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GENERALIZED WARPED PRODUCT MANIFOLDS AND BIHARMONIC MAPS

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Abstract. In this paper, we present some new properties for biharmonic and conformal biharmonic maps between generalized warped product manifolds.

1. Introduction

Biharmonic maps are critical points of bi-energy functional defined on the space of smooth maps between Riemannian manifolds, introduced by Eells and Sampson in 1964, which is a generalization of harmonic maps [**7**].
If $\varphi\colon (M,g) \to (N,h)$ is a smooth map between Riemannian manifolds then the tension field of φ is defined as

$$\tau(\varphi) = \operatorname{trace}_g \nabla \mathrm{d}\varphi.$$

Then φ is called harmonic if the tension field vanishes. The equivalent definition is that φ is a critical point of the energy functional

$$E(\varphi) = \int_M e(\varphi) v_g,$$

where $e(\varphi) = \frac{1}{2}\operatorname{trace}_g(\varphi^* h)$ is called energy density of φ. If M is not compact then the energy $E(\varphi)$ may be defined on its compact subsets.

Definition 1. A map $\varphi\colon (M,g) \to (N,h)$ between Riemannian manifolds is called *biharmonic* if it is a critical point of the *bi-energy* functional:

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

(or over any compact subset $K \subset M$).

The Euler-Lagrange equation attached to bienergy is given by the vanishing of the bi-tension field

$$\tau_2(\varphi) = -J_\varphi(\tau(\varphi)) = -\big(\Delta^\varphi \tau(\varphi) + \operatorname{trace}_g R^N(\tau(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi\big), \tag{1}$$

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where R^N is the curvature tensor field on N and J_φ is the Jacobi operator defined by

$$\begin{aligned} J_\varphi\colon \Gamma(\varphi^{-1}(TN) &\to \Gamma(\varphi^{-1}(TN) \\ V &\mapsto \Delta^\varphi V + \operatorname{trace}_g R^N(V, \mathrm{d}\varphi)\mathrm{d}\varphi. \end{aligned} \tag{2}$$

(One can refer to [**1**] [**5**] [**6**] [**9**] [**11**] for more details)

2. Some results on generalized warped product manifolds

In this section, we give the definition and some geometric properties of generalized warped product manifolds. For more detail see [**3**] [**4**] [**8**] [**13**].

Definition 2 ([**4**]). Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and $f\colon M \times N \to \mathbb{R}$ be a smooth positive function. The generalized warped metric on $M \times_f N$ is defined by

$$G_f = \pi^* g + (f)^2 \eta^* h \tag{3}$$

where $\pi\colon (x, y) \in M \times N \to x \in M$ and $\eta\colon (x, y) \in M \times N \to y \in N$ are the canonical projections. For all $X, Y \in T(M \times N)$, we have

$$G_f(X, Y) = g(\mathrm{d}\pi(X), \mathrm{d}\pi(Y)) + (f)^2 h(\mathrm{d}\eta(X), \mathrm{d}\eta(Y)).$$

By $X \wedge_{G_{f^2}} Y$, we denote the linear map

$$Z \in \mathcal{H}(M) \times \mathcal{H}(N) \to \Big(X \wedge_{G_{f^2}} Y\Big)Z = G_{f^2}\Big(Z, Y\Big)X - G_{f^2}\Big(Z, X\Big)Y. \tag{4}$$

Proposition 1 ([**13**]). *Let (M^m, g) and (N^n, h) be two Riemannian manifolds. If $\overline{\nabla}$ denotes the Levi-Civita connection and $\overline{R}$ the curvature tensor on $(M \times_f N, G_f)$, then for all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$, we have*

$$\begin{aligned} \overline{\nabla}_X Y - \nabla_X Y = {} & X(\ln f)(0, Y_2) + Y(\ln f)(0, X_2) \\ & - \frac{1}{2} h(X_2, Y_2)(\operatorname{grad}_M f^2, \frac{1}{f^2}\operatorname{grad}_N f^2) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \overline{R}(X, Y)Z - R(X, Y)Z = {} & (\nabla^M_{Y_1} \operatorname{grad}_M \ln f + Y_1(\ln f)\operatorname{grad}_M \ln f, 0) \wedge_{G_f} (0, X_2)Z \\ & - (\nabla^M_{X_1} \operatorname{grad}_M \ln f + X_1(\ln f)\operatorname{grad}_M \ln f, 0) \wedge_{G_f} (0, Y_2)Z \\ & + \frac{1}{f^2}\Big[(0, \nabla^N_{Y_2} \operatorname{grad}_N \ln f - Y_2(\ln f)\operatorname{grad}_N \ln f) \wedge_{G_f} (0, X_2) \\ & - (0, \nabla^N_{X_2} \operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f) \wedge_{G_f} (0, Y_2) \\ & - (f^2 |\operatorname{grad}_M \ln f|^2 + |\operatorname{grad}_N \ln f|^2)(0, X_2) \wedge_{G_f} (0, Y_2)\Big]Z \\ & + \Big[X_1(Z_2(\ln f)) + X_2(Z_1(\ln f))\Big](0, Y_2) \\ & - \Big[Y_1(Z_2(\ln f)) + Y_2(Z_1(\ln f))\Big](0, X_2) \end{aligned} \tag{6}$$

where $X = (X_1, X_2)$, $\quad Y = (Y_1, Y_2)$, $\quad \nabla_X Y = (\nabla^M_{X_1} Y^1, \nabla^N_{X_2} Y^2)$ and $R(X,Y)Z = (R^M(X_1,Y_1)Z_1, R^N(X_2,Y_2)Z_2)$.

Proposition 2 ([**8**]). *Let (M^m, g) and (N^n, h) be two Riemannian manifolds and $f\colon M \times N \to \mathbb{R}$ be smooth positive function. The Ricci curvature of the generalized warped product manifolds $(M \times_f N, G_f)$ is given by the following formulas:*

$$
\begin{aligned}
\mathrm{Ric}((X_1,0),(Y_1,0)) &= \mathrm{Ric}^M(X_1,Y_1) - \mathrm{ng}(\nabla^M_{X_1} \operatorname{grad}_M \ln f + X_1(\ln f) \operatorname{grad}_M \ln f, Y_1)\\
\mathrm{Ric}((X_1,0),(0,Y_2)) &= -\, nX_1(Y_2(\ln f))\\
\mathrm{Ric}((0,X_2),(Y_1,0)) &= h(X_2, \operatorname{grad}_N(Y_1(\ln f))) - nX_2(Y_1(\ln f))\\
\mathrm{Ric}((0,X_2),(0,Y_2)) &= \mathrm{Ric}^N(X_2,Y_2) + (2-n)h(\nabla^N_{X_2} \operatorname{grad}_N \ln f, Y_2)\\
&\quad + (2-n)\big[h(X_2,Y_2) \mid \operatorname{grad}_N \ln f\,|^2 - X_2(\ln f)h(\operatorname{grad}_N \ln f, Y_2)\big]\\
&\quad + h(X_2,Y_2)\big[nf^2 \mid \operatorname{grad}_M \ln f\ |^2 - \Delta_N(\ln f) - f^2 \Delta_M(\ln f)\big]
\end{aligned}
$$

for all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$.

Proposition 3 ([**3**]). *If $\varphi\colon P \to M$ and $\psi\colon P \to N$ are regular maps. Then the tension field of $\phi\colon x \in (P^p, \ell) \longrightarrow (\phi(x) = (\varphi(x), \psi(x)) \in (M \times_f N, G_f)$ is given by the following relation*

$$
\begin{aligned}
\tau(\phi) &= \Big(\tau(\varphi), \tau(\psi)\Big) + 2\Big(0, \mathrm{d}\psi(\operatorname{grad}_P(\ln f \circ \phi))\Big)\\
&\qquad - e(\psi)\Big(\operatorname{grad}_M f^2, \frac{1}{f^2} \operatorname{grad}_N f^2\Big).
\end{aligned}
\tag{7}
$$

Corollary 1 ([**3**]). *Let (M^m, g) be a Riemannian manifold and $f\colon (x,y) \in M \times M \to f(x,y) \in \mathbb{R}$ be smooth positive function. Then the tension field of the map*

$$
\begin{aligned}
\phi\colon (M,g) &\longrightarrow (M \times_f M, G_f)\\
x &\longmapsto (x,x)
\end{aligned}
$$

is given by

$$
\begin{aligned}
\tau(\phi) &= \big\{ -\frac{m}{2}(\operatorname{grad}_x f^2, 0) + 2(0, \operatorname{grad}_x \ln f)\\
&\qquad + (2-m)(0, \operatorname{grad}_y \ln f)\big\} \circ \phi\\
&= \{-m.f^2(\operatorname{grad}_x \ln f, 0) + 2(0, \operatorname{grad}_x \ln f)\\
&\qquad + (2-m)(0, \operatorname{grad}_y \ln f)\} \circ \phi.
\end{aligned}
\tag{8}
$$

Proposition 4 ([**3**]). *The tension field of $\phi\colon (M \times_f N, G_f) \longrightarrow (P, k)$ is given by*

$$
\begin{aligned}
\tau(\phi) &= \tau(\phi_M) + nd\phi_M(\operatorname{grad}_M \ln f)\\
&\qquad + \frac{1}{f^2}\{\tau(\phi_N) + (n-2)d\phi_N(\operatorname{grad}_N \ln f)\}
\end{aligned}
\tag{9}
$$

where $\phi_M\colon x \in M \to \phi_M(x) = \phi(x,y) \in P$ and $\phi_N\colon y \in N \to \phi_N(y) = \phi(x,y) \in P$.

Proposition 5 ([**3**]). *If $\varphi\colon M \to M$ and $\psi\colon N \to N$ are harmonic maps, then the tension fields of*

$$\begin{aligned}\phi\colon (M \times_f N, G_f) &\longrightarrow (M \times N, G)\\ (x,y) &\longmapsto (\varphi(x), \psi(y))\end{aligned}$$

is given by the following formula

$$\tau(\phi) = n(\mathrm{d}\varphi(\operatorname{grad}_M \ln f), 0) + \frac{(n-2)}{f^2}(0, \mathrm{d}\psi(\operatorname{grad}_N \ln f)). \tag{10}$$

3. Biharmonic maps on generalized warped product manifolds

3.1. Biharmonicity conditions of the inclusion $\bar\phi\colon (N,h) \longrightarrow (M^m \times_f N^n, G_f)$

Theorem 1. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds and x_0 be an arbitrary point of M. Then the tension and the bitension fields of the inclusion*

$$\begin{aligned}\bar\phi\colon (N,h) &\longrightarrow (M \times_f N, G_f)\\ y &\longmapsto (x_0, y)\end{aligned} \tag{11}$$

are given by:

i) $\tau(\bar\phi) = \{-n\,\mathrm{e}^{2\gamma}(\operatorname{grad}_M \gamma, 0) + (2-n)(0, \operatorname{grad}_N \gamma)\} \circ \bar\phi.$

ii)
$$\begin{aligned}\tau_2(\bar\phi) = \Big\{ &-\frac{n^2\,\mathrm{e}^{4\gamma}}{2}(\operatorname{grad}_M(|\operatorname{grad}_M \gamma|^2), 0)\\ &+ (n-2)(0, \operatorname{grad}_N(\Delta_N(\gamma)) + 2\operatorname{Ricci}_N(\operatorname{grad}_N \gamma))\\ &- \mathrm{e}^{2\gamma}\Big[2n^2\,\mathrm{e}^{2\gamma}\,|\operatorname{grad}_M \gamma|^2 - 4\Delta_N\gamma\Big](\operatorname{grad}_M \gamma, 0)\\ &- \mathrm{e}^{2\gamma}\Big[(n^2-4n-4)\,|\operatorname{grad}_N \gamma|^2\Big](\operatorname{grad}_M \gamma, 0)\\ &+ \Big[(2-n)^2\,|\operatorname{grad}_N \gamma|^2 - 2(2-n)\Delta_N\gamma\Big](0, \operatorname{grad}_N \gamma)\\ &+ \Big[2n(n-4)\,\mathrm{e}^{2\gamma}\,|\operatorname{grad}_M \gamma|^2\Big](0, \operatorname{grad}_N \gamma)\\ &+ n\,\mathrm{e}^{2\gamma}\Big[(0, \operatorname{grad}_N(|\operatorname{grad}_M \gamma|^2) + \operatorname{trace}_N\big((\operatorname{grad}_M \gamma)(\star(\gamma))(0,\star)\big)\Big]\\ &+ \frac{(n-2)(6-n)}{2}(0, \operatorname{grad}_N(|\operatorname{grad}_N \gamma|^2)\Big\} \circ \bar\phi\end{aligned} \tag{12}$$

where $f(x,y) = \mathrm{e}^{\gamma(x,y)}$.

Proof. From Proposition 3, we obtain

i) $\tau(\bar\phi) = \{-n\,\mathrm{e}^{2\gamma}(\operatorname{grad}_M \gamma, 0) + (2-n)(0, \operatorname{grad}_N \gamma)\} \circ \bar\phi.$

ii) Let $y \in N$ and $(F_i)_i$ be a local orthonormal frame on (N^n, h) such that

$$(\nabla_{F_i} F_j)_y = 0 \quad (1 \le i, j \le n).$$

Using the general formula of bitension field

$$\tau_2(\bar\phi) = -\operatorname{tr}_h(\nabla^{\bar\phi})^2\tau(\bar\phi) - \operatorname{tr}_h \overline{R}(\tau(\bar\phi), d\bar\phi)d\bar\phi. \tag{13}$$

and Proposition 1, we have:

- $\nabla^{\bar{\phi}}_{F_i}\tau(\bar{\phi})$

$$
\begin{aligned}
&= -n\,\mathrm{e}^{2\gamma}\Big[2F_i(\gamma)(\operatorname{grad}_M\gamma,0)+\mid \operatorname{grad}_M\gamma\mid^2(0,F_i)\Big]\\
&\quad + (2-n)\Big[(0,\nabla^N_{F_i}\operatorname{grad}_N\gamma)+\mid \operatorname{grad}_N\gamma\mid^2(0,F_i)\Big]\\
&\quad - (2-n)\,\mathrm{e}^{2\gamma}\,F_i(\gamma)(\operatorname{grad}_M\gamma,0).
\end{aligned}
\tag{14}
$$

- $tr_h(\nabla^{\bar{\phi}})^2\tau(\bar{\phi})$

$$
\begin{aligned}
&= -n\,\mathrm{e}^{2\gamma}\Big[(0,\operatorname{grad}_N(\mid \operatorname{grad}_M\gamma\mid^2)+(6-n)\mid \operatorname{grad}_M\gamma\mid^2\operatorname{grad}_N\gamma)\\
&\quad + (2-n)\big(0,tr_h(\nabla^N)^2\operatorname{grad}_N\gamma+2\operatorname{grad}_N(\mid \operatorname{grad}_N\gamma\mid^2)\big)\\
&\quad + (2-n)\Big[(2-n)\mid \operatorname{grad}_N\gamma\mid^2-\mathrm{e}^{2\gamma}\mid \operatorname{grad}_M\gamma\mid^2-\Delta(\gamma)\Big](0,\operatorname{grad}_N\gamma)\\
&\quad - \mathrm{e}^{2\gamma}\Big[4\Delta_N(\gamma)-n^2\,\mathrm{e}^{2\gamma}\mid \operatorname{grad}_M\gamma\mid^2-(n^2-4n-4)\mid \operatorname{grad}_N\gamma\mid^2\Big].
\end{aligned}
\tag{15}
$$

- $\sum_i \overline{R}((\mathrm{e}^{2\gamma}\operatorname{grad}_M\gamma,0),(0,F_i))(0,F_i)$

$$
\begin{aligned}
&= -n\,\mathrm{e}^{4\gamma}\left(\frac{1}{2}\operatorname{grad}_M(\mid \operatorname{grad}_M\gamma\mid^2)+\mid \operatorname{grad}_M\gamma\mid^2\operatorname{grad}_M\gamma,0\right)\\
&\quad + \mathrm{e}^{2\gamma}\sum_i(\operatorname{grad}_M\gamma)(F_i(\gamma))(0,F_i).
\end{aligned}
\tag{16}
$$

- $\sum_i \overline{R}((0,\operatorname{grad}_N\gamma),(0,F_i))(0,F_i)$

$$
\begin{aligned}
&= (0,\operatorname{Ricci}_N(\operatorname{grad}_N\gamma)+\frac{2-n}{2}\operatorname{grad}_N(\mid \operatorname{grad}_N\gamma\mid^2))\\
&\quad + \Big[(1-n)\,\mathrm{e}^{2\gamma}\mid \operatorname{grad}_M\gamma\mid^2-\Delta_N(\gamma)\Big](0,\operatorname{grad}_N\gamma).
\end{aligned}
\tag{17}
$$

Substituting (15), (16) and (17) in (13), we obtain the formula (12). □

Remarks.

1) If $\dim N = 2$, then
$\tau(\bar{\phi}) = -2\,\mathrm{e}^{2\gamma}(\operatorname{grad}_M\gamma,0)$
and

$$
\begin{aligned}
\tau_2(\bar{\phi}) = &-2\,\mathrm{e}^{4\gamma}(\operatorname{grad}_M(\mid \operatorname{grad}_M\gamma\mid^2),0)+8\,\mathrm{e}^{2\gamma}\mid \operatorname{grad}_M\gamma\mid^2(0,\operatorname{grad}_N\gamma)\\
&-\mathrm{e}^{2\gamma}\Big[8\,\mathrm{e}^{2\gamma}\mid \operatorname{grad}_M\gamma\mid^2-4\Delta_N(\gamma)-8\mid \operatorname{grad}_N\gamma\mid^2\Big](\operatorname{grad}_M\gamma,0)\\
&+2\,\mathrm{e}^{2\gamma}(0,\operatorname{grad}_N(\mid \operatorname{grad}_M\gamma\mid^2)).
\end{aligned}
$$

2) If $\gamma\in C^\infty(M)$, $\big(\gamma(x,y)=\gamma(x),\quad \forall(x,y)\in M\times N\big)$, then
$\tau(\bar{\phi}) = -n\,\mathrm{e}^{2\gamma}(\operatorname{grad}_M\gamma,0)$

and

$\tau_2(\bar{\phi}) = \; - 2\,\mathrm{e}^{4\gamma}\left(\mathrm{grad}_M(\mid \mathrm{grad}_M \gamma \mid^2) - 4 \mid \mathrm{grad}_M \gamma \mid^2 (\mathrm{grad}_M \gamma, 0)\right.$.

The results coincide with the formulas obtained in [**1**].

3) If $\gamma \in C^\infty(N)$, $\quad\big(\gamma(x,y) = \gamma(y), \quad \forall (x,y) \in M \times N\big)$, then
$\tau(\bar{\phi}) = (2-n)(0, \mathrm{grad}_N \ln f)$
and

$$\begin{aligned}
\tau_2(\bar{\phi}) = &\,(n-2)\big(0, \mathrm{grad}_N(\Delta(\gamma)) + 2\,\mathrm{Ricci}(\mathrm{grad}_N \gamma\big)\\
&+ (n-2)\Big[(2-n) \mid \mathrm{grad}_N \gamma \mid^2 - 2\Delta_N(\gamma)\Big](0, \mathrm{grad}_N \gamma)\\
&+ \frac{(n-2)(6-n)}{2}(0, \mathrm{grad}_N(\mid \mathrm{grad}_N \gamma \mid^2))
\end{aligned}$$

3.2. Biharmonicity conditions of ϕ: $(M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$

Lemma 1. *Let $\lambda \in C^\infty(M \times N)$ be a smooth function and $\sigma \in \Gamma(\phi^{-1}TP)$. Then*

$$\begin{aligned}
J_\phi(\lambda\sigma) = &\,\lambda J_{\phi_M}(\sigma) + \Delta_M(\lambda)\sigma + 2\nabla^{\phi_M}_{\mathrm{grad}_M \lambda}\sigma\\
&+ n\Big[(\mathrm{grad}_M \gamma)(\lambda)\sigma + \lambda\nabla^{\phi_M}_{\mathrm{grad}_M \gamma}\sigma\Big]\\
&+ \mathrm{e}^{-2\gamma}\Big[\lambda J_{\phi_N}(\sigma) + \Delta_N(\lambda)\sigma + 2\nabla^{\phi_N}_{\mathrm{grad}_N \lambda}\sigma\Big]\\
&+ (n-2)\,\mathrm{e}^{-2\gamma}\Big[(\mathrm{grad}_N \gamma)(\lambda)\sigma + \lambda\nabla^{\phi_N}_{\mathrm{grad}_N \gamma}\sigma\Big]
\end{aligned} \tag{18}$$

where $f(x,y) = \mathrm{e}^{\gamma(x,y)}$.

Proof. Let $(E_i)_{i=1}^m$ and $(F_j)_{j=1}^n$ be a local orthonormal frame on M and N, respectively. From the expression of Jacobi operator (formula (2)), we have

$$J_\phi(\lambda\sigma) \quad = \quad \mathrm{trace}_{G_f}(\nabla^\phi)^2(\lambda\sigma) + \mathrm{trace}_{G_f} R^p(\lambda\sigma, \mathrm{d}\phi)\mathrm{d}\phi. \tag{19}$$

By calculating each term, we obtain:

$$\begin{aligned}
\mathrm{trace}_{G_f}(\nabla^\phi)^2(\lambda\sigma) = &\sum_{i=1}^m \Big[\nabla^\phi_{(E_i,0)}\nabla^\phi_{(E_i,0)}\lambda\sigma - \nabla^\phi_{\overline{\nabla}_{(E_i,0)}(E_i,0)}\lambda\sigma\Big]\\
&+ \sum_{j=1}^n \Big[\frac{1}{f}\nabla^\phi_{(0,F_j)}\frac{1}{f}\nabla^\phi_{(0,F_j)}\lambda\sigma - \nabla^\phi_{\overline{\nabla}_{\frac{1}{f}(0,F_j)}\frac{1}{f}(0,F_j)}\lambda\sigma\Big],
\end{aligned} \tag{20}$$

$$\sum_{i=1}^m \nabla^\phi_{(E_i,0)}\nabla^\phi_{(E_i,0)}\lambda\sigma = \Delta_M(\lambda)\sigma + 2\nabla^{\phi_M}_{\mathrm{grad}_M \lambda}\sigma + \lambda\nabla^{\phi_M}_{E_i}\nabla^{\phi_M}_{E_i}\sigma, \tag{21}$$

(22)

$$\begin{aligned}
\sum_{j=1}^n \frac{1}{f}\nabla^\phi_{(0,F_j)}\frac{1}{f}\nabla^\phi_{(0,F_j)}\lambda\sigma = &\,\frac{1}{f^2}\Big[\Delta_N(\ln f)\sigma - (\mathrm{grad}_N \ln f)(\lambda)\sigma\\
&- \lambda\nabla^{\phi_N}_{\mathrm{grad}_N \ln f}\sigma + 2\nabla^{\phi_N}_{\mathrm{grad}_N \lambda}\sigma + \lambda\nabla^{\phi_N}_{F_j}\nabla^{\phi_N}_{F_j}\sigma\Big],
\end{aligned}$$

$$\sum_{j=1}^{n} \overline{\nabla}_{\frac{1}{f}(0,F_j)} \frac{1}{f}(0,F_j) = \frac{1-n}{f^2}(0, \mathrm{grad}_N \ln f) - n(\mathrm{grad}_M \ln f, 0), \tag{23}$$

$$\begin{aligned} -\sum_{j=1}^{n} \nabla^{\phi}_{\overline{\nabla}_{\frac{1}{f}(0,F_j)} \frac{1}{f}(0,F_j)} \lambda\sigma = \frac{n-1}{f^2}\Big[(\mathrm{grad}_N \ln f)(\lambda)\sigma + \lambda \nabla^{\phi_N}_{\mathrm{grad}_N \ln f}\sigma\Big] \\ + n\Big[(\mathrm{grad}_M \ln f)(\lambda)\sigma + \lambda\nabla^{\phi_M}_{\mathrm{grad}_M \ln f}\sigma\Big], \end{aligned} \tag{24}$$

$$\begin{aligned} &\mathrm{trace}_{G_f} R^p(\lambda\sigma, \mathrm{d}\phi)\mathrm{d}\phi \\ &\qquad = \lambda\, \mathrm{trace}_g R^p(\sigma, \mathrm{d}\phi_M)\mathrm{d}\phi_M + \frac{\lambda}{f^2} \mathrm{trace}_h R^p(\sigma, \mathrm{d}\phi_N)\mathrm{d}\phi_N. \end{aligned} \tag{25}$$

Substuting (21), (22) and (24) in (20), and summing with (25), we obtain the formula (18). □

Theorem 2. *Let (M^m, g), (N^n, h) and (P^p, k) be Riemannian manifolds and $f\colon M \times N \to \mathbb{R}$ be a smooth positive function. Then the bitension fields of $\phi\colon (M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$ is given by the following*

$$\begin{aligned} \tau_2(\phi) = {}& \tau_2(\phi_M) - nJ_{\phi_M}(\mathrm{d}\phi_M(\mathrm{grad}_M \gamma)) - n\nabla^{\phi_M}_{\mathrm{grad}_M \gamma} V \\ & + \mathrm{e}^{-4\gamma}\Big[\tau_2(\phi_N) - (n-2)J_{\phi_N}(\mathrm{d}\phi_N(\mathrm{grad}_N \gamma)) - (n-6)\nabla^{\phi_N}_{\mathrm{grad}_N \gamma} W \\ & - \big(2(4-n) \mid \mathrm{grad}_N \gamma \mid^2 - 2\Delta_N(\gamma)\big)W\Big] \\ & - \mathrm{e}^{-2\gamma}\Big[J_{\phi_N}(V) + (n-2)\nabla^{\phi_N}_{\mathrm{grad}_N \gamma} V + J_{\phi_M}(W) + (n-4)\nabla^{\phi_M}_{\mathrm{grad}_M \gamma} W \\ & + \big(2(2-n) \mid \mathrm{grad}_M \gamma \mid^2 - 2\Delta_M(\gamma)\big)W\Big] \end{aligned} \tag{26}$$

where $V = \tau(\phi_M) + n\mathrm{d}\phi_M(\mathrm{grad}_M \gamma)$, $W = \tau(\phi_N) + (n-2)\mathrm{d}\phi_N(\mathrm{grad}_N \gamma)$ and $f = \mathrm{e}^{\gamma}$.

Proof. From formulas (1) and (2), we have

$$\begin{aligned} J_\phi(V) = {}& \tau_2(\phi_M) + nJ_{\phi_M}(\mathrm{d}\phi_M(\mathrm{grad}_M \gamma)) - n\nabla^{\phi_M}_{\mathrm{grad}_M \gamma} V \\ & + \mathrm{e}^{-2\gamma} J_{\phi_N}(V) - (n-2)\,\mathrm{e}^{-2\gamma}\, \nabla^{\phi_N}_{\mathrm{grad}_N \gamma} V. \end{aligned} \tag{27}$$

From Lemma 18, we obtain

$$\begin{aligned} J_\phi(\mathrm{e}^{-2\gamma} W) = {}& \mathrm{e}^{-4\gamma}\Big[\tau_2(\phi_N) + (n-2)J_{\phi_N}(d\phi_N(\mathrm{grad}_N \gamma)) \\ & - (n-6)\nabla^{\phi_N}_{\mathrm{grad}_N \gamma} W - 2\big((4-n) \mid \mathrm{grad}_N \gamma \mid^2 - \Delta_N(\gamma)\big)W\Big] \\ & + \mathrm{e}^{-2\gamma}\Big[J_{\phi_M}(W) - (n-4)\nabla^{\phi_M}_{\mathrm{grad}_M \gamma} W\Big] \\ & - 2\,\mathrm{e}^{-2\gamma}\Big[(2-n) \mid \mathrm{grad}_M \gamma \mid^2 - \Delta_M(\gamma)\Big]W \end{aligned} \tag{28}$$

using Proposition 4 and summing the formulas (27) and (28), Theorem 2 follows. □

Particular cases

- If $f \in C^\infty(M)$, then

$$\begin{aligned}\tau_2(\phi) = \tau_2(\phi_M) &+ \mathrm{e}^{-4\gamma}\,\tau_2(\phi_N) - nJ_{\phi_M}(\mathrm{d}\phi_M(\mathrm{grad}_M\,\gamma)) - n\nabla^{\phi_M}_{\mathrm{grad}_M\,\gamma}V\\ &- (n-4)\,\mathrm{e}^{-2\gamma}\,\nabla^{\phi_M}_{\mathrm{grad}_M\,\gamma}\tau(\phi_N) + 4\,\mathrm{e}^{-2\gamma}\,\Delta_M(\gamma)\tau(\phi_N)\\ &- \mathrm{e}^{-2\gamma}\Big[J_{\phi_N}(V) + J_{\phi_M}(\tau(\phi_N)) + 2(2-n)\mid \mathrm{grad}_M\,\gamma\mid^2\tau(\phi_N)\Big]\end{aligned}$$

- If $f \in C^\infty(M)$ and $\phi\colon (x,y)\in M\times N \to x \in M$ is the first projection, then $V = n\cdot \mathrm{grad}(\gamma)$ and $\tau_2(\phi) = -n\Big(J_{\phi_M}(\mathrm{grad}(\gamma)) + \dfrac{n}{2}\,\mathrm{grad}(|\,\mathrm{grad}\,\gamma|^2\Big)\circ\phi,$ we find the result obtained in [**1**]
- If $f \in C^\infty(N)$, then

$$\begin{aligned}\tau_2(\phi) = \tau_2(\phi_M) &- \mathrm{e}^{-2\gamma}\Big[J_{\phi_M}(W) + J_{\phi_N}(\tau(\phi_M)) + (n-2)\nabla^{\phi_N}_{\mathrm{grad}_N\,\gamma}\tau(\phi_M)\Big]\\ &+ \mathrm{e}^{-4\gamma}\Big[\tau_2(\phi_N) - (n-2)J_{\phi_N}(\mathrm{d}\phi_N(\mathrm{grad}_N\,\gamma)) - (n-6)\nabla^{\phi_N}_{\mathrm{grad}_N\,\gamma}W\\ &- (2(4-n)\mid \mathrm{grad}_N\,\gamma\mid^2 - 2\Delta_N(\gamma))W\Big].\end{aligned}$$

- If $\varphi\colon (M,g)\longrightarrow (P,k)$ be regular map and $\phi(x,y)=\varphi(x)$, then

$$\tau_2(\phi) = \tau_2(\varphi) - nJ_\varphi(\mathrm{d}\varphi(\mathrm{grad}_M\,\gamma)) - n\nabla^{\varphi}_{\mathrm{grad}_M\,\gamma}V. \tag{29}$$

From Proposition 4 and Lemma 1, we deduce the following.

Theorem 3. *Let $\varphi\colon (M,g)\longrightarrow (P,\ell)$ be a conformal map with dilation λ. Then the bitension field of $\phi\colon (x,y)\in(M\times_f N, G_f)\longrightarrow \phi(x,y)=\varphi(x)\in(P,\ell)$, is given by*

$$\tau_2(\phi) = -J_\varphi(\mathrm{d}\varphi(\mathrm{grad}_M\ln(\mu))) - n\nabla^{\varphi}_{\mathrm{grad}_M\ln f}\mathrm{d}\varphi(\mathrm{grad}_M\ln(\mu)) \tag{30}$$

where $\mu = \lambda^{2-m}f^n$.

Theorem 4. *Let $f \in C^\infty(M)$, thus the domain of ϕ is a warped product, and $\varphi\colon (M^m,g)\longrightarrow (P^m,\ell)$ $(m\geq 3)$ be a conformal map with dilation λ. Then $\phi\colon (x,y)\in(M\times_f N, G_f)\longrightarrow \phi(x,y)=\varphi(x)\in(P,\ell)$ is biharmonic map if and only if the following equation is verified*

$$\begin{aligned}0 = \;&\mathrm{grad}(\Delta\ln\lambda^{2-m}f^n) + 2\,\mathrm{Ricci}^M(\mathrm{grad}\ln\lambda^{2-m}f^n)\\ &+ 2n(2-m)\nabla_{\mathrm{grad}\ln f}\,\mathrm{grad}\ln\lambda + 4n\nabla_{\mathrm{grad}\ln\lambda}\,\mathrm{grad}\ln f\\ &+ \frac{n^2}{2}\,\mathrm{grad}(\mid\mathrm{grad}\ln f\mid^2) + \frac{(6-m)(2-m)}{2}\,\mathrm{grad}(\mid\mathrm{grad}\ln\lambda\mid^2)\\ &+ \big[(2-m)^2\mid\mathrm{grad}\ln\lambda\mid^2 - n^2\mid\mathrm{grad}\ln f\mid^2 + 2\Delta(\ln\lambda^{2-m}f^n)\big]\,\mathrm{grad}\ln\lambda\\ &+ 2n\big[(2-m)\mid\mathrm{grad}\ln\lambda\mid^2 + nd\ln f(\mathrm{grad}\ln\lambda)\big]\,\mathrm{grad}\ln f\end{aligned} \tag{31}$$

where grad, Δ *and* ∇ *are evaluated on M.*

For the proof of Theorem 4, we need the following two lemmas.

Lemma 2. *Let* $\varphi\colon (M,g) \longrightarrow (P,\ell)$ *be a conformal map with dilation* λ *and* $f \in C^\infty(M)$. *Then for any vector field* $X, Y \in \Gamma(TM)$, *we have*

$$\begin{aligned}\ell(\nabla_X \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(Y)) = \lambda^2 \mathrm{d}f(\operatorname{grad}\ln\lambda)g(X,Y) + \lambda^2 g(\nabla_X \operatorname{grad} f, Y)\\ + \lambda^2[X(\ln\lambda)Y(f) - X(f)Y(\ln\lambda)].\end{aligned} \tag{32}$$

Proof.

$$\begin{aligned}&\ell(\nabla_X \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(Y)) - \ell(\nabla_Y \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(X))\\
&= X(\lambda^2 g(\operatorname{grad} f, Y)) - \ell(\mathrm{d}\varphi(\operatorname{grad} f), \nabla_X \mathrm{d}\varphi(Y)) - Y(\lambda^2 g(\operatorname{grad} f, X))\\
&\quad + \ell(\mathrm{d}\varphi(\operatorname{grad} f), \nabla_Y \mathrm{d}\varphi(X))\\
&= X(\lambda^2)g(\operatorname{grad} f,Y) + \lambda^2 g(\nabla_X \operatorname{grad} f,Y) + \lambda^2 g(\operatorname{grad} f, \nabla_X Y) - Y(\lambda^2)g(\operatorname{grad} f,X)\\
&\quad - \lambda^2 g(\nabla_Y \operatorname{grad} f, X) - \lambda^2 g(\operatorname{grad} f, \nabla_Y X) - \lambda^2 g(\operatorname{grad} f, [X,Y])\end{aligned}$$

from which we have

$$\begin{aligned}&\ell(\nabla_X \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(Y)) =\\ &\ell(\nabla_Y \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(X)) + 2\lambda^2[X(\ln\lambda)Y(f) \quad - Y(\ln\lambda)X(f)]\end{aligned} \tag{33}$$

On the other hand,

$$\begin{aligned}\ell(\nabla_Y \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(X)) &= \ell(\nabla \mathrm{d}\varphi(\operatorname{grad} f, Y), \mathrm{d}\varphi(X)) + \lambda^2 g(\nabla_Y \operatorname{grad} f, X)\\
&= \ell(\nabla_{\operatorname{grad} f} \mathrm{d}\varphi(Y), \mathrm{d}\varphi(X)) - \lambda^2 g(\nabla_{\operatorname{grad} f} Y, X)\\
&\quad + \lambda^2 g(\nabla_Y \operatorname{grad} f, X)\\
&= \operatorname{grad} f(\lambda^2 g(X,Y)) - \ell(\mathrm{d}\varphi(Y), \nabla_{\operatorname{grad} f}\mathrm{d}\varphi(X))\\
&\quad - \lambda^2 g(\nabla_{\operatorname{grad} f} Y, X) + \lambda^2 g(\nabla_Y \operatorname{grad} f, X)\\
&= 2\lambda^2 df(\operatorname{grad}\lambda)g(X,Y) + \lambda^2 g(\nabla_{\operatorname{grad} f} X, Y)\\
&\quad + \lambda^2 g(\nabla_Y \operatorname{grad} f, X) - \ell(\mathrm{d}\varphi(Y), \nabla \mathrm{d}\varphi(X, \operatorname{grad} f))\\
&\quad - \lambda^2 g(Y, \nabla_{\operatorname{grad} f} X)\end{aligned}$$

$$\begin{aligned}\ell(\nabla_Y \mathrm{d}\varphi(\operatorname{grad} f),\mathrm{d}\varphi(X)) = \ & 2\lambda^2 df(\operatorname{grad}\lambda)g(X,Y)\\ &+ 2\lambda^2 g(\nabla_Y \operatorname{grad} f, X) - \ell(\mathrm{d}\varphi(Y), \nabla_X \mathrm{d}\varphi(\operatorname{grad} f))\end{aligned} \tag{34}$$

Substituting (34) in (33) we obtain

$$\begin{aligned}h(\nabla_X \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(Y)) = \lambda^2 df(\operatorname{grad}\ln\lambda)g(X,Y) + \lambda^2 g(\nabla_X \operatorname{grad} f, Y)\\ + \lambda^2[X(\ln\lambda)Y(f) - X(f)Y(\ln\lambda)]\end{aligned}$$

□

Lemma 3. *Let* $\varphi\colon (M,g) \longrightarrow (P,\ell)$ *be conformal map with dilation* λ *and* $f \in C^\infty(M)$. *Then for any vector field* $X \in \Gamma(TM)$, *we have*

$$\begin{aligned}h(\langle \nabla \mathrm{d}\varphi, \nabla df\rangle, \mathrm{d}\varphi(X)) = \ & 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X)\\ &- \lambda^2 \Delta(f) g(\operatorname{grad}\ln\lambda, X)\end{aligned} \tag{35}$$

where

$$\langle \nabla \mathrm{d}\varphi, \nabla \mathrm{d}f\rangle = \mathrm{trace}_g \nabla \mathrm{d}\varphi(*, \nabla_* \operatorname{grad} f)) = \sum_i \nabla \mathrm{d}\varphi(e_i, \nabla_{e_i} \operatorname{grad} f))$$

$(e_i)_{i=1}^m$ *is a local orthonormal frame on* M.

Proof. For any vector field $X \in \Gamma(TM)$, summing over the index i, we obtain

$$\begin{aligned}
&h(\langle \nabla \mathrm{d}\varphi, \nabla df\rangle, \mathrm{d}\varphi(X)) \\
&\quad = h(\nabla_{e_i} \mathrm{d}\varphi(\nabla_{e_i} \operatorname{grad} f), \mathrm{d}\varphi(X)) - h(\mathrm{d}\varphi(\nabla_{e_i}\nabla_{e_i} \operatorname{grad} f), \mathrm{d}\varphi(X)) \\
&\quad = e_i(\lambda^2 g(\nabla_{e_i} \operatorname{grad} f, X)) - \lambda^2 g(\nabla_{e_i}\nabla_{e_i} \operatorname{grad} f, X) \\
&\qquad - h(\mathrm{d}\varphi(\nabla_{e_i} \operatorname{grad} f), \nabla_{e_i} \mathrm{d}\varphi(X)) \\
&\quad = 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) + \lambda^2 g(\nabla_{e_i}\nabla_{e_i} \operatorname{grad} f, X) \\
&\qquad + \lambda^2 g(\nabla_{e_i} \operatorname{grad} f, \nabla_{e_i} X) - h(\mathrm{d}\varphi(\nabla_{e_i} \operatorname{grad} f), \nabla \mathrm{d}\varphi(e_i, X)) \\
&\qquad - \lambda^2 g(\nabla_{e_i} \operatorname{grad} f, \nabla_{e_i} X) - \lambda^2 g(\nabla_{e_i}\nabla_{e_i} \operatorname{grad} f, X) \\
&\quad = 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) + h(\nabla_X \mathrm{d}\varphi(\nabla_{e_i} \operatorname{grad} f), \mathrm{d}\varphi(e_i)) \\
&\qquad - X(\lambda^2 \Delta(f)) \\
&\quad = 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) + h(\nabla \mathrm{d}\varphi(X, \nabla_{e_i} \operatorname{grad} f), \mathrm{d}\varphi(e_i)) \\
&\qquad - X(\lambda^2 \Delta(f)) + \lambda^2 g(\nabla_X \nabla_{e_i} \operatorname{grad} f, e_i) \\
&\quad = 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) - X(\lambda^2 \Delta(f)) + \lambda^2 g(\nabla_X \nabla_{e_i} \operatorname{grad} f, e_i) \\
&\qquad + h(\nabla_{\nabla_{e_i} \operatorname{grad} f} \mathrm{d}\varphi(X), \mathrm{d}\varphi(e_i)) - \lambda^2 g(\nabla_{\nabla_{e_i} \operatorname{grad} f} X, e_i) \\
&\quad = 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) - X(\lambda^2 \Delta(f)) + \lambda^2 g(\nabla_X \nabla_{e_i} \operatorname{grad} f, e_i) \\
&\qquad + \nabla_{e_i} \operatorname{grad} f(\lambda^2 g(X, e_i)) - h(\mathrm{d}\varphi(X), \nabla_{\nabla_{e_i} \operatorname{grad} f} \mathrm{d}\varphi(e_i)) \\
&\qquad - \lambda^2 g(\nabla_{\nabla_{e_i} \operatorname{grad} f} X, e_i) \\
&\quad = 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) - X(\lambda^2 \Delta(f)) + \lambda^2 g(\nabla_X \nabla_{e_i} \operatorname{grad} f, e_i) \\
&\qquad - h(\mathrm{d}\varphi(X), \nabla \mathrm{d}\varphi(e_i, \nabla_{e_i} \operatorname{grad} f)) + \nabla_{e_i} \operatorname{grad} f(\lambda^2) g(X, e_i)
\end{aligned}$$

from which

$$2h(\langle \nabla \mathrm{d}\varphi, \nabla df\rangle, \mathrm{d}\varphi(X)) = 4\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) - 2\lambda^2 \Delta(f) g(\operatorname{grad}\ln\lambda, X)$$

□

Proof of Theorem 4. From formula (30), the bitension field of ϕ is given by

$$\tau_2(\phi) = -J_\varphi(\mathrm{d}\varphi(\operatorname{grad}_M \ln(\mu))) - n\nabla^\varphi_{\operatorname{grad}_M \ln f} \mathrm{d}\varphi(\operatorname{grad}_M \ln(\mu))$$

where $\mu = \lambda^{2-m} f^n$. Then ϕ is a biharmonic map if and only if

$$\ell(\tau_2(\phi), d\phi(X)) = 0$$

for each $X \in \Gamma(T(M \times N))$. We have

$$\begin{aligned}
J_\varphi(\mathrm{d}\varphi(\operatorname{grad}_M \ln(\mu))) &= \mathrm{d}\varphi(\operatorname{grad}_M \Delta_M(\ln\mu)) + 2\mathrm{d}\varphi(\mathrm{Ricci}^M(\operatorname{grad}_M \ln\mu)) \\
&\quad + \nabla^M_{\operatorname{grad}_M \ln\mu} \tau(\varphi) + 2\langle \nabla^M \mathrm{d}\varphi, \nabla(d^M \ln\mu)\rangle
\end{aligned}$$

(see [**12**, formula (2.47)]), hence

$$
\begin{aligned}
&\ell(\tau_2(\phi), d\phi(X))\\
&\qquad = \underbrace{\ell(\mathrm{d}\varphi(\operatorname{grad}\Delta(\ln\mu)), \mathrm{d}\phi(X))}_{T_1} + 2\underbrace{\ell(\mathrm{d}\varphi(\mathrm{Ricci}^M(\operatorname{grad}\ln\mu)), \mathrm{d}\phi(X))}_{T_2}\\
&\qquad + \underbrace{\ell(\nabla_{\operatorname{grad}\ln\mu}\tau(\varphi), \mathrm{d}\phi(X))}_{T_3} + \underbrace{2\ell(\langle\nabla\mathrm{d}\varphi, \nabla d\ln\mu\rangle, \mathrm{d}\phi(X))}_{T_4}\\
&\qquad + \underbrace{n.\ell(\nabla^{\varphi}_{\operatorname{grad}_M \ln f}\mathrm{d}\varphi(\operatorname{grad}_M \ln(\mu)), \mathrm{d}\phi(X))}_{T_5}.
\end{aligned}
\tag{36}
$$

Calculating each term of the above equation, we get

$$
\begin{aligned}
T_1 &= \lambda^2 g(\operatorname{grad}\Delta(\ln\mu), X_1) = \lambda^2 g(\operatorname{grad}\Delta(\ln\lambda^{2-m}f^n), X_1)\\
T_2 &= 2\lambda^2 g(\mathrm{Ricci}^M(\operatorname{grad}\ln\lambda^{2-m}f^n), X_1).
\end{aligned}
$$

From formula (32) of Lemma 2, we obtain

$$
\begin{aligned}
T_3 &= \ell(\nabla_{\operatorname{grad}\ln\mu}\tau(\varphi), \mathrm{d}\phi(X))\\
&= (2-m)\ell(\nabla_{\operatorname{grad}\ln\mu}\mathrm{d}\varphi(\operatorname{grad}\ln\lambda), \mathrm{d}\phi(X))\\
&= \lambda^2(2-m) \mid \operatorname{grad}\ln\lambda \mid^2 g(\operatorname{grad}\ln\mu, X_1)\\
&\quad + n(2-m)g(\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda)\\
&\quad + \frac{(2-m)^2}{2}g(\operatorname{grad}(\mid \operatorname{grad}\ln\lambda\mid^2), X_1)
\end{aligned}
$$

and

$$
\begin{aligned}
T_5 = \lambda^2\Big[&n[(2-m) \mid \operatorname{grad}\ln\lambda\mid^2 + 2d^M \ln f(\operatorname{grad}\ln\lambda)]g(\operatorname{grad}\ln f, X_1)\\
&+ \frac{n^2}{2}g(\operatorname{grad}(\mid\operatorname{grad}\ln f\mid^2), X_1) + n(2-m)g(\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda, X_1)\\
&- n^2 \mid \operatorname{grad}\ln f\mid^2 g(\operatorname{grad}\ln\lambda, X_1),
\end{aligned}
$$

using formula (35) of Lemma 3, we deduce

$$
\begin{aligned}
T_4 &= 2\ell(\langle\nabla\mathrm{d}\varphi, \nabla d\ln\mu\rangle, \mathrm{d}\phi(X))\\
&= 4n\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f, X_1) + 2(2-m)\lambda^2 g(\operatorname{grad}(\mid\operatorname{grad}\ln\lambda\mid^2), X_1)\\
&\quad - 2\lambda^2\Delta(\ln\mu)g(\operatorname{grad}\ln\lambda, X_1)
\end{aligned}
$$

Substituting T_1, T_2, T_3, T_4 and T_5 in (36), we obtain

$$
\begin{aligned}
0 = {} & \operatorname{grad}(\Delta\ln\lambda^{2-m}f^n) + 2\,\mathrm{Ricci}^M(\operatorname{grad}\ln\lambda^{2-m}f^n)\\
&+ 2n(2-m)\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda + 4n\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f\\
&+ \frac{n^2}{2}\operatorname{grad}(\mid\operatorname{grad}\ln f\mid^2) + \frac{(6-m)(2-m)}{2}\operatorname{grad}(\mid\operatorname{grad}\ln\lambda\mid^2)\\
&+ \big[(2-m)^2 \mid\operatorname{grad}\ln\lambda\mid^2 - n^2 \mid\operatorname{grad}\ln f\mid^2 + 2\Delta(\ln\lambda^{2-m}f^n)\big]\operatorname{grad}\ln\lambda\\
&+ 2n\big[(2-m) \mid\operatorname{grad}\ln\lambda\mid^2 + nd^M\ln f(\operatorname{grad}\ln\lambda)\big]\operatorname{grad}\ln f
\end{aligned}
$$

□

From Theorem 4, we deduce the following corollary.

Corollary 2. *Let* $\varphi\colon (M^m, g) \longrightarrow (P^m, \ell)$ $(m \geq 3)$ *be a conformal map with dilation* λ. *If* φ *is a biharmonic, not harmonic map, then*

$$\phi\colon (x, y) \in (M \times_f N, G_f) \longrightarrow \phi(x, y) = \varphi(x) \in (P, \ell)$$

is biharmonic if and only if the following equation

$$\begin{aligned} 0 = {} & \operatorname{grad}(\Delta \ln f) + 2\operatorname{Ricci}^M(\operatorname{grad}\ln f) + 2(2-m)\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda \\ & + 2(2-m) \mid \operatorname{grad}\ln\lambda \mid^2 \operatorname{grad}\ln f - 2\Delta(\ln f)\operatorname{grad}\ln\lambda \\ & + 2nd\ln f(\operatorname{grad}\ln\lambda)\operatorname{grad}\ln f - n \mid \operatorname{grad}\ln f \mid^2 \operatorname{grad}\ln\lambda \\ & + 4\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f + \frac{n}{2}\operatorname{grad}(\mid \operatorname{grad}\ln f \mid^2) \end{aligned}$$

is verified.

Example 1. We consider the inversion map $\varphi\colon \mathbb{R}^m - \{0\} \longrightarrow \mathbb{R}^m - \{0\}$ defined by

$$\varphi(x) = \frac{x}{\mid x \mid^2}$$

φ is a conformal map with the dilation

$$\lambda(x) = \frac{1}{\mid x \mid^2} = \frac{1}{r^2}.$$

φ is biharmonic not harmonic map if and only if $m = 4$ (see [**12**]). Let

$$\begin{aligned} \phi\colon (\mathbb{R}^4 - \{0\}) \times_f N^n & \longrightarrow (\mathbb{R}^4 - \{0\}) \\ (x, y) & \longmapsto \frac{x}{\mid x \mid^2} \end{aligned}$$

and $f = \mathrm{e}^{\alpha(r)}$, where $r = |x|$ and $\alpha \in C^\infty([0, +\infty[, \mathbb{R})$. We have:

$$\begin{aligned} \operatorname{grad}\ln f &= \alpha' \frac{\partial}{\partial r} \\ \mid \operatorname{grad}\ln f \mid^2 &= (\alpha')^2 \\ \operatorname{grad}(\mid \operatorname{grad}\ln f \mid^2) &= 2\alpha'\alpha'' \frac{\partial}{\partial r} \\ \Delta \ln f &= \alpha'' + \frac{3}{r}\alpha' \\ \operatorname{grad}(\Delta \ln f) &= \left(\alpha''' + \frac{3}{r}\alpha'' - \frac{3}{r^2}\alpha'\right)\frac{\partial}{\partial r}. \end{aligned}$$

Let $\ln\lambda = \beta(r)$. So ϕ is biharmonic if and only if α satisfies the following ordinary differential equation

$$\alpha''' + n\alpha'\alpha'' - \frac{1}{r}\alpha'' - \frac{15}{r^2}\alpha' - \frac{2n}{r}(\alpha')^2 = 0. \tag{37}$$

From which we obtain

$$f(x) = |x|^{-\frac{4}{n}}.$$

From Theorem 2, we deduce the following theorem.

Theorem 5. *Let $\psi\colon (N,h) \longrightarrow (P,\ell)$ be a regular map and $f\colon M \times N \to \mathbb{R}$ be a smooth positive function, then the bitension field of*

$$\phi\colon (x,y) \in (M \times_f N, G_f) \longrightarrow \phi(x,y) = \psi(y) \in (P,\ell)$$

is given by

$$\begin{aligned}\tau_2(\phi) = &+ \frac{1}{f^4}\Big[\tau_2(\psi) - (n-2)J_\psi(\mathrm{d}\psi(\mathrm{grad}_N \ln f)) - (n-6)\nabla^\psi_{\mathrm{grad}_N \ln f} W \\ &- \big(2(4-n) \mid \mathrm{grad}_N \ln f \mid^2 - 2\Delta_N(\ln f)\big)W\Big] \\ &- \frac{2}{f^2}\Big[(2-n) \mid \mathrm{grad}_M \ln f \mid^2 - \Delta_M(\ln f)\Big]W.\end{aligned}$$

Corollary 3. *If ψ is a conformal map with dilation μ, then*

$$\begin{aligned}\tau_2(\phi) = &- \frac{n-2}{f^4}\left[J_\psi\left(\mathrm{d}\psi\left(\mathrm{grad}_N\left(\ln\frac{f}{\mu}\right)\right)\right)\right. \\ &+ (n-6)\nabla^{\phi_N}_{\mathrm{grad}_N \ln f}\mathrm{d}\psi\left(\mathrm{grad}_N\left(\ln\frac{f}{\mu}\right)\right) \\ &\left.+ \big(2(4-n) \mid \mathrm{grad}_N \ln f \mid^2 - 2\Delta_N(\ln f)\big)\mathrm{d}\psi\left(\mathrm{grad}_N\left(\ln\frac{f}{\mu}\right)\right)\right] \\ &- \frac{2(n-2)}{f^2}\Big[(2-n) \mid \mathrm{grad}_M \ln f \mid^2 - \Delta_M(\ln f)\Big]\mathrm{d}\psi\left(\mathrm{grad}_N\left(\ln\frac{f}{\mu}\right)\right).\end{aligned}$$

Corollary 4. *The tension and bitension fields of the second projection η are given by the following formulae*

$$\tau(\eta) = \frac{n-2}{f^2}\,\mathrm{grad}_N \ln f$$

and

$$\begin{aligned}\tau_2(\eta) = &- \frac{n-2}{f^4}\Big[\mathrm{grad}_N(\Delta_N(\ln f)) + 2\,\mathrm{Ricci}(\mathrm{grad}_N)\ln f \\ &+ \frac{n-6}{2}\,\mathrm{grad}_N(\mid \mathrm{grad}_N \ln f \mid^2) + \big((4-n) \mid \mathrm{grad}_N \ln f \mid^2\ \Big] \\ &+ \frac{n-2}{f^4}\Delta_N(\ln f)\big)\,\mathrm{grad}_N \ln f + \frac{2(n-2)^2}{f^2} \mid \mathrm{grad}_M \ln f \mid^2 \\ &+ \frac{2(n-2)}{f^2}\Delta_M(\ln f)\,\mathrm{grad}_N \ln f.\end{aligned}$$

Theorem 6. *Let $\psi\colon N \to N$ be a harmonic map, then the bitension field of $\phi\colon (x,y) \in (M \times_f N, G_f) \longrightarrow (x,\psi(y)) \in (M \times N, G)$ is given by the following* *formula*

(38)

$$\begin{aligned}\tau_2(\phi) = & -\Big(n.\operatorname{grad}_M(\Delta(\ln f)) + 2n\operatorname{Ricci}^M(\operatorname{grad}_M \ln f), 0\Big)\\ & -\frac{n^2}{2}\Big(\operatorname{grad}_M(\mid \operatorname{grad}_M \ln f\mid^2), 0\Big)\\ & +\frac{n-2}{f^4}\Big(0, J_\psi(\mathrm{d}\psi(\operatorname{grad}_N \ln f)) - (n-6)\nabla^\psi_{\operatorname{grad}_N \ln f}\mathrm{d}\psi(\operatorname{grad}_N \ln f)\Big)\\ & +\frac{2(n-2)}{f^4}\Big[\Delta_N(\ln f) + f^2\Delta_M(\ln f) + (n-4)\mid \operatorname{grad}_N \ln f\mid^2\\ & +(n-2)\mid \operatorname{grad}_M \ln f\mid^2\Big](0, \mathrm{d}\psi(\operatorname{grad}_N \ln f))\end{aligned}$$

Proof. From Proposition 5, we obtain

$$\tau(\phi) = n(\operatorname{grad}_M \ln f, 0) + \frac{n-2}{f^2}(0, \mathrm{d}\psi(\operatorname{grad}_N \ln f)).$$

$$\begin{aligned}&\operatorname{trace}_{G_f}(\nabla^\phi)^2(\tau(\phi))\\ &\quad = n.\operatorname{trace}_g(\nabla^M)^2\Big(\operatorname{grad}_M \ln f, 0\Big)\\ &\qquad +\frac{n^2}{2}\Big(\operatorname{grad}_M(\mid \operatorname{grad}_M \ln f\mid^2), 0\Big)\\ &\qquad +\frac{n-2}{f^4}\Big(0, \operatorname{trace}_h(\nabla^\psi)^2\mathrm{d}\psi(\operatorname{grad}_N \ln f)\Big)\\ &\qquad +\frac{(n-2)(n-6)}{f^4}\Big(0, \nabla^\psi_{\operatorname{grad}_N \ln f}\mathrm{d}\psi(\operatorname{grad}_N \ln f)\Big)\\ &\qquad -\frac{2(n-2)}{f^4}\Big[\Delta_N(\ln f) + (n-4)\mid \operatorname{grad}_N \ln f\mid^2\\ &\qquad + f^2\Delta_M(\ln f) + (n-2)\mid \operatorname{grad}_M \ln f\mid^2\Big](0, \mathrm{d}\psi(\operatorname{grad}_N \ln f))\end{aligned} \tag{39}$$

and

$$\begin{aligned}trG_f\widetilde{R}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi = & \; n(\operatorname{Ricci}^M(\operatorname{grad}_M \ln f), 0)\\ & + \frac{n-2}{f^4}(0, tr_h R^N(\mathrm{d}\psi(\operatorname{grad}_N \ln f))).\end{aligned} \tag{40}$$

Substituting (39) and (40) in Jacobi formula

$$J_\phi(\tau(\phi)) = -\operatorname{trace}_{G_f}(\nabla^\phi)^2(\tau(\phi)) - \operatorname{trace} G_f\widetilde{R}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi,$$

we deduce formula (5). □

Corollary 5. *If N is a surface of dimension 2 ($\dim N = 2$), then the bitension field of ϕ is given by*

$$\tau_2(\phi) = -2\Big(\operatorname{grad}_M(\Delta_M(\ln f)) + 2\operatorname{Ricci}^M(\operatorname{grad}_M \ln f) + \operatorname{grad}_M(\mid \operatorname{grad}_M \ln f\mid^2), 0\Big).$$

Example 2. Let $M = \mathbb{R}^n - \{0\}$, $\dim N = 2$ and $\psi\colon N \to N$ be a harmonic map. Then the tension and the bitension fields of

$$\begin{aligned}\phi\colon \mathbb{R}^n - \{0\} \times_f N &\longrightarrow \mathbb{R}^n - \{0\} \times N\\ (x,y) &\longmapsto (x, \psi(y))\end{aligned}$$

are given by the following equations:

$$\begin{aligned}\tau(\phi) &= 2(\operatorname{grad}_M \ln f, 0)\\ \tau_2(\phi) &= 2(\operatorname{grad}_M(\Delta_M(\ln f)) + \operatorname{grad}_M(|\operatorname{grad}_M \ln f\,|^2)),\end{aligned}$$

hence ϕ is biharmonic not harmonic if and only if

$$\begin{cases}\Delta_M(\ln f) + |\operatorname{grad}_M \ln f\,|^2 = \beta(y) \quad \text{(independent of } x\text{)},\\ \operatorname{grad}_M \ln f \neq 0\end{cases}.$$

If $f \in C^\infty(\mathbb{R}^n - \{0\})$, thus the domain of ϕ is a warped product, such as $\ln f$ is a radial function ($\ln f = \alpha(|x|)$, then ϕ is biharmonic not harmonic if and only if

$$f(x) = k|x|^{(2-n)}, \quad (k \neq 0)$$

where $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$

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