Acta Math. Univ. Comenianae
Vol. LXXXI, 1 (2012), pp. 127–139

SUMMATION FORMULAE FOR THE LEGENDRE POLYNOMIALS

SUBUHI KHAN and A. A. AL-GONAH

Abstract. In this paper, summation formulae for the 2-variable Legendre polynomials in terms of certain multi-variable special polynomials are derived. Several summation formulae for the classical Legendre polynomials are also obtained as applications. Further, Hermite-Legendre polynomials are introduced and summation formulae for these polynomials are also established.

1. Introduction and preliminaries

We recall that the 2-variable Legendre polynomials (2VLeP) $R_n(x,y)$ are defined by the series [**13**]

$$R_n(x,y) = (n!)^2 \sum_{k=0}^{n} \frac{(-1)^{n-k} y^k x^{n-k}}{(k!)^2[(n-k)!]^2} \tag{1.1}$$

and specified by the following generating function

$$C_0(-yt)\ C_0(xt) = \sum_{n=0}^{\infty} R_n(x,y) \frac{t^n}{(n!)^2}\ , \tag{1.2}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as [**20**]

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}. \tag{1.3}$$

The 2VLeP $R_n(x,y)$ are linked to the classical Legendre polynomials $P_n(x)$ [**1**] by the following relation

$$R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right) = P_n(x). \tag{1.4}$$

Received September 29, 2011.
2010 *Mathematics Subject Classification.* Primary 33B10, 33C45, 33C47.
Key words and phrases. Legendre polynomials; Generating functions; Summation formulae; Hermite-Legendre polynomials.

Further, we recall a second form of the 2-variable Legendre polynomials (2VLeP) $S_n(x,y)$ which are defined by the series [**4**, p. 158] (see also [**9**])

$$S_n(x,y) = n!\sum_{k=0}^{n} \frac{(-1)^k x^k y^{n-2k}}{(k!)^2(n-2k)!} \tag{1.5}$$

and specified by the following generating function

$$\exp(yt)\ C_0(xt^2) = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!}. \tag{1.6}$$

Next, we recall that the higher-order Hermite polynomials, some times called the Kampé de Fériet or the Gould-Hopper polynomials (GHP) $H_n^{(m)}(x,y)$, are defined as [**18**, p. 58, (6.2)] (see also [**3**])

$$g_n^m(x,y) = H_n^{(m)}(x,y) = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{y^k x^{n-mk}}{k!(n-mk)!}, \tag{1.7}$$

where m is a positive integer. These polynomials are specified by the generating function

$$\exp(xt + yt^m) = \sum_{n=0}^{\infty} H_n^{(m)}(x,y)\frac{t^n}{n!}. \tag{1.8}$$

In particular, we note that

$$H_n^{(1)}(x,y) = (x+y)^n, \tag{1.9}$$

$$H_n^{(2)}(x,y) = H_n(x,y), \tag{1.10}$$

where $H_n(x,y)$ denotes the 2-variable Hermite-Kampé de Fériet polynomials (2VHKdFP) [**2**], defined by the generating function

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.11}$$

We note the following link between the 2VHKdFP $H_n(x,y)$ and the 2VLeP $S_n(x,y)$ [**9**, p. 613]

$$H_n(y, -D_x^{-1}) = S_n(x,y), \tag{1.12}$$

where D_x^{-1} denotes the inverse of the derivative operator $D_x := \frac{\partial}{\partial x}$ and is defined in such a way that

$$D_x^{-n}\{f(x)\} = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1} f(\xi)d\xi, \tag{1.13}$$

so that for $f(x) = 1$, we have

$$D_x^{-n}\{1\} = \frac{x^n}{n!}. \tag{1.14}$$

In view of equations (1.8) and (1.11), we note the following link

$$H_n^{(2)}\left(x, -\frac{1}{2}\right) = H_n\left(x, -\frac{1}{2}\right) = He_n(x), \tag{1.15}$$

where $He_n(x)$ denotes the classical Hermite polynomials [**1**].

Also, we recall that the 2-variable generalized Laguerre polynomials (2VgLP) ${}_mL_n(x,y)$ are defined by the series [**12**, p. 213]

$$ {}_mL_n(x,y) = n!\sum_{r=0}^{[\frac{n}{m}]} \frac{x^r y^{n-mr}}{(r!)^2(n-mr)!} \tag{1.16} $$

and by the following generating function

$$ \exp(yt)\ C_0(-xt^m) = \sum_{n=0}^{\infty} {}_mL_n(x,y)\frac{t^n}{n!}. \tag{1.17} $$

We note the following link between the 2VgLP ${}_mL_n(x,y)$ and the GHP $H_n^{(m)}(x,y)$ [**12**, p. 213]

$$ {}_mL_n(x,y) = H_n^{(m)}(y, D_x^{-1}). \tag{1.18} $$

In particular, we note that

$$ {}_2L_n(-x,y) = S_n(x,y), \tag{1.19} $$

$$ {}_1L_n(-x,y) = L_n(x,y), \tag{1.20} $$

where $L_n(x,y)$ denotes the 2-variable Laguerre polynomials (2VLP) [**14**] (see also [**16**]), defined by means of the generating function

$$ \exp(yt)\ C_0(xt) = \sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!}. \tag{1.21} $$

In terms of classical Laguerre polynomials $L_n(x)$ [**1**], it is easily seen from definition (1.16) and relation (1.20) that

$$ {}_1L_n(-x,1) = L_n(x,1) = L_n(x). \tag{1.22} $$

Again, we recall that the 2-variable generalized Laguerre type polynomials (2VgLtP) ${}_{[m]}L_n(x,y)$ are defined by the series [**7**, p. 603]

$$ {}_{[m]}L_n(x,y) = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{y^k(-x)^{n-mk}}{k![(n-mk)!]^2} \tag{1.23} $$

and by the following generating function

$$ \exp(yt^m)\ C_0(xt) = \sum_{n=0}^{\infty} {}_{[m]}L_n(x,y)\frac{t^n}{n!}. \tag{1.24} $$

For $m = 2$ and $x \to -x$, the polynomials ${}_{[m]}L_n(x,y)$ reduce to the 2-variable Hermite type polynomials (2VHtP) $G_n(x,y)$ [**9**], i.e., we have

$$ {}_{[2]}L_n(-x,y) = G_n(x,y). \tag{1.25} $$

In view of equations (1.21) and (1.24), we note the following link

$$ {}_{[1]}L_n(x,y) = L_n(x,y). \tag{1.26} $$

Further, we recall that the 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x,y,z)$ are defined by the series [**15**, p. 241]

$$(1.27)\qquad {}_LH_n(x,y,z) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{z^k L_{n-2k}(x,y)}{k!(n-2k)!}$$

and specified by the following generating function

$$(1.28)\qquad \exp(yt+zt^2)\ C_0(xt) = \sum_{n=0}^{\infty} {}_LH_n(x,y,z)\frac{t^n}{n!}.$$

In particular, we note that

$$(1.29)\qquad {}_LH_n\left(x,y,-\frac{1}{2}\right) = {}_LH_n^*(x,y),$$

$$(1.30)\qquad {}_LH_n(x,1,-1) = {}_LH_n(x),$$

where ${}_LH_n^*(x,y)$ denotes the 2-variable Laguerre-Hermite polynomials (2VLHP) [**16**] and ${}_LH_n(x)$ denotes the Laguerre-Hermite polynomials (LHP) [**17**], respectively.

Furthermore, we recall that the 3-variable Hermite-Laguerre polynomials (3VHLP) ${}_HL_n(x,y,z)$ are defined by the series [**11**, p. 58]

$$(1.31)\qquad {}_HL_n(x,y,z) = n!\sum_{k=0}^{n} \frac{(-1)^k z^{n-k} H_k(x,y)}{(k!)^2(n-k)!}$$

and by the following generating function

$$(1.32)\qquad \exp(zt)\ {}_HC_0(x,y;t) = \sum_{n=0}^{\infty} {}_HL_n(x,y,z)\frac{t^n}{n!},$$

where ${}_HC_0(x,y;t)$ denotes the Hermite-Tricomi functions defined by the following operational definition [**11**, p. 58]

$$(1.33)\qquad {}_HC_0(x,y;t) = \exp\left(y\frac{\partial^2}{\partial x^2}\right)\Big\{C_0(xt)\Big\}.$$

The special polynomials of more than one variable provide new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems. It happens very often that the solution of a given problem in physics or applied mathematics requires the evaluation of infinite sums, involving special functions. Problems of this type arise, for example, in the computation of the higher-order moments of a distribution or in evaluation of transition matrix elements in quantum mechanics. In [**5**], Dattoli showed that the summation formulae of special functions, often encountered in applications ranging from electromagnetic processes to combinatorics, can be written in terms of Hermite polynomials of more than one variable.

In this paper, we derive the explicit summation formulae for the 2VLeP $R_n(x,y)$ in terms of the product of certain multi-variable special polynomials. Also, we derive the implicit summation formula for the 2VLeP $S_n(x,y)$. Summation formulae for the classical Legendre polynomials $P_n(x)$ are obtained as special cases of the summation formulae for the 2VLeP $R_n(x,y)$. Further, the Hermite-Legendre polynomials ${}_HR_n(x,y,z)$ are introduced and summation formulae for these polynomials are also obtained.

2. Summation formulae for the 2-variable Legendre polynomials

First, we prove the following explicit summation formula for the 2VLeP $R_n(x,y)$ by using generating functions (1.8), (1.21) and (1.24) of GHP $H_n^{(m)}(x,y)$, 2VLP $L_n(x,y)$ and 2VgLtP ${}_{[m]}L_n(x,y)$, respectively.

Theorem 2.1. *The following explicit summation formula for the 2VLeP $R_n(x,y)$ in terms of the product of GHP $H_n^{(m)}(x,y)$, 2VgLtP ${}_{[m]}L_n(x,y)$ and 2VLP $L_n(x,y)$ holds true*

$$\begin{aligned} &R_n(x,z) \\ &= n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k^{(m)}(-y,-w)\ {}_{[m]}L_r(-z,w)\ L_{n-k-r}(x,y). \end{aligned} \tag{2.1}$$

Proof. Consider the product of 2VLP generating function (1.21) and 2VgLtP generating function (1.24) in the following form

$$\begin{aligned} &\exp(yt)\ C_0(xt)\ \exp(wt^m)\ C_0(zt) \\ &\qquad = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} L_n(x,y)\ {}_{[m]}L_r(z,w)\ \frac{t^{n+r}}{n!r!}. \end{aligned} \tag{2.2}$$

Replacing n by $n-r$ in the r.h.s. of equation (2.2) and then using the lemma [**20**, p. 100]

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A(r,n) = \sum_{n=0}^{\infty}\sum_{r=0}^{n} A(r,n-r), \tag{2.3}$$

we find

$$\begin{aligned} &C_0(xt)\ C_0(zt)\ \exp(yt+wt^m) \\ &\qquad = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}{}_{[m]}L_r(z,w)\ L_{n-r}(x,y)\ \frac{t^n}{n!}, \end{aligned} \tag{2.4}$$

which on shifting the exponential to the r.h.s. and then using the generating function (1.8) in the r.h.s. becomes

$$\begin{aligned} &C_0(xt)\ C_0(zt) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}H_k^{(m)}(-y,-w)\ {}_{[m]}L_r(z,w)\ L_{n-r}(x,y)\ \frac{t^{n+k}}{n!k!}. \end{aligned} \tag{2.5}$$

Again, replacing n by $n-k$ in the r.h.s. of equation (2.5), we get

$$\begin{aligned} & C_0(zt)\ C_0(xt) \\ & = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} H_k^{(m)}(-y,-w)\ {}_{[m]}L_r(z,w)\ L_{n-k-r}(x,y)\ \frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

Finally, using generating function (1.2) in the l.h.s. of equation (2.6) and then equating the coefficients of like powers of t in the resultant equation, we get assertion (2.1) of Theorem 2.1. □

Remark 2.1. Taking $m = 2$ in assertion (2.1) of Theorem 2.1 and using relations (1.10) and (1.25), we deduce the following consequence of Theorem 2.1.

Corollary 2.1. *The following summation formula for the 2VLeP $R_n(x,y)$ involving product of 2VHKdFP $H_n(x,y)$, 2VHtP $G_n(x,y)$ and 2VLP $L_n(x,y)$ holds true*

$$\begin{aligned} & R_n(x,z) \\ & = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} H_k(-y,-w)\ G_r(z,w)\ L_{n-k-r}(x,y). \end{aligned} \tag{2.7}$$

Note. For $y = 1$, equation (2.7) yields to the following summation formula

$$\begin{aligned} & R_n(x,z) \\ & = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} H_k(-1,-w)\ G_r(z,w)\ L_{n-k-r}(x). \end{aligned} \tag{2.8}$$

Again, for $w = \frac{1}{2}$, equation (2.7) yields to the following summation formula

$$\begin{aligned} & R_n(x,z) \\ & = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} He_k(-y)\ G_r\left(z,\frac{1}{2}\right)\ L_{n-k-r}(x,y). \end{aligned} \tag{2.9}$$

Remark 2.2. Taking $m = 1$ in assertion (2.1) of Theorem 2.1 and using relations (1.9) and (1.26), we deduce the following consequence of Theorem 2.1.

Corollary 2.2. *The following explicit summation formula for the 2VLeP $R_n(x,y)$ in terms of product of the 2VLP $L_n(x,y)$ holds true*

$$\begin{aligned} & R_n(x,z) \\ & = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} (-1)^k(y+w)^k\ L_r(-z,w)\ L_{n-k-r}(x,y). \end{aligned} \tag{2.10}$$

Note. For $w = -y$, equation (2.10) yields to the following summation formula

$$R_n(x,z) = n!\sum_{r=0}^{n}\binom{n}{r}\ L_r(-z,-y)\ L_{n-r}(x,y), \tag{2.11}$$

which for $y = 1$ gives the following summation formula

$$R_n(x,z) = n!\sum_{r=0}^{n}\binom{n}{r}\ L_r(-z,-1)\ L_{n-r}(x). \tag{2.12}$$

Again, for $y = w = 1$, equation (2.10) yields to the following summation formula:

$$R_n(x,z) = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-2)^k\ L_r(-z)\ L_{n-k-r}(x). \tag{2.13}$$

Remark 2.3. Using generating functions (1.11), (1.21) and (1.28) of 2VHKdFP $H_n(x,y)$, 2VLP $L_n(x,y)$ and 3VLHP ${}_LH_n(x,y,z)$ respectively and proceeding on the same lines of proof of Theorem 2.1, we get the following result.

Theorem 2.2. *The following explicit summation formula for the 2VLeP $R_n(x,y)$ in terms of the product of 2VHKdFP $H_n(x,y)$, 3VLHP ${}_LH_n(x,y,z)$ and 2VLP $L_n(x,y)$ holds true*

$$\begin{aligned} &R_n(x,z)\\ &= n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k(-y-w,-v)\ {}_LH_r(-z,w,v)\ L_{n-k-r}(x,y). \end{aligned} \tag{2.14}$$

Note. For $y = 1$ and $v = -\frac{1}{2}$, equation (2.14) yields to the following summation formula

$$\begin{aligned} &R_n(x,z)\\ &= n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k\left(-1-w,\frac{1}{2}\right)\ {}_LH_r^*(-z,w)\ L_{n-k-r}(x). \end{aligned} \tag{2.15}$$

Again, for $y = 1$ and $v = \frac{1}{2}$, equation (2.14) yields to the following summation formula

$$\begin{aligned} &R_n(x,z)\\ &= n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}He_k(-1-w)\ {}_LH_r\left(-z,w,\frac{1}{2}\right)\ L_{n-k-r}(x). \end{aligned} \tag{2.16}$$

Further, for $w = 1$ and $v = -1$, equation (2.14) yields to the following summation formula

$$\begin{aligned} &R_n(x,z)\\ &= n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k(-y-1,1)\ {}_LH_r(-z)\ L_{n-k-r}(x,y). \end{aligned} \tag{2.17}$$

Next, we prove the following result involving the 2VLeP $S_n(x,y)$.

Theorem 2.3. *The following implicit summation formula for the 2VLeP $S_n(x,y)$ holds true*

$$S_{k+l}(y,w)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}S_{k+l-n-r}(y,x). \tag{2.18}$$

Proof. We start by a recently derived summation formula for the 2VHKdFP $H_n(x,y)$ [**19**, p. 1539]

$$H_{k+l}(w,y)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}H_{k+l-n-r}(x,y). \tag{2.19}$$

Replacing y by $-D_y^{-1}$ in the above equation, we have

$$H_{k+l}(w,-D_y^{-1})=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}H_{k+l-n-r}(x,-D_y^{-1}), \tag{2.20}$$

which using relation (1.12) gives assertion (2.18) of Theorem 2.3. □

Alternate proof. Replacing y by D_y^{-1} in the following result [**19**, p. 1538]

$$H^{(m)}_{k+l}(w,y)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}H^{(m)}_{k+l-n-r}(x,y) \tag{2.21}$$

and then using link (1.18), we get the following summation formula for 2VgLP ${}_mL_n(x,y)$

$${}_mL_{k+l}(y,w)=\sum_{n,r=0}^{k,l}\binom{k}{n}\binom{l}{r}(w-x)^{n+r}{}_mL_{k+l-n-r}(y,x). \tag{2.22}$$

Now, taking $m=2$ and replacing y by $-y$ in equation (2.22) and using relation (1.19), we get assertion (2.18) of Theorem 2.3. □

Remark 2.4. Taking $l=0$ in assertion (2.18) of Theorem 2.3, we deduce the following consequence of Theorem 2.3

Corollary 2.3. *The following implicit summation formula for the 2VLeP $S_n(x,y)$ holds true*

$$S_k(y,w)=\sum_{n=0}^{k}\binom{k}{n}(w-x)^n\ S_{k-n}(y,x). \tag{2.23}$$

3. Applications

In this section, we derive the summation formulae for the classical Legendre polynomials $P_n(x)$ as applications of the results derived in the previous section.

I. Replacing x by $\frac{1-x}{2}$ and z by $\frac{1+x}{2}$ in equations (2.1), (2.7), (2.10) and (2.11) and using relation (1.4), we get the following explicit summation formulae for the classical Legendre polynomials $P_n(x)$

(3.1)

$$\begin{aligned}P_n(x)\\ &= n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k^{(m)}(-y,-w)\,{}_{[m]}L_r\left(\frac{-1-x}{2},w\right)L_{n-k-r}\left(\frac{1-x}{2},y\right),\end{aligned}$$

(3.2)

$$P_n(x) = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k(-y,-w)G_r\left(\frac{1+x}{2},w\right)L_{n-k-r}\left(\frac{1-x}{2},y\right),$$

(3.3)

$$P_n(x) = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k(y+w)^k L_r\left(\frac{-1-x}{2},w\right)L_{n-k-r}\left(\frac{1-x}{2},y\right),$$

(3.4)

$$P_n(x) = n!\sum_{r=0}^{n}\binom{n}{r}\;L_r\left(\frac{-1-x}{2},-y\right)\;L_{n-r}\left(\frac{1-x}{2},y\right).$$

Next, taking $y=1$ in equations (3.2) and (3.4) and using relation (1.22), we get

(3.5)

$$P_n(x) = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k(-1,-w)\;G_r\left(\frac{1+x}{2},w\right)\;L_{n-k-r}\left(\frac{1-x}{2}\right),$$

(3.6)

$$P_n(x) = n!\sum_{r=0}^{n}\binom{n}{r}\;L_r\left(\frac{-1-x}{2},-1\right)\;L_{n-r}\left(\frac{1-x}{2}\right).$$

Further, taking $y=w=1$ in equation (3.3) and using relation (1.22), we get

$$P_n(x) = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-2)^k\;L_r\left(\frac{-1-x}{2}\right)\;L_{n-k-r}\left(\frac{1-x}{2}\right). \tag{3.7}$$

Furthermore, taking $w=\frac{1}{2}$ in equation (3.2) and using relation (1.15), we get

(3.8)

$$P_n(x) = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}He_k(-y)\;G_r\left(\frac{1+x}{2},\frac{1}{2}\right)\;L_{n-k-r}\left(\frac{1-x}{2},y\right).$$

II. Replacing x by $\frac{1-x}{2}$ and z by $\frac{1+x}{2}$ in equation (2.14) and using relation (1.4), we get the following explicit summation formulae for the classical Legendre polynomials $P_n(x)$

$$\begin{aligned} P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} \\ \times\ H_k(-y-w,-v)\ {}_LH_r\left(\frac{-1-x}{2}, w, v\right)\ L_{n-k-r}\left(\frac{1-x}{2}, y\right). \end{aligned} \tag{3.9}$$

Next, taking $y = 1$ and $v = -\frac{1}{2}$ in equation (3.9) and using relations (1.22) and (1.29), we get

$$\begin{aligned} P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} \\ \times H_k\left(-1-w, \frac{1}{2}\right)\ {}_LH_r^*\left(\frac{-1-x}{2}, w\right)\ L_{n-k-r}\left(\frac{1-x}{2}\right). \end{aligned} \tag{3.10}$$

Further, taking $y = 1$ and $v = \frac{1}{2}$ in equation (3.9) and using relations (1.15) and (1.22), we get

$$P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} He_k(-1-w)\ {}_LH_r\left(\frac{-1-x}{2}, w, \frac{1}{2}\right) L_{n-k-r}\left(\frac{1-x}{2}\right). \tag{3.11}$$

Furthermore, taking $w = 1$ and $v = -1$ in equation (3.9) and using relation (1.30), we get

$$P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k(-y-1,1)\ {}_LH_r\left(\frac{-1-x}{2}\right) L_{n-k-r}\left(\frac{1-x}{2}, y\right). \tag{3.12}$$

4. Concluding remarks

Operational methods can be used to simplify the derivation of the properties associated with ordinary and generalized special functions and to define new families of functions. We recall that the 2VHKdFP $H_n(x,y)$ have the following operational definition

$$H_n(x,y) = \exp\left(y\frac{\partial^2}{\partial x^2}\right)\Big\{x^n\Big\}. \tag{4.1}$$

Now, in view of definition (4.1) the 3VHLP ${}_HL_n(x,y,z)$ are specified by the following operational definition [**11**, p. 58]

$${}_HL_n(x,y,z) = \exp\left(y\frac{\partial^2}{\partial x^2}\right)\Big\{L_n(x,z)\Big\}. \tag{4.2}$$

In order to introduce Hermite-Legendre polynomials (HLeP) ${}_HR_n(x,y,z)$, we replace y by z in generating function (1.2) and then operate $\exp\left(y\frac{\partial^2}{\partial x^2}\right)$ on both sides of the resultant equation. Now, using operational definition (1.33) in the l.h.s. of the resultant equation, we get the following generating function of the HLeP ${}_HR_n(x,y,z)$

$$C_0(-zt)\ {}_HC_0(x,y;t)=\sum_{n=0}^{\infty}{}_HR_n(x,y,z)\frac{t^n}{(n!)^2}, \tag{4.3}$$

where ${}_HR_n(x,y,z)$ are defined as

$$ {}_HR_n(x,y,z)=\exp\left(y\frac{\partial^2}{\partial x^2}\right)\Big\{R_n(x,z)\Big\}. \tag{4.4}$$

It is worthy to note that the method adopted in this paper can be exploited to establish further consequences regarding other families of special polynomials. Here, we establish summation formulae for the HLeP ${}_HR_n(x,y,z)$. To this aim, we consider the product of generating functions (1.24) and (1.32) of the 2VgLtP and 3VHLP respectively, in the following form

$$\begin{aligned}&\exp(zt)_HC_0(x,y;t)\ \exp(vt^m)\ C_0(wt)\\ &\qquad=\sum_{n,r=0}^{\infty}{}_HL_n(x,y,z)\ {}_{[m]}L_r(w,v)\ \frac{t^{n+r}}{n!r!}.\end{aligned} \tag{4.5}$$

Now, following the same lines of proof of Theorem 2.1 and in view of generating function (4.3), we get the following summation formula for the HLeP ${}_HR_n(x,y,z)$

$$\begin{aligned}&{}_HR_n(x,y,w)\\ &=n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k^{(m)}(-z,-v)\ {}_{[m]}L_r(-w,v)\ {}_HL_{n-k-r}(x,y,z),\end{aligned} \tag{4.6}$$

which for $m=2$ gives the following summation formula

$$\begin{aligned}&{}_HR_n(x,y,w)\\ &\quad=n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}H_k(-z,-v)\ G_r(-w,v)\ {}_HL_{n-k-r}(x,y,z).\end{aligned} \tag{4.7}$$

Again, for $m=1$, equation (4.6) yields to the following summation formula

$$\begin{aligned}&{}_HR_n(x,y,w)\\ &\quad=n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-z-v)^k\ L_r(-w,v)\ {}_HL_{n-k-r}(x,y,z).\end{aligned} \tag{4.8}$$

We remark that the summation formula (4.6) can also be obtained after replacing y by z, z by w and w by v in assertion (2.1) of Theorem 2.1 and operating

$\exp\left(y\frac{\partial^2}{\partial x^2}\right)$ on the resultant equation and then using operational definitions (4.2) and (4.4).

Similarly, by considering the product of generating functions (1.28) and (1.32) of the 2VgLtP and 3VHLP, respectively, and following the same method, we get another summation formula for the HLeP ${}_HR_n(x,y,z)$

$$\begin{aligned} {}_HR_n(x,y,w) = n!\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r} \\ \times H_k(-z-v,-u)\ {}_LH_r(-w,v,u)\ {}_HL_{n-k-r}(x,y,z), \end{aligned} \tag{4.9}$$

which can also be obtained after replacing y by z, z by w, w by v and v by u in assertion (2.14) of Theorem 2.2 and operating $\exp\left(y\frac{\partial^2}{\partial x^2}\right)$ on the resultant equation and then using operational definitions (4.2) and (4.4).

Very recently Dattoli et al. [**8**] introduced a two-variable extension of the Legendre polynomials $P_n(x,y)$, defined by the generating function

$$\frac{1}{\sqrt{1+xt+yt^2}} = \sum_{n=0}^{\infty} P_n(x,y)\ t^n. \tag{4.10}$$

To give another example of the method adopted in this paper, we derive a summation formula for the 2-variable Chebyshev polynomials (2VCP) $U_n(x,y)$ [**6**] in terms of the product of the polynomials $P_n(x,y)$. To this aim, we consider the product of generating function (4.10) in the following form

$$\frac{1}{(1+xt+yt^2)} = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} P_n(x,y)P_r(x,y)\ t^{n+r}. \tag{4.11}$$

Replacing x by $-2x$ in equation (4.11) and then replacing n by $n-r$ in the r.h.s. of the resultant equation, we find

$$\frac{1}{(1-2xt+yt^2)} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} P_{n-r}(-2x,y)P_r(-2x,y)\ t^{n}. \tag{4.12}$$

Now, using the generating function [**10**, p. 43] (see also [**6**])

$$\frac{1}{(1-2xt+yt^2)} = \sum_{n=0}^{\infty} U_n(x,y)\ t^n \tag{4.13}$$

of the 2VCP $U_n(x,y)$ in the l.h.s. of equation (4.12), we get the following summation formula

$$U_n(x,y) = \sum_{r=0}^{n} P_{n-r}(-2x,y)\ P_r(-2x,y). \tag{4.14}$$

The above examples prove the usefulness of the method adopted in this paper. Further, to bolster up the contention of using operational techniques, certain new families of special polynomials will be introduced in a forthcoming investigation.

References

1. Andrews L. C., *Special Functions for Engineers and Applied Mathematicians*, Macmillan Co., New York 1985.

2. Appell P. and Kampé de Fériet J., *Fonctions hypergéométriques et hypersphériques: Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

3. Dattoli G., *Generalized polynomials, operational identities and their applications*, J. Comput. Appl. Math. **118** (2000), 111-123.

4. ______, *HermiteBessel and LaguerreBessel functions: A by product of the monomiality principle*, In: Advanced Special Functions and Applications, Melfi, 1999, in: Proc. Melfi Sch. Adv. Top. Math. Phys. **1** (2000), 147-164.

5. ______, *Summation formulae of special functions and multivariable Hermite polynomials.* Nuovo Cimento Soc. Ital. Fis. **B 119(5)** (2004), 479-488.

6. Dattoli G. and Cesarano C., *On a new family of Hermite polynomials associated to parabolic cylinder functions.* Appl. Math. Comput. **141** (2003), 143–149.

7. Dattoli G., Cesarano C. and Sacchetti D., *A note on truncated polynomials.* Appl. Math. Comput. **134** (2003), 595–605.

8. Dattoli G., Germano B., Martinelli M. R. and Ricci P.E., *A novel theory of Legendre polynomials.* Math. Comput. Modelling **54** (2011), 80-87.

9. ______, *Monomiality, orthogonal and pseudo-orthogonal polynomials.* Inter. Math. Forum **1(13)** (2006), 603–616..

10. Dattoli G., Lorenzutta S. and Cesarano C., *From Hermite to Humbert polynomials.* Rend. Istit. Mat. Univ. Trieste **XXXV** (2003), 37–48.

11. ______, *Generalized polynomials and new families of generating functions.* Ann. Univ. Ferrara Sez. VII (N.S.) **47** (2001), 57–61.

12. Dattoli G., Lorenzutta S., Mancho A. M. and Torre A., *Generalized polynomials and associated operational identities.* J. Comput. Appl. Math. **108** (1999), 209-218.

13. Dattoli G. and Ricci P.E., *A note on Legendre polynomials.* Int. J. Nonlinear Sci. Numer. Simul. **2** (2001), 365–370.

14. Dattoli G. and Torre A., *Operational methods and two variable Laguerre polynomials.* Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **132** (1998), 3-9.

15. Dattoli G., Torre A., Lorenzutta S. and Cesarano C., *Generalized polynomials and operational identities*, Atti. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **134** (2000), 231–249.

16. Dattoli G., Torre A. and Mancho A. M., *The generalized Laguerre polynomials, the associated Bessel functions and applications to propagation problems.* Radiat. Phys. Chem. **59** (2000), 229-237.

17. Dattoli G., Torre A. and Mazzacurati G., *Monomiality and integrals involving Laguerre polynomials*, Rend. Mat. (VII) **18** (1998), 565–574.

18. Gould H. W. and Hopper A. T., *Operational formulas connected with two generalizations of Hermite polynomials.* Duke. Math. J. **29** (1962), 51–63.

19. Subuhi Khan and Al-Saad M. W., *Summation formulae for Gould-Hopper generalized Hermite polynomials.* Comput. Math. Appl. **61** (2011), 1536–1541.

20. Srivastava H. M. and Manocha H. L., *A Treatise on Generating Functions*, Halsted Press, New York 1984.

Subuhi Khan, Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India, *e-mail*: `subuhi2006@gmail.com`

A. A. Al-Gonah, Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India, *e-mail*: `gonah1977@yahoo.com`