)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}$$
$$\le d(x,0)^{p_{mn}}+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$
$$\le d(x,0)^{p_{mn}}+\varepsilon.$$

□

Proposition 3.2. *For every* $p=(p_{mn})$,

$$\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}=\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\alpha}=\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\gamma}=\left[Ces_p^q\left(\eta_f^2\right)\right]^{\beta},$$

where

$$\left[Ces_p^q\left(\eta_f^2\right)\right]$$
$$=\bigcap_{N\in\mathbb{N}-\{1\}}\left\{x=x_{mn}:\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(|x_{mn}|\,N^{\frac{m+n}{p_{mn}}}\right)\right)^{\frac{1}{p_{mn}}}<\infty\right\}.$$

Proof. First we show that $\left[Ces_p^q\left(\eta_f^2\right)\right]\subset\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}$.

Let $x \in \left[Ces_p^q\left(\eta_f^2\right)\right]$ and $y \in \left[Ces_p^q\left(\Lambda_f^2\right)\right]^\beta$. Then we can find a positive integer N such that

$$\left(|y_{mn}|^{1/m+n}\right)$$
$$< \max\left(1, \sup_{m,n\geq 1}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(|y_{mn}|^{\frac{1}{m+n}}\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right) < N$$

for all m, n.

Hence we may write

$$\left|\sum_{m,n} x_{mn}y_{mn}\right| \leq \sum_{m,n}|x_{mn}y_{mn}| \leq \sum_{mn}\left(f\left(|x_{mn}y_{mn}|\right)\right) \leq \sum_{m,n}\left(f\left(|x_{mn}|\, N^{m+n}\right)\right).$$

Since $x \in Ces_p^q\left(\eta_f^2\right)$. The series on the right side of the above inequality is convergent, whence $x \in \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$. Hence $\left[\mathrm{Ces}_p^q\left(\eta_f^2\right)\right] \subset \left[Ces_p^q\left(\Lambda_f^2\right)\right]^\beta$.

Now we show that $\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^\beta \subset \left[\mathrm{Ces}_p^q\left(\eta_f^2\right)\right]$.

For this, let $x \in \left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^\beta$ and suppose that $x \notin \left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]$. Then there exists a positive integer $N > 1$ such that $\sum_{m,n}\left(f\left(|x_{mn}|\, N^{m+n}\right)\right) = \infty$.

If we define $y_{mn} = N^{m+n}\mathrm{Sgn}x_{mn}\quad m, n = 1, 2, \cdots$, then $y \in \left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]$. But, since

$$\left|\sum_{m,n} x_{mn}y_{mn}\right| = \sum_{mn}\left(f\left(|x_{mn}y_{mn}|\right)\right) = \sum_{m,n}\left(f\left(|x_{mn}|\, N^{m+n}\right)\right) = \infty,$$

we get $x \notin \left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^\beta$, which contradicts the assumption $x \in \left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^\beta$. Therefore $x \in \left[\mathrm{Ces}_p^q\left(\eta_f^2\right)\right]$ and $\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^\beta = \left[\mathrm{Ces}_p^q\left(\eta_f^2\right)\right]$.

(ii)and (iii) can be shown in a similar way of (i). Therefore, we omit it. □

Proposition 3.3. *Let $p = (p_{mn})$ be a Cesàro space of double analytic modulus sequence of strictly positive real numbers p_{mn}. Then*

(i) $\mathrm{Ces}_p^q\left(\Lambda_f^2\right)$ *is a paranormed space with*

$$g(x) = \sup\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(|x_{mn}|^{\frac{1}{m+n}}\right)\right)^{\frac{p_{mn}}{M}}\right)^{\frac{1}{p_{mn}}} \tag{3.1}$$

if and only if $h = \inf p_{mn} > 0$, where $M = \max(1, H)$ and $H = \sup p_{mn}$.

(ii) $\mathrm{Ces}_p^q\left(\Lambda_f^2\right)$ *is a complete paranormed linear metric space if the condition p in* (3.1) *is satisfied.*

Proof. The proof of (i). *Sufficiency.* Let $h > 0$. It is trivial that $g(\theta) = 0$ and $g(-x) = g(x)$.

The inequality $g(x+y) \leq g(x) + g(y)$ follows from the inequality (3.1), since $p_{mn}/M \leq 1$ for all positive integers m, n. We also may write $g(\lambda x) \leq \max\left(|\lambda|, |\lambda|^{h/M}\right) g(x)$, since $|\lambda|^{p_{mn}} \leq \max\left(|\lambda|^h, |\lambda|^M\right)$ for all positive integers m, n and for any $\lambda \in C$, the set of complex numbers. Using this inequality, it can be proved that $\lambda x \to \theta$, when x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

Necessity. Let $\mathrm{Ces}_p^q\left(\Lambda_f^2\right)$ be a paranormed space with the paranorm

$$g(x) = \sup\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(|x_{mn}|^{\frac{1}{m+n}}\right)\right)^{\frac{p_{mn}}{M}}\right)^{\frac{1}{p_{mn}}}$$

and suppose that $h = 0$. Since $|\lambda|^{p_{mn}/M} \leq |\lambda|^{h/M} = 1$ for all positive integers m, n and $\lambda \in \mathbb{C}$ such that $0 < |\lambda| \leq 1$, we have

$$\sup\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f(|\lambda|)\right)^{\frac{p_{mn}}{M}}\right)^{\frac{1}{p_{mn}}} = 1.$$

Hence it follows that

$$g(\lambda x) = \sup\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f(|\lambda|)\right)^{\frac{p_{mn}}{M}}\right)^{\frac{1}{p_{mn}}} = 1$$

for $x = (\alpha) \in \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$ as $\lambda \to 0$. But this contradicts the assumption $\mathrm{Ces}_p^q\left(\Lambda_f^2\right)$ is a paranormed space with $g(x)$.

The proof of (ii) is clear. □

Corollary 3.4. $\mathrm{Ces}_p^q\left(\Lambda_f^2\right)$ *is a complete paranormed space with the natural paranorm if and only if* $\mathrm{Ces}_p^q\left(\Lambda_f^2\right) = \mathrm{Ces}^q\left(\Lambda_f^2\right)$.

Proposition 3.5. *For every* $p = (p_{mn})$, $\mathrm{Ces}_p^q\left(\eta_f^2\right) \subset \left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^\beta \subsetneqq \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$.

Proof. The proof of (i). First, we show that $\mathrm{Ces}_p^q\left(\eta_f^2\right) \subset \left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^\beta$. We know that $\left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right] \subset \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$.
$\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^\beta \subset \left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^\beta$. But $\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^\beta = \mathrm{Ces}_p^q\left(\eta_f^2\right)$ by Proposition 3.2. Therefore,

$$\mathrm{Ces}_p^q\left(\eta_f^2\right) \subset \left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^\beta. \tag{3.2}$$

The proof of (ii). Now we show that $\left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^\beta \subsetneqq \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$.

Let $y = (y_{mn})$ be an arbitrary point $\left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^\beta$. If y is not $\mathrm{Ces}_p^q\left(\Lambda_f^2\right)$, then for each natural number q, we can find an index $m_p n_q$ such that

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{m_q n_q} f\left(\left((m_q+n_q)!\left|y_{m_q n_q}\right|\right)^{\frac{1}{m_q+n_q}}\right)\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}} > q$$

for $(1,2,3,\cdots)$. Define $x = \{x_{mn}\}$ by

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)!\left|x_{mn}\right|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}} >\right)\right) q^{\frac{1}{m+n}}$$

for $(mn) = (m_q n_q)$ and some $q \in \mathbb{N}$; and

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)!\left|x_{mn}\right|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right) = 0, \quad \text{otherwise.}$$

Then x is $\mathrm{Ces}_p^q\left(\chi_f^2\right)$, but for infinitely mn,

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)!\left|y_{mn}x_{mn}\right|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right) > 1. \tag{3.3}$$

Consider the sequence $z = \{z_{mn}\}$, where

$$Q_{11}\left(q_{11} f\left(2! z_{11}\right)^{p_{mn}}\right)^{p_{mn}} = Q_{11}\left(q_{11} f\left(2! x_{11}\right)^{p_{mn}}\right)^{p_{mn}} - s$$

with

$$s = \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)!\left|x_{mn}\right|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right);$$

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)!\left|z_{mn}\right|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right)$$
$$= \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)!\left|x_{mn}\right|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right).$$

The z is a point of $\mathrm{Ces}_p^q\left(\chi_f^2\right)$. Also

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)!\left|z_{mn}\right|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right) = 0.$$

Hence z is in $\mathrm{Ces}_p^q\left(\chi_f^2\right)$. But, by the equation (3.3),

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn} f\left(\left((m+n)!\left|z_{mn}y_{mn}\right|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right)$$

does not converge and so $\sum\sum x_{mn}y_{mn}$ diverges. Thus, the sequence y would not be $\left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^{\beta}$. This contradiction proves that

$$\left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^{\beta} \subset \mathrm{Ces}_p^q\left(\Lambda_f^2\right). \tag{3.4}$$

If we now choose $f = \mathrm{id}$, where id is the identity and

$$\frac{1}{Q_{1j}}\left(q_{1n}\left((m+n)!y_{1n}\right)\right) = \frac{1}{Q_{1j}}\left(q_{1n}\left((m+n)!x_{1n}\right)\right)$$

and

$$\frac{1}{Q_{ij}}\left(q_{mn}\left((m+n)!y_{mn}\right)\right) = \frac{1}{Q_{ij}}\left(q_{mn}\left((m+n)!x_{mn}\right)\right) = 0, \qquad (m, i > 1)$$

for all n, j, then obviously $x \in \mathrm{Ces}_p^q\left(\chi_f^2\right)$ and $y \in \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$, but

$$\sum\sum x_{mn}y_{mn} = \infty. \tag{3.5}$$

Hence $y \notin \left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^{\beta}$.

From (3.4) and (3.5), we are granted $\left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^{\beta} \subsetneqq \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$. □

Proposition 3.6. *In* $\mathrm{Ces}_p^q\left(\chi_f^2\right)$ *weak convergence does not imply strong convergence.*

Proof. Assume that weak convergence implies strong convergence $\mathrm{Ces}_p^q\left(\chi_f^2\right)$. Then, we would have $\left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^{\beta\beta} = \mathrm{Ces}_p^q\left(\chi_f^2\right)$ [see Wilansky]. But

$$\left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^{\beta\beta} \subsetneqq \left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^{\beta} = \mathrm{Ces}_p^q\left(\eta_f^2\right).$$

Thus $\left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^{\beta\beta} \neq \mathrm{Ces}_p^q\left(\chi_f^2\right)$. Hence weak convergence does not imply strong convergence in $\mathrm{Ces}_p^q\left(\chi_f^2\right)$. □

Proposition 3.7. *Let f be an modulus function which satisfies the Δ_2-condition. Then* $\mathrm{Ces}_p^q\left(\chi^2\right) \subset \mathrm{Ces}_p^q\left(\chi_f^2\right)$.

Proof. Let

$$x \in \mathrm{Ces}_p^q\left(\chi^2\right). \tag{3.6}$$

Then

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left(\left((m+n)!\,|x_{mn}|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right)\leq\varepsilon$$

for sufficiently large m, n and every $\varepsilon > 0$.

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(\left((m+n)!\,|x_{mn}|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right)\leq f(\varepsilon)$$

(because f is non-decreasing). This implies

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(\left((m+n)!\,|x_{mn}|\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right)\right)\leq Kf(\varepsilon)<(\varepsilon)$$

(by the Δ_2-condition, for some $K > 0$ and by defining $f(\varepsilon) < \frac{\varepsilon}{K}$).

$$\lim_{m,n\to\infty}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}f\left(\left((m+n)!\,|x_{mn}|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}=0. \tag{3.7}$$

Hence

$$x\in \mathrm{Ces}_p^q\left(\chi_f^2\right). \tag{3.8}$$

From (3.6) and (3.8), we get $\mathrm{Ces}_p^q\left(\chi^2\right)\subset \mathrm{Ces}_p^q\left(\chi_f^2\right)$. □

Proposition 3.8. $\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^{\beta}\subsetneqq \mathrm{Ces}_p^q\left(\chi_f^2\right)$.

Proof. Let $(x_{mn})\in\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^{\beta}$

$$\sum\sum|x_{mn}y_{mn}|<\infty \tag{3.9}$$

for all $(y_{mn})\in\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^{\beta}$. Assume that $(x_{mn})\notin \mathrm{Ces}_p^q\left(\chi_f^2\right)$. Then there exists a sequence of positive integers

$$f\left(|x_{m_r+n_r}|\right)>\frac{1}{\left((m_r+n_r!2)^{(m_r+n_r)}\right)},\qquad (r=1,2,3,\cdots).$$

Take

$$y_{m_r+n_r}=\begin{cases}\left(2\left(m_r+n_r\right)!\right)^{m_r+n_r} & \text{for } r=1,2,3,\cdots,\\ y_{m_r+n_r}=0 & \text{otherwise.}\end{cases}$$

Then $(y_{mn})\in\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]$. But

$$\sum\sum|x_{mn}y_{mn}|=\sum_{r=1}^{\infty}|x_{m_r+n_r}y_{m_r+n_r}|=f\left(\sum_{r=1}^{\infty}|x_{m_r+n_r}y_{m_r+n_r}|\right)$$
$$>1+1+1+\cdots.$$

We know that the infinite series $1+1+1+\ldots$ diverges. Hence $\sum\sum |x_{mn}y_{mn}|$ diverges. This contradicts (3.9). Hence $(x_{mn}) \in \mathrm{Ces}_p^q\left(\chi_f^2\right)$. Therefore,

$$\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^\beta \subset \mathrm{Ces}_p^q\left(\chi_f^2\right). \tag{3.10}$$

If we now choose $p=(p_{mn})$, it is a constant $f=\mathrm{id}$, where id is the identity and

$$\frac{1}{Q_{1j}}\left(q_{1n}\left((1+n)!y_{1n}\right)\right)=\frac{1}{Q_{1j}}\left(q_{1n}\left((1+n)!x_{1n}\right)\right) \qquad \text{and}$$
$$\frac{1}{Q_{ij}}\left(q_{mn}\left((m+n)!y_{mn}\right)\right)=\frac{1}{Q_{ij}}\left(q_{mn}\left((m+n)!x_{mn}\right)\right)=0$$

where $(m,i>1)$ for all n,j, then obviously $x \in \mathrm{Ces}_p^q\left(\chi_f^2\right)$ and $y \in \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$, but

$$\sum\sum x_{mn}y_{mn}=\infty. \tag{3.11}$$

Hence $y \notin \left[\mathrm{Ces}_p^q\left(\chi_f^2\right)\right]^\beta$.

From (3.10) and (3.1), we are granted $\left[\mathrm{Ces}_p^q\left(\Lambda_f^2\right)\right]^\beta \subsetneqq \mathrm{Ces}_p^q\left(\chi_f^2\right)$. $\square$

Proposition 3.9. *Let* $\left(\mathrm{Ces}_p^q\left(\chi_f^2\right)\right)^*$ *denote the dual space of* $\mathrm{Ces}_p^q\left(\chi_f^2\right)$. *Then we have* $\left(\mathrm{Ces}_p^q\left(\chi_f^2\right)\right)^*=\mathrm{Ces}_p^q\left(\Lambda_f^2\right)$.

Proof. We recall that

$$x=\Im_{mn}=\begin{pmatrix} 0, & 0, & \ldots & 0, & 0, & \ldots & 0 \\ 0, & 0, & \ldots & 0, & 0, & \ldots & 0 \\ \vdots & & & & & & \\ 0, & 0, & \ldots & \frac{1}{(m+n)!}, & 0, & \ldots & 0 \\ 0, & 0, & \ldots & 0, & 0, & \ldots & 0 \end{pmatrix}$$

with $\frac{1}{(m+n)!}$ in the $(m,n)^{th}$ position and zero otherwise, with

$$x=\Im_{mn}\left\{\left(\sum_{i=1}^\infty\sum_{j=1}^\infty\left(\frac{1}{Q_{ij}}\sum_{m=1}^i\sum_{n=1}^j q_{mn}f\left(\left((m+n)!\,|x_{mn}|\right)^{\frac{1}{m+n}}\right)\right)^{p_{mn}}\right)^{\frac{1}{p_{mn}}}\right\}$$
$$=\begin{pmatrix} 0, & 0, & \ldots & 0, & 0, & \ldots & 0 \\ \vdots & & & & & & \\ 0, & 0, & \ldots & \left(\frac{(m+n)!}{(m+n)!}\right), & 0, & \ldots & 0 \\ 0, & 0, & \ldots & 0, & 0, & \ldots & 0 \end{pmatrix}=\begin{pmatrix} 0, & 0, & \ldots & 0, & 0, & \ldots & 0 \\ 0, & 0, & \ldots & 0, & 0, & \ldots & 0 \\ \vdots & & & & & & \\ 0, & 0, & \ldots & 1^{1/m+n}, & 0, & \ldots & 0 \\ 0, & 0, & \ldots & 0, & 0, & \ldots & 0 \end{pmatrix}$$

which is a $\mathrm{Ces}_p^q\left(\chi_f^2\right)$ sequence. Hence $\Im_{mn} \in \mathrm{Ces}_p^q\left(\chi_f^2\right)$. Let us take $f(x)=\sum_{m=1}^\infty\sum_{n=1}^\infty x_{mn}y_{mn}$ with $x \in \mathrm{Ces}_p^q\left(\chi_f^2\right)$ and $f \in \left(\mathrm{Ces}_p^q\left(\chi_f^2\right)\right)^*$. Take $x=$

$(x_{mn}) = \Im_{mn} \in \mathrm{Ces}_p^q\left(\chi_f^2\right)$. Then

$$|y_{mn}| \leq \|f\|\, d\left(\Im_{mn}, 0\right) < \infty \qquad \text{for each } m, n.$$

Thus (y_{mn}) is a bounded sequence and hence an Cesàro double analytic sequence of modulus. In other words $y \in \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$. Therefore $\left(\mathrm{Ces}_p^q\left(\chi_f^2\right)\right)^* = \mathrm{Ces}_p^q\left(\Lambda_f^2\right)$. □

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