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A-STATISTICAL KOROVKIN-TYPE APPROXIMATION THEOREM FOR FUNCTIONS OF TWO VARIABLES ON AN INFINITE INTERVAL

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Abstract. In this paper, using the concept of A-statistical convergence for double sequences, we provide a Korovkin-type approximation theorem for positive linear operators on the space of all real-valued uniform continuous functions on $[0,\infty)\times[0,\infty)$ with the property that have a finite limit at the infinity. Then, we display an application which shows that our new result is stronger than its classical version.

1. Introduction

For a sequence (L_n) of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [**15**] established first the sufficient conditions for the uniform convergence of $L_n(f)$ to a function f by using the test function 1, x, x^2 (see, for instance, [**5**]). Later many researchers have investigated these conditions for various operators defined on different spaces (see, for instance, [**1**], [**10**]). Using the concept of statistical convergence in approximation theory provides us with many advantages. In particular, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence of various sequences of linear operators such as the interpolation operator of Hermite-Fejér [**3**], because these types of operators do not converge at points of simple discontinuity. Furthermore, in recent years, with the help of the concept of uniform statistical convergence, which is a regular (non-matrix) summability transformation, various statistical approximation results were proved [**2**], [**7**], [**8**], [**9**], [**14**]. Then, it was demonstrated that those results are more powerful than the classical Korovkin theorem. In this paper, using the concept of A-statistical convergence for double sequences and test functions 1, e^{-x}, e^{-y} and $\mathrm{e}^{-2x}+\mathrm{e}^{-2y}$, we provide a Korovkin-type approximation for positive linear operators on the space $UC_*(D)$, the Banach space of all real-valued uniform continuous functions on $D:=[0,\infty)\times[0,\infty)$ with the property that $\lim_{(x,y)\to(\infty,\infty)} f(x,y)$ exists and is finite, endowed with the supremum norm $\|f\|=\sup_{(x,y)\in D}|f(x,y)|$

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for $f \in UC_*(D)$. Then, we display an application which shows that our new result is stronger than its classical version.

We now recall some basic definitions and notations used in the paper.

A double sequence $x = \{x_{m,n}\}$, $m, n \in \mathbb{N}$, is convergent in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. Then, L is called the Pringsheim limit of x and is denoted by $P - \lim x = L$ (see [**18**]). In this case, we say that $x = \{x_{m,n}\}$ is "P-convergent to L". Also, if there exists a positive number M such that $|x_{m,n}| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = \{x_{m,n}\}$ is said to be bounded. Recall that if a single sequence is convergent, then it is also bounded. But, this case does not hold for a double sequence, i.e., the convergence in Pringsheim's sense of a double sequence does not imply the boundedness of the double sequence.

Now let $A = [a_{j,k,m,n}]$, $j, k, m, n \in \mathbb{N}$, be a four-dimensional summability matrix. For a given double sequence $x = \{x_{m,n}\}$, the A-transform of x, denoted by $Ax := \{(Ax)_{j,k}\}$, is given by

$$(Ax)_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \qquad j, k \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(j, k) \in \mathbb{N}^2$. In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization of regularity for two dimensional matrix transformations is known as Silverman-Toeplitz conditions (see, for instance, [**13**]). In 1926, Robison [**19**] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption can be made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four dimensional matrices is known as Robison-Hamilton conditions, or briefly, RH-regularity (see, [**12**], [**19**]).

Recall that a four dimensional matrix $A = [a_{j,k,m,n}]$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = [a_{j,k,m,n}]$ is RH-regular if and only if

(i) $P - \lim\limits_{j,k} a_{j,k,m,n} = 0$ for each $(m, n) \in \mathbb{N}^2$,

(ii) $P - \lim\limits_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} = 1$,

(iii) $P - \lim\limits_{j,k} \sum_{m\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,

(iv) $P - \lim\limits_{j,k} \sum_{n\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{j,k,m,n}|$ is P-convergent for each $(j, k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that $\sum_{m,n>B} |a_{j,k,m,n}| < A$ holds for every $(j, k) \in \mathbb{N}^2$.

Now let $A = [a_{j,k,m,n}]$ be a nonnegative RH-regular summability matrix and let $K \subset \mathbb{N}^2$. Then, a real double sequence $x = \{x_{m,n}\}$ is said to be A-statistically convergent to a number L if for every $\varepsilon > 0$,

$$P - \lim_{j,k} \sum_{(m,n) \in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}.$$

In this case we write $\mathrm{st}^2_{(A)} - \lim_{m,n} x_{m,n} = L$. Observe that a P-convergent double sequence is A-statistically convergent to the same value, but the converse is not always true.

We should note that if we take $A = C(1,1)$ which is the double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for double sequence which was introduced in [**16**], [**17**]. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence.

2. A Korovkin-Type Theorem

Let L be a linear operator from $UC_* (D)$ into itself. Then, as usual, we say that L is a positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $(x,y) \in D$ by $L(f; x, y)$.

For single sequence Boyanov and Veselinov [**4**] proved the Korovkin theorem on $C_* [0,\infty)$ which is the Banach space of all real-valued continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and finite, endowed with the supremum norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$ for $f \in C_*[0,\infty)$, by using the test function 1, e^{-x}, e^{-2x}. Then, using the concept of A-statistical convergence for single sequences, Duman, Demirci and Karakuş [**6**] have obtained the following theorem on $UC_*[0,\infty)$ which is the Banach space of all real-valued uniform continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and finite, endowed with the supremum norm $\|f\|_* = \sup_{x\in[0,\infty)} |f(x)|$ for $f \in UC_*[0,\infty)$.

Theorem 2.1 ([**6**]). *Let $A = (a_{jn})$ be a nonnegative regular summability matrix and let $\{L_n\}$ be a sequence of positive linear operators mapping from $UC_*[0,\infty)$ into itself. Then, for all $f \in UC_*[0,\infty)$,*

$$\mathrm{st}_A - \lim_{n\to\infty} \|L_n(f) - f\|_* = 0$$

if and only if the following statements hold

$$\mathrm{st}_A - \lim_{n\to\infty} \left\|L_n\left(\mathrm{e}^{-kt}\right) - \mathrm{e}^{-kx}\right\|_* = 0, \qquad k = 0,1,2.$$

We note that Boyanov and Veselinov [**4**] considered the usual continuity instead of uniform continuity. In this case, δ may depend on the points x, t, the uniform approximation in [**4**, Theorem 2] may be invalid.

Now we have the following main result.

Theorem 2.2. *Let $A = [a_{j,k,m,n}]$ be a nonnegative RH-regular summability matrix. Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from*

$UC_*(D)$ into itself. Then, for all $f \in UC_*(D)$

$$\mathrm{st}^2_{(A)} - \lim_{m,n} \|L_{m,n}(f) - f\| = 0$$

if and only if the following statements hold

a) $\mathrm{st}^2_{(A)} - \lim_{m,n} \|L_{m,n}(1) - 1\| = 0,$

b) $\mathrm{st}^2_{(A)} - \lim_{m,n} \|L_{m,n}(\mathrm{e}^{-u}) - \mathrm{e}^{-x}\| = 0,$

c) $\mathrm{st}^2_{(A)} - \lim_{m,n} \|L_{m,n}(\mathrm{e}^{-v}) - \mathrm{e}^{-y}\| = 0,$

d) $\mathrm{st}^2_{(A)} - \lim_{m,n} \left\|L_{m,n}\left(\mathrm{e}^{-2u} + \mathrm{e}^{-2v}\right) - \left(\mathrm{e}^{-2x} + \mathrm{e}^{-2y}\right)\right\| = 0.$

Proof. Since the necessity is clear, then it is enough to prove sufficiency. Assume that the conditions (a), (b), (c) and (d) are satisfied. Let $f \in UC_*(D)$. There exists a constant M such that $|f(x,y)| \leq M$ for each $(x,y) \in D$. Let ε be an arbitrary positive number. By hypothesis we may find $\delta := \delta(\varepsilon) > 0$ such that if $|\mathrm{e}^{-u} - \mathrm{e}^{-x}| < \delta$ and $|\mathrm{e}^{-v} - \mathrm{e}^{-y}| < \delta$ for every (x,y), $(u,v) \in D$, then $|f(u,v) - f(x,y)| < \varepsilon$ (Here, we should remark that the number δ just depends on ε due to uniform continuity). Then the following inequality holds

$$|f(u,v) - f(x,y)| < \varepsilon + \frac{2M}{\delta^2}\left[\left(\mathrm{e}^{-u} - \mathrm{e}^{-x}\right)^2 + \left(\mathrm{e}^{-v} - \mathrm{e}^{-y}\right)^2\right]$$

for all (x,y), $(u,v) \in D$. Using the linearity and the positivity of the operators $L_{m,n}$, we get for any $(m,n) \in \mathbb{N}^2$ that

$$\begin{aligned}
&|L_{m,n}(f;x,y) - f(x,y)| \\
&\quad \leq L_{m,n}(|f(u,v) - f(x,y)|;x,y) + |f(x,y)|\,|L_{m,n}(1;x,y) - 1| \\
&\quad \leq L_{m,n}\left(\varepsilon + \frac{2M}{\delta^2}\left[\left(\mathrm{e}^{-u} - \mathrm{e}^{-x}\right)^2 + \left(\mathrm{e}^{-v} - \mathrm{e}^{-y}\right)^2\right];x,y\right) \\
&\qquad + |f(x,y)|\,|L_{m,n}(1;x,y) - 1| \\
&\quad \leq \varepsilon + (\varepsilon + M)\,|L_{m,n}(1;x;y) - 1| \\
&\qquad + \frac{2M}{\delta^2} L_{m,n}\left(\left[\left(\mathrm{e}^{-u} - \mathrm{e}^{-x}\right)^2 + \left(\mathrm{e}^{-v} - \mathrm{e}^{-y}\right)^2\right];x;y\right) \\
&\quad \leq \varepsilon + (\varepsilon + M)\,|L_{m,n}(1;x,y) - 1| + \frac{2M}{\delta^2}\left|\mathrm{e}^{-2x} + \mathrm{e}^{-2y}\right|\,|L_{m,n}(1;x,y) - 1| \\
&\qquad + \frac{2M}{\delta^2}\left|L_{m,n}\left(\mathrm{e}^{-2u} + \mathrm{e}^{-2v};x,y\right) - \left(\mathrm{e}^{-2x} + \mathrm{e}^{-2y}\right)\right| \\
&\qquad + \frac{4M}{\delta^2}\left|\mathrm{e}^{-x}\right|\left|L_{m,n}\left(\mathrm{e}^{-u};x,y\right) - \mathrm{e}^{-x}\right| + \frac{4M}{\delta^2}\left|\mathrm{e}^{-y}\right|\left|L_{m,n}\left(\mathrm{e}^{-v};x,y\right) - \mathrm{e}^{-y}\right| \\
&\quad \leq \varepsilon + \left(\varepsilon + M + \frac{4M}{\delta^2}\right)|L_{m,n}(1;x,y) - 1| \\
&\qquad + \frac{4M}{\delta^2}\left|L_{m,n}\left(\mathrm{e}^{-u};x,y\right) - \mathrm{e}^{-x}\right| + \frac{4M}{\delta^2}\left|L_{m,n}\left(\mathrm{e}^{-v};x,y\right) - \mathrm{e}^{-y}\right| \\
&\qquad + \frac{2M}{\delta^2}\left|L_{m,n}\left(\mathrm{e}^{-2u} + \mathrm{e}^{-2v};x,y\right) - \left(\mathrm{e}^{-2x} + \mathrm{e}^{-2y}\right)\right|
\end{aligned}$$

where $\left|e^{-kt}\right| \leq 1$ for all $t \in [0,\infty)$ and $k \in \mathbb{N}$. Then taking the supremum over $(x,y) \in D$, we have

$$\begin{aligned} \|L_{m,n}(f) - f\| \leq \varepsilon + K \{ & \|L_{m,n}(1) - 1\| \\ & + \left\|L_{m,n}\left(e^{-u}\right) - e^{-x}\right\| + \left\|L_{m,n}\left(e^{-v}\right) - e^{-y}\right\| \\ & + \left\|L_{m,n}\left(e^{-2u} + e^{-2v}\right) - \left(e^{-2x} + e^{-2y}\right)\right\| \} \end{aligned} \tag{2.1}$$

where $K := \max\left\{\varepsilon + M + \frac{4M}{\delta^2}, \frac{4M}{\delta^2}, \frac{2M}{\delta^2}\right\}$. For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. Define the following sets:

$$D := \left\{(m,n) \in \mathbb{N}^2 \colon \|L_{m,n}(f) - f\| \geq r\right\},$$

$$D_1 := \left\{(m,n) \in \mathbb{N}^2 \colon \|L_{m,n}(1) - 1\| \geq \frac{r-\varepsilon}{4K}\right\},$$

$$D_2 := \left\{(m,n) \in \mathbb{N}^2 \colon \left\|L_{m,n}\left(e^{-u}\right) - e^{-x}\right\| \geq \frac{r-\varepsilon}{4K}\right\},$$

$$D_3 := \left\{(m,n) \in \mathbb{N}^2 \colon \left\|L_{m,n}\left(e^{-v}\right) - e^{-y}\right\| \geq \frac{r-\varepsilon}{4K}\right\},$$

$$D_4 := \left\{(m,n) \in \mathbb{N}^2 \colon \left\|L_{m,n}\left(e^{-2u} + e^{-2v}\right) - \left(e^{-2x} + e^{-2y}\right)\right\| \geq \frac{r-\varepsilon}{4K}\right\}.$$

It follows from (2.1) that $D \subset D_1 \cup D_2 \cup D_3 \cup D_4$. Therefore, for each $(m,n) \in \mathbb{N}^2$, we may write

$$\begin{aligned} \sum_{(m,n)\in D} a_{j,k,m,n} \leq \sum_{(m,n)\in D_1} a_{j,k,m,n} & + \sum_{(m,n)\in D_2} a_{j,k,m,n} \\ & + \sum_{(m,n)\in D_3} a_{j,k,m,n} + \sum_{(m,n)\in D_4} a_{j,k,m,n}. \end{aligned} \tag{2.2}$$

From (2.2), using (a), (b), (c) and (d), we conclude that

$$P - \lim_{j,k} \sum_{(m,n)\in D} a_{j,k,m,n} = 0$$

whence the result. □

If we replace the matrix A in Theorem 2.2 by identity double matrix, then we immediately get the following classical result:

Theorem 2.3. *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $UC_*(D)$ into itself. Then, for all $f \in UC_*(D)$,*

$$P - \lim_{m,n} \|L_{m,n}(f) - f\| = 0$$

if and only if the following statements hold:

a) $P - \lim_{m,n} \|L_{m,n}(1) - 1\| = 0,$

b) $P - \lim_{m,n} \|L_{m,n}(e^{-u}) - e^{-x}\| = 0,$

c) $P - \lim_{m,n} \|L_{m,n}(e^{-v}) - e^{-y}\| = 0,$

d) $P-\lim\limits_{m,n}\left\|L_{m,n}\left(\mathrm{e}^{-2u}+\mathrm{e}^{-2v}\right)-\left(\mathrm{e}^{-2x}+\mathrm{e}^{-2y}\right)\right\|=0.$

Remark 2.1. Now, we exhibit an example of a double sequence of positive linear operators of two variables satisfying the conditions of Theorem 2.2, but that does not satisfy the conditions of Theorem 2.3. We consider the following Baskakov operators (see [**11**])

$$B_{m,n}\left(f;x,y\right)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}f\left(\frac{j}{m},\frac{k}{n}\right)\binom{m-1+j}{j}\binom{n-1+k}{k} \cdot\left(1+x\right)^{-m-j}\left(1+y\right)^{-n-k}x^{j}y^{k}, \tag{2.3}$$

where $(x,y)\in D$, $f\in UC_{*}\left(D\right)$. Also, observe that

$$\begin{aligned}
B_{m,n}\left(1;x,y\right)&=1,\\
B_{m,n}\left(\mathrm{e}^{-u};x,y\right)&=\left(1+x-x\,\mathrm{e}^{-\frac{1}{m}}\right)^{-m},\\
B_{m,n}\left(\mathrm{e}^{-v};x,y\right)&=\left(1+y-y\,\mathrm{e}^{-\frac{1}{n}}\right)^{-n},\\
B_{m,n}\left(\mathrm{e}^{-2u}+\mathrm{e}^{-2v};x,y\right)&=\left(1+x-x\,\mathrm{e}^{-\frac{1}{m}}\right)^{-m}+\left(1+y-y\,\mathrm{e}^{-\frac{1}{n}}\right)^{-n}.
\end{aligned}$$

Then, by Theorem 2.3, we get that for any $f\in UC_{*}\left(D\right)$,

$$P-\lim_{m,n}\left\|L_{m,n}\left(f\right)-f\right\|=0.$$

Now we take $A=C\left(1,1\right)$ and define a double sequence $(\alpha_{m,n})$ by

$$\alpha_{m,n}=\begin{cases}1, & \text{if } m \text{ and } n \text{ are squares,}\\ 0, & \text{otherwise.}\end{cases} \tag{2.4}$$

It is clear that

$$\mathrm{st}^{2}_{(C(1,1))}-\lim_{m,n}\alpha_{m,n}=0. \tag{2.5}$$

Now using (2.3) and (2.4), we define the following positive linear operators on $UC_{*}\left(D\right)$ as follows:

$$L_{m,n}\left(f;x,y\right)=\left(1+\alpha_{m,n}\right)B_{m,n}\left(f;x,y\right). \tag{2.6}$$

So, by the Theorem 2.2 and (2.5), we see that

$$\mathrm{st}^{2}_{(C(1,1))}-\lim_{m,n}\left\|L_{m,n}\left(f\right)-f\right\|=0.$$

Also, since $(\alpha_{m,n})$ is not $P-$convergent, we say that the Theorem 2.3 does not work for our operators defined by (2.6).

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