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ON INTEGERS EXPRESSIBLE BY SOME SPECIAL LINEAR FORM

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Abstract. Let $E(4)$ be the set of positive integers expressible by the form $4M-d$, where M is a multiple of the product ab and d is a divisor of the sum $a+b$ of two positive integers a, b. We show that the set $E(4)$ does not contain perfect squares and three exceptional positive integers $288, 336, 4545$ and verify that $E(4)$ contains all other positive integers up to $2 \cdot 10^9$. We conjecture that there are no other exceptional integers. This would imply the Erdős-Straus conjecture asserting that each number of the form $4/n$, where $n \geq 2$ is a positive integer, is the sum of three unit fractions $1/x+1/y+1/z$. We also discuss similar problems for sets $E(t)$, where $t \geq 3$, consisting of positive integers expressible by the form $tM-d$. The set $E(5)$ is related to a conjecture of Sierpiński, whereas the set $E(t)$, where t is any integer greater than or equal to 4, is related to the most general in this context conjecture of Schinzel.

1. Introduction

Let t be a fixed positive integer. In this paper we consider the set of positive integers

$$E(t) := \{n \,:\, n = tM - d\},$$

where M is a positive multiple of the product and d is a positive divisor of the sum of two positive integers, namely,

$$ab|M \quad \text{and} \quad d|(a+b)$$

for some $a, b \in \mathbb{N}$. Evidently,

$$E(t') \subseteq E(t) \quad \text{whenever} \quad t|t'.$$

It is easy to see that

$$E(1) = E(2) = \mathbb{N}. \tag{1}$$

Indeed, suppose first that $t = 1$. Then, for each $n \in \mathbb{N}$ selecting $a = 2n+1$, $b = 1$, $M = ab = 2n+1$ and $d = (a+b)/2 = n+1$, we find that

$$n = 2n + 1 - (n+1) = M - d,$$

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giving $E(1) = \mathbb{N}$. In case $t = 2$, for each $n \in \mathbb{N}$ we may choose $a = n + 1$, $b = 1$, $M = ab = n + 1$ and $d = a + b = n + 2$. Then $2M - d = 2(n + 1) - (n + 2) = n$, so that $E(2) = \mathbb{N}$.

Apart form (1) the situation with $t \geq 3$ is not clear. In this context, the sets $E(4)$ and $E(5)$ are of special interest because an integer n belongs to the set $E(t)$ if and only if

$$n = tM - d = tuab - (a + b)/v$$

with some $a, b, u, v \in \mathbb{N}$. Therefore, $n \in E(t)$ yields the representation

$$\frac{t}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

with positive integers

$$x := uab, \quad y := uvna, \quad z := uvnb.$$

Thus if $n \in E(t)$, then the fraction t/n is expressible by the sum of three unit fractions. In particular, if every prime number p belongs to the set $E(4)$, then the Erdős-Straus conjecture (asserting that for each integer $n \geq 2$, the fraction $4/n$ is expressible by the sum $1/x + 1/y + 1/z$ with $x, y, z \in \mathbb{N}$) is true, whereas if every prime number p belongs to $E(5)$, then the corresponding conjecture of Sierpiński (asserting that for each $n \geq 4$, the fraction $5/n$ is expressible by the sum $1/x + 1/y + 1/z$) is true [**10**]. In this context, the most general Schinzel's conjecture asserts that the fraction t/n for each $n \geq n(t)$ is expressible by the sum $1/x + 1/y + 1/z$. This clearly holds for $t \leq 3$, but is open for each fixed $t \geq 4$. Conjecture 5 given in Section 3 implies that there is an integer $C(t)$ such that each prime number $p > C(t)$ belongs to $E(t)$. This would imply Schinzel's conjecture as well.

Yamamoto [**12**], [**13**] and Mordell [**8**] observed that it is sufficient to prove the Erdős-Straus conjecture for those prime numbers p which modulo 840 are 1, 121, 169, 289, 361 or 529. Vaughan [**11**] showed that the Erdős-Straus conjecture is true for almost all positive integers n. See also the list of references in D11 for the literature concerning the conjectures of Erdős-Straus, Sierpiński and Schinzel on Egyptian fractions. More references on the Erdős-Straus (including recent ones) can be found in a paper of Elsholtz and Tao [**4**] on the average number of solutions of the equation $4/p = 1/x+1/y+1/z$ with prime numbers p. At the computational side the calculations of Swett `http://math.uindy.edu/swett/esc.htm` show that the Erdős-Straus conjecture holds for integers n up to 10^{14}.

In this note we observe that the following holds

Theorem 1. *The set $E(4)$ does not contain perfect squares and the numbers 288, 336, 4545.*

Suppose $k^2 \in E(4)$, i.e., there exist $u, v, a, b, k \in \mathbb{N}$ such that

$$v(4uab - k^2) = a + b. \tag{2}$$

To show that $k^2 \notin E(4)$, we shall use the following fact

Lemma 2. *The equation (2) has no solutions in positive integers u, v, a, b, k.*

Lemma 2 implies that $-d$ is a quadratic nonresidue modulo $4ab$ if $d|(a+b)$. Indeed, if the number $-d$ were a quadratic residue modulo $4ab$, then by selecting the positive integer $v := (a+b)/d$, we would see that the equation $k^2 = -d + 4uab$ with $u \in \mathbb{N}$ has a solution $k \in \mathbb{N}$, which is impossible in view of Lemma 2. Note that the set of divisors of $a+b$, when $a < b$ both run through the set $\{1, 2, \ldots, n\}$, contains the set $\{1, 2, \ldots, 2n-1\}$. Thus, by Lemma 2, it holds

Corollary 3. *For each positive integer n the $2n-1$ consecutive integers*

$$4n! - 2n + 1, 4n! - 2n + 2, \ldots, 4n! - 1$$

are quadratic nonresidues modulo $4n!$.

Corollary 3 gives the example of at least $(2-\varepsilon)\log m/\log\log m$ consecutive quadratic nonresidues modulo $m = 4n!$ (by a completely elementary method). In this direction, the most interesting problem is to determine how many consecutive quadratic residues and consecutive quadratic nonresidues modulo m may occur for prime numbers m. See, e.g., [**3**], [**5**], where it is shown that we have at least $c_1 \log m \log\log\log m$ consecutive quadratic residues modulo m for infinitely many primes m, and [**7**], where the factor $\log\log\log m$ is replaced by $\log\log m$ under assumption of the generalized Riemann hypothesis.

A set of positive integers which is a subset of $\cup_{q=0}^{\infty} E(4q+3)$ was recently considered in [**1**]. For $M = ab$ and $d = a+b$, where a, b are positive integers and $b \equiv 3 \pmod 4$, put

$$E^*(t) := \{n \,:\, n = tab - a - b\}.$$

Evidently, $E^*(t) \subseteq E(t)$. In [**1**] it was shown that the set $E := \cup_{q=0}^{\infty} E^*(4q+3)$ does not contain perfect squares and that all prime numbers of the form $4s+1$ less than 10^{10} belong to E.

2. Proof of Theorem 1

Lemma 2 was apparently first proved by Yamamoto [**13**]. See also [**9**, Lemma 2] and [**4**, Proposition 1.6]. Here is a short proof.

Since $a = vd - b$, equality (2) yields

$$k^2 = 4u(vd-b)b - d = (4buv-1)d - 4b^2u.$$

So if (2) has a solution in positive integers, then the Jacobi symbol $\left(\frac{-4b^2u}{4buv-1}\right) = \left(\frac{k^2}{4buv-1}\right)$ must be equal to 1. Indeed, since $-4b^2u$ and $4buv-1$ are relatively prime, we have $\left(\frac{-4b^2u}{4buv-1}\right) \neq 0$, and so $\left(\frac{k^2}{4buv-1}\right) = 1$. We will show, however, that the Jacobi symbol $\left(\frac{-4b^2u}{4buv-1}\right)$ is equal to -1. Indeed, write $u \in \mathbb{N}$ in the form $u = 2^r u_0$, where $r \geq 0$ is an integer and $u_0 \geq 1$ is an odd integer. Using $\left(\frac{-1}{4buv-1}\right) = -1$ and also $\left(\frac{2}{4buv-1}\right) = 1$ in case u is even, i.e., $r \geq 1$, we find that

$$\left(\frac{-4b^2u}{4buv-1}\right) = \left(\frac{-2^{r+2}b^2u_0}{4buv-1}\right) = -\left(\frac{2^r u_0}{4buv-1}\right) = -\left(\frac{u_0}{4buv-1}\right).$$

Further, by the quadratic reciprocity law, in view of $u_0|u$ we conclude that

$$-\left(\frac{u_0}{4buv-1}\right) = -(-1)^{(u_0-1)/2}\left(\frac{4buv-1}{u_0}\right) = -(-1)^{(u_0-1)/2}\left(\frac{-1}{u_0}\right) = -1.$$

Lemma 2 implies that $k^2 \notin E(4)$. To complete the proof of Theorem 1 we need to show that $288, 336, 4545 \notin E(4)$.

The case $n = 288$ can be easily checked 'by hand'. Observe that $288 = 4M - d$ implies that $d = 4s$ and $M = (288 + 4s)/4 = 72 + s$. Furthermore, from

$$72 + s = M \geq ab \geq a + b - 1 \geq d - 1 = 4s - 1$$

we find that $1 \leq s \leq 24$. So for each $s = 1, 2, \ldots, 24$, it remains to check that there are no positive integers a, b for which $4s|(a + b)$ and $ab|(72 + s)$.

Note first that for $s \geq 11$ we must have $a + b = 4s$ and $ab = 72 + s$. Indeed, if $a + b > 4s$, then $a + b \geq 8s$ and so

$$72 + s = M \geq ab \geq a + b - 1 \geq 8s - 1,$$

which is impossible, because $s \geq 11$. If $ab < 72 + s$, then $2ab \leq 72 + s$, so that

$$72 + s \geq 2ab \geq 2(a + b - 1) \geq 2(d - 1) = 2(4s - 1) = 8s - 2,$$

which is a contradiction again. However, from $a+b = 4s$ and $ab = 72+s$ it follows that

$$(4s)^2 - 4(72 + s) = 4(4s^2 - s - 72)$$

is a perfect square. So $4s^2 - s - 72$ must be a perfect square. It remains to check the values of s between 11 and 24 which modulo 4 are 0 or 3, namely, $s = 11, 12, 15, 16, 19, 20, 23, 24$. For none these values, $4s^2 - s - 72$ is a perfect square.

The values of s between 1 and 10 can also be excluded, because there are no a, b with $ab|(72 + s)$ for which $4s$ divides $a + b$; see the Table 1 below.

To complete the proof of the theorem, observe that if $n = tM - d$, then

$$n \geq tab - a - b \geq ta^2 - 2a$$

in case $a \leq b$. Hence $(at - 1)^2 \leq nt + 1$ and $b \leq (a + n)/(ta - 1)$. Therefore, all values from 1 to 10000 which do not belong to $E(4)$ can be found with Maple as follows:

```
for n from 1 to 10000 do s := true;
   for a from 1 by 1 while (s and (at − 1)^2 ≤ tn + 1) do
      B := (a + n)/(ta − 1);
      for b from a by 1 while (s and b ≤ B) do
         if a + b (mod tab − n (mod tab)) = 0 then s := false endif;
      endfor;
   endfor;
   if s then k := k + 1;
      print(n);
   endif;
endfor;
print(k):
```

s	1	2	3	4	5	6	7	8	9	10
$4s$	4	8	12	16	20	24	28	32	36	40
$72+s$	73	$2\cdot 37$	$3\cdot 5^2$	$2^2\cdot 19$	$7\cdot 11$	$2\cdot 3\cdot 13$	79	$2^4\cdot 5$	3^4	$2\cdot 41$

Table 1

For every particular value of n from 1 to 10000, we check all the pairs (a, b) which satisfy the above inequalities for the existence of an appropriate value of d, i.e., for the divisibility of $a+b$ by some positive integer of the form $d = tuab - n$. However, $d \leq a + b \leq tab$ which means that there is only one possible such integer d. Take a unique integer in the interval $[1, tab]$ which equals $-n$ modulo tab. It can be expressed as $tab - n \pmod{tab}$, as in the pseudocode describing our algorithm above. To obtain a code for Maple, one only needs to change both 'endfor' and 'endif' to 'end'.

As a result (in less than three seconds), we got that only 100 perfect squares and three exceptional numbers 288, 336, 4545 less than 10000 do not lie in $E(4)$. This completes the proof of Theorem 1.

The calculation to the bound 10^6 with Maple took us almost 40 minutes, so all the calculations of the next section to the bound $2 \cdot 10^9$ have been performed with C++.

3. Some speculations concerning the sets $E(t)$

As we already observed in (1), the sets $\mathbb{N} \setminus E(1)$ and $\mathbb{N} \setminus E(2)$ are empty. By Lemma 2, the equation $v(4uab - k^2) = a + b$ has no solutions in positive integers u, v, a, b, k. In particular, if t is a positive integer divisible by 4 and $s \in \mathbb{N}$ is such that $4s|t$, then the equation $vs(4(t/4s)uab - k^2) = a + b$ has no solutions in positive integers u, v, a, b, k. The latter is equivalent to the equation $v(tuab - sk^2) = a + b$. Consequently, we obtain the following corollary

Corollary 4. *The set $E(t)$, where $4|t$, does not contain the numbers of the form sk^2, where $s \in \mathbb{N}$ satisfies $4s|t$ and $k \in \mathbb{N}$.*

In particular, this implies that the set $\mathbb{N} \setminus E(t)$ is infinite when $4|t$. We conjecture that all other sets, namely, $\mathbb{N} \setminus E(t)$ with $t \in \mathbb{N}$ which is not a multiple of 4 are finite. More precisely, we get next conjecture

Conjecture 5. *There exists an integer $C(t) \in \mathbb{N} \cup \{0\}$ such that the set $E(t)$ contains all integers greater than or equal to $C(t) + 1$ if 4 does not divide t and all integers greater than or equal to $C(t) + 1$ except for sk^2, where $4s|t$ and $k \in \mathbb{N}$, if $4|t$.*

By (1), we have $C(1) = C(2) = 0$. It is known that the total number of representations of t/n by the sum $1/x + 1/y + 1/z$ does not exceed $c(\varepsilon)(n/t)^{2/3}n^{\varepsilon}$, where $\varepsilon > 0$ (see [**2**]). We know that if $n \in E(t)$ then t/n is expressible by the sum of three unit fractions, so this bound also holds for the number of representations of n in the form $tM - d$. On the other hand, by the above mentioned result of

Vaughan [**11**], almost all positive integers are expressible by the sum of three unit fractions. It is easy to see that for each fixed $t \geq 3$, almost all positive integers belong to the set $E(t)$.

In fact, one can easily show a much stronger statement that almost all positive integers can be written in the form $pa-1$ with some prime number $p \equiv -1 \pmod{t}$ and some $a \in \mathbb{N}$. If $n \in \mathbb{N}$ can be written in this way, then

$$n = pa - 1 = (p+1)a - a - 1 = tM - d \in E(t)$$

with $b = 1$, $d = a + 1$ and $M = (p+1)a/t$. By the above, it suffices to show that the density of positive integers n that have no prime divisors of the form $p \equiv -1 \pmod{t}$ is zero. This can be easily done using a standard sieve argument. Let $p_1 < p_2 < p_3 < \dots$ denote consecutive primes in the arithmetic progression $kt-1$, $k = 1, 2, 3, \dots$. By Dirichlet's theorem, the sum $\sum_{j=1}^{\infty} 1/p_j$ diverges. Thus for each $\varepsilon > 0$, we can pick $s \in \mathbb{N}$ for which $\prod_{j=1}^{s}(1 - 1/p_j) < \varepsilon/2$. Further, for each $N \geq P := p_1 p_2 \dots p_s$, select a unique $k \in \mathbb{N}$ for which $kP \leq N < (k+1)P$. The number of positive integers $n \leq N$ without prime divisors in the set $\{p_1, \dots, p_s\}$ does not exceed the number of such positive integers in the interval $[1, (k+1)P]$. The latter, by the inclusion-exclusion principle, is equal to

$$(k+1)P \prod_{j=1}^{s} \Big(1 - \frac{1}{p_j}\Big) \leq \frac{(k+1)P\varepsilon}{2} \leq \frac{(1+1/k)N\varepsilon}{2} \leq \frac{2N\varepsilon}{2} = N\varepsilon.$$

This implies the claim.

Coming back to Conjecture 5, by calculation with C++, in the range $[1, 2 \cdot 10^9]$ we found only three exceptional integers 6, 36, 3600 which do not belong to the set $E(3)$. So we conjecture that

$$E(3) = \mathbb{N} \setminus \{6, 36, 3600\} \quad \text{and} \quad C(3) = 3600.$$

For $t = 4$, we have

$$288,\ 336,\ 4545,\ \mathbb{N}^2 \in \mathbb{N} \setminus E(4),$$

and we conjecture that $C(4) = 4545$.

There are much more integers which do not lie in $E(5)$. In the range $[1, 2 \cdot 10^9]$ there are 48 such integers:

1, 2, 5, 6, 10, 12, 20, 21, 30, 32, 45, 46, 50, 60, 92, 102, 105, 126, 141, 182, 192, 210, 282, 320, 330, 366 ,406, 600, 650, 726, 732, 842, 846, 920, 992, 1020, 1446, 1452, 1905, 1920, 2100, 2250, 2262, 3962, 7320, 9050, 11520, 40500.

We conjecture that this list is full, i.e., $C(5) = 40500$. The list of integers in $[1, 2 \cdot 10^9]$ which do not lie in $E(6)$ contains 108 numbers, the largest one being 684450. We are more cautious to claim that $C(6) = 684450$ since this number is quite large compared to the computation bound $2 \cdot 10^9$. Here is a result of our calculations with C++ for $3 \leq t \leq 9$.

In Table 2 for $t = 4$, all squares k^2 are excluded, whereas for $t = 8$, all squares k^2 and all numbers of the form $2k^2$ are excluded (see Corollary 4 and Conjecture 5).

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t	computation bound	number of exceptions	largest exception
3	$2 \cdot 10^9$	3	3600
4	$2 \cdot 10^9$	3	4545
5	$2 \cdot 10^9$	48	40500
6	$2 \cdot 10^9$	108	684450
7	10^9	270	9673776
8	10^9	335	3701376
9	10^9	932	18481050

Table 2

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