Acta Math. Univ. Comenianae
Vol. LXXXI, 2 (2012), pp. 247–254

CONGRUENCES OF STRONGLY MORITA EQUIVALENT POSEMIGROUPS

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ABSTRACT. We prove that congruence lattices of strongly Morita equivalent posemigroups with common joint weak local units are isomorphic. Moreover, the quotient posemigroups by the congruences that correspond to each other under this isomorphism are also strongly Morita equivalent.

1. INTRODUCTION

Morita theories have been studied for many different structures: for rings with or without identity, monoids, categories, etc. Our work belongs to the Morita theory of semigroups without identity, the study of which was initiated by Talwar ([**7**], [**8**]). Recently Tart (see [**9**]) initiated a research of Morita equivalent partially ordered semigroups (shortly posemigroups). One ingredient in Morita theories is the study of Morita invariants, these are the properties shared by all Morita equivalent structures. For example, a classical result about rings (see [**1**, Proposition 21.11]) states that Morita equivalent rings with identity have isomorphic ideal lattices. In [**10**] Tart considers Morita invariants of posemigroups. In [**6**], Morita invariants for unordered semigroups were considered, in particular it was proven that if two semigroups with certain kind of local units are strongly Morita equivalent then their congruence lattices are isomorphic. In this article we prove the analogue of that result for the ordered case.

Let S be a posemigroup. A left *S-poset* is a poset A together with a mapping (action) $S \times A \to A, (s,a) \mapsto sa$ such that (i) $(ss')a = s(s'a)$, (ii) $s \leq s'$ implies $sa \leq s'a$, (iii) $a \leq a'$ implies $sa \leq sa'$ for all $s, s' \in S$ and $a, a' \in A$. Right T-posets are defined similarly. A left S-poset and right T-poset A is called an *(S,T)-biposet* (and denoted by ${}_SA_T$) if $(sa)t = s(at)$ for all $s \in S$, $a \in A$ and $t \in T$. A *biposet morphism* has to preserve both actions and the order. A biposet ${}_SA_T$ is said to be *unitary* if $SA = A$ and $AT = A$.

The *tensor product* $A \otimes_T B$ of a right T-poset A and a left T-poset B is the quotient poset $(A \times B)/\sim$, where $(a,b) \sim (a',b')$ if $(a,b) \preceq (a',b')$ and $(a',b') \preceq (a,b)$,

Received February 8, 2012; revised May 7, 2012.
2010 *Mathematics Subject Classification.* Primary 06F05.
Key words and phrases. Posemigroup; Morita equivalence; congruence lattice.
Research of the second author was supported by the Estonian Science Foundation grant No. 8394 and Estonian Targeted Financing Project SF0180039s08.

and $(a,b) \preceq (a',b')$ iff there exist $t_1,\dots,t_n,w_1,\dots,w_n \in T^1$, $a_1,\dots,a_n \in A$ and $b_2,\dots,b_n \in B$ such that

$$\begin{array}{rclrcl} a & \leq & a_1t_1 & & & \\ a_1w_1 & \leq & a_2t_2 & t_1b & \leq & w_1b_2 \\ a_2w_2 & \leq & a_3t_3 & t_2b_2 & \leq & w_2b_3 \\ & \dots & & & \dots & \\ a_nw_n & \leq & a' & t_nb_n & \leq & w_nb' \end{array} \tag{1}$$

where $xu = x$ for every element $x \in \{a_1,\dots,a_n\}$ and $uy = y$ for every element $y \in \{b,b'\} \cup \{b_2,\dots,b_n\}$ if $u \in T^1$ is the externally adjoined identity. For $(a,b) \in A \times B$, the equivalence class $[(a,b)]_\sim$ is denoted by $a \otimes b$. The order relation on $A \otimes_T B$ is defined by setting

$$a \otimes b \leq a' \otimes b' \iff (a,b) \preceq (a',b')$$

for $a \otimes b, a' \otimes b' \in A \otimes_T B$.

If A is an (S,T)-biposet, then $A \otimes_T B$ is a left S-poset, where the action is defined by $s(a \otimes b) = (sa) \otimes b$. Similarly, if B is a (T,S)-biposet, then $A \otimes_T B$ is a right S-poset.

Definition 1 ([**8**], [**9**])**.** A *unitary Morita context* is a six-tuple $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$, where S and T are posemigroups, ${}_SP_T$ and ${}_TQ_S$ are unitary biposets and

$$\theta \colon {}_S(P \otimes_T Q)_S \to {}_SS_S, \quad \phi \colon {}_T(Q \otimes_S P)_T \to {}_TT_T$$

are biposet morphisms such that for every $p,p' \in P$ and $q,q' \in Q$,

$$\theta(p \otimes q)p' = p\phi(q \otimes p'), \quad q\theta(p \otimes q') = \phi(q \otimes p)q'. \tag{2}$$

Definition 2 ([**8**], [**9**])**.** Posemigroups S and T are called *strongly Morita equivalent* if there exists a unitary Morita context $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ such that the mappings θ and ϕ are surjective.

Let ρ be a reflexive binary relation on a poset A. For $a,a' \in A$ we write $a \underset{\rho}{\leq} a'$ if there exist $a_1,\dots,a_n \in A$ such that

$$a \leq a_1 \;\rho\; a_2 \leq a_3 \;\rho\; \dots \;\rho\; a_n \leq a'.$$

We note that the relation $\underset{\rho}{\leq}$ is reflexive and transitive.

Definition 3 ([**3**])**.** A *congruence* on a posemigroup S is an equivalence relation ρ on S such that

1. $s\rho s'$ implies $sx\rho s'x$ and $xs\rho xs'$ for every $s,s',x \in S$;
2. $s \underset{\rho}{\leq} s'$ and $s' \underset{\rho}{\leq} s$ implies $s\rho s'$ (the closed chains condition).

The multiplication of the *quotient posemigroup* S/ρ is defined as usual and the order is given by

$$[s]_\rho \leq [s']_\rho \iff s \underset{\rho}{\leq} s'.$$

Similarly, a *biposet congruence* is an equivalence relation that is compatible with both actions and satisfies the closed chains condition. We shall need biposet congruences induced by a binary relation. Our construction will be an analogue of the one given in [**2**]. Let ${}_SA_T$ be an (S,T)-biposet and let $H \subseteq A \times A$. Define a relation $\alpha(H)$ on A by $a\,\alpha(H)\,a'$ if and only if $a = a'$ or there exist a natural number n and $(x_i, x_i') \in H$, $u_i \in S^1$, $v_i \in T^1$, $i = 1, \dots, n$, such that

$$\begin{array}{llllllll} a = u_1x_1v_1 & & u_2x_2'v_2 = u_3x_3v_3 & \dots & & & u_nx_n'v_n = a' \\ & u_1x_1'v_1 = u_2x_2v_2 & & & & u_{n-1}x_{n-1}'v_{n-1} = u_nx_nv_n & \end{array}.$$

Note that the relation $\alpha(H)$ is reflexive, transitive and compatible with both actions. Therefore, the relation $\nu(H)$ defined on A by

$$a\,\nu(H)\,a' \Longleftrightarrow a \underset{\alpha(H)}{\leq} a' \text{ and } a' \underset{\alpha(H)}{\leq} a$$

is an (S,T)-biposet congruence. The relation $\nu(H)$ is called the (S,T)-*biposet congruence on* ${}_SA_T$ *induced by* H. We consider the quotient set $A/\nu(H)$ as an (S,T)-biposet with respect to the order given by

$$[a]_{\nu(H)} \leq [a']_{\nu(H)} \Longleftrightarrow a \underset{\alpha(H)}{\leq} a'$$

and naturally defined actions.

Definition 4 ([**6**])**.** A posemigroup S is said to have *common joint weak local units* if

$$(\forall s, s' \in S)(\exists u, v \in S)(s = usv \wedge s' = us'v).$$

As examples of semigroups with common joint weak local units we mention monoids, lower semilattices where every pair of elements has an upper bound (in particular lattices) and multiplicative semigroups of s-unital rings (in particular of rings with local units). Also, an ordinal sum of any set of semigroups with common joint weak local units is again a semigroup with common joint weak local units and a direct product of two semigroups with common joint weak local units is a semigroup with common joint weak local units.

2. The result

Theorem 1. *If S and T are strongly Morita equivalent posemigroups with common joint weak local units, then there exists an isomorphism $\Pi : \mathsf{Con}(S) \to \mathsf{Con}(T)$ of their congruence lattices. Moreover, if $\rho \in \mathsf{Con}(S)$, then the posemigroups S/ρ and $T/\Pi(\rho)$ are strongly Morita equivalent.*

Proof. Define the mappings $\Pi\colon \mathsf{Con}(S) \to \mathsf{Con}(T)$ and $\Omega\colon \mathsf{Con}(T) \to \mathsf{Con}(S)$ as follows:

$$x\Pi(\rho)y \Longleftrightarrow x \underset{\Pi_\rho}{\leq} y \text{ and } y \underset{\Pi_\rho}{\leq} x,$$

$$x\Omega(\tau)y \Longleftrightarrow x \underset{\Omega_\tau}{\leq} y \text{ and } y \underset{\Omega_\tau}{\leq} x,$$

where $\rho \in \mathsf{Con}(S)$, $\tau \in \mathsf{Con}(T)$ and

$$\begin{aligned}
\Pi_\rho &= \{(\phi(q \otimes sp), \phi(q \otimes s'p)) \mid (s, s') \in \rho, p \in P, q \in Q\} \quad &\subseteq T \times T,\\
\Omega_\tau &= \{(\theta(p \otimes tq), \theta(p \otimes t'q)) \mid (t, t') \in \tau, p \in P, q \in Q\} \quad &\subseteq S \times S.
\end{aligned}$$

We first show that the relation Π_ρ is reflexive and compatible with multiplication.

Let $t \in T$ be an arbitrary element and let $t', t'' \in T$ be such that $t = t't''$. Since ϕ is surjective, there exist $p', p'' \in P$ and $q', q'' \in Q$ such that $t' = \phi(q' \otimes p')$ and $t'' = \phi(q'' \otimes p'')$. Hence $t = \phi(q' \otimes p')\phi(q'' \otimes p'') = \phi(q' \otimes p'\phi(q'' \otimes p'')) = \phi(q' \otimes \theta(p' \otimes q'')p'')$. Because ρ is reflexive, $(\theta(p' \otimes q''), \theta(p' \otimes q'')) \in \rho$ and therefore $(t, t) = (\phi(q' \otimes \theta(p' \otimes q'')p''), \phi(q' \otimes \theta(p' \otimes q'')p'')) \in \Pi_\rho$. Thus Π_ρ is reflexive.

Let now $(\phi(q \otimes sp), \phi(q \otimes s'p)) \in \Pi_\rho$, where $(s, s') \in \rho$, and let $t = \phi(q_t \otimes p_t) \in T$. Then

$$\begin{aligned}
(\phi(q \otimes sp)t, \phi(q \otimes s'p)t) &= (\phi(q \otimes sp\phi(q_t \otimes p_t)), \phi(q \otimes s'p\phi(q_t \otimes p_t)))\\
&= (\phi(q \otimes s\theta(p \otimes q_t)p_t), \phi(q \otimes s'\theta(p \otimes q_t)p_t)) \in \Pi_\rho,
\end{aligned}$$

because $(s\theta(p \otimes q_t), s'\theta(p \otimes q_t)) \in \rho$. Similarly we can prove compatibility with multiplication from the left.

Analogously one can show that Ω_τ is reflexive and compatible with multiplication.

Now we show that the relations $\Pi(\rho)$ and $\Omega(\tau)$ are posemigroup congruences. Symmetricity and transitivity are clear from the definition. Reflexivity and compatibility with multiplication follow from the fact that the relations Π_ρ, Ω_τ and $\leq$ are reflexive and compatible with multiplication. Let us check the closed chains condition. First we note that

$$\begin{aligned}
t \underset{\Pi(\rho)}{\leq} t' &\Leftrightarrow \exists t_1, \ldots, t_n \in T : t \leq t_1 \Pi(\rho) t_2 \leq t_3 \Pi(\rho) \ldots \Pi(\rho) t_n \leq t'\\
&\Rightarrow t \leq t_1 \underset{\Pi_\rho}{\leq} t_2 \leq t_3 \underset{\Pi_\rho}{\leq} \ldots \underset{\Pi_\rho}{\leq} t_n \leq t'\\
&\Rightarrow t \underset{\Pi_\rho}{\leq} t'.
\end{aligned}$$

Analogously $t' \underset{\Pi(\rho)}{\leq} t$ implies $t' \underset{\Pi_\rho}{\leq} t$ and consequently, if $t \underset{\Pi(\rho)}{\leq} t' \underset{\Pi(\rho)}{\leq} t$, then $t \Pi(\rho) t'$. Similarly it can be proven that $s \underset{\Omega(\tau)}{\leq} s' \underset{\Omega(\tau)}{\leq} s$ implies $s \Omega(\tau) s'$. Thus we have seen that $\Pi(\rho)$ and $\Omega(\tau)$ are congruences.

Obviously Π and Ω preserve order. So it remains to prove that Π and Ω are inverses of each other. To prove that $\rho \subseteq (\Omega\Pi)(\rho)$, it suffices to show that $\rho \subseteq \Omega_{\Pi(\rho)}$. Let $(s, s') \in \rho$ and let $u, v \in S$ be such that $usv = s$ and $us'v = s'$. If $u = \theta(p_u \otimes q_u)$ and $v = \theta(p_v \otimes q_v)$, $p_u, p_v \in P, q_u, q_v \in Q$, then $(\phi(q_u \otimes sp_v), \phi(q_u \otimes s'p_v)) \in \Pi_\rho \subseteq \Pi(\rho)$. Hence

$$\begin{aligned}
(s, s') = (usv, us'v) &= (\theta(p_u \otimes q_u s\theta(p_v \otimes q_v)), \theta(p_u \otimes q_u s'\theta(p_v \otimes q_v)))\\
&= (\theta(p_u \otimes \phi(q_u \otimes sp_v)q_v), \theta(p_u \otimes \phi(q_u \otimes s'p_v)q_v)) \in \Omega_{\Pi(\rho)}.
\end{aligned}$$

Conversely, to prove the inclusion $(\Omega\Pi)(\rho) \subseteq \rho$ we first show that $\Omega_{\Pi(\rho)} \subseteq \rho$. Suppose that $(\theta(p \otimes tq), \theta(p \otimes t'q)) \in \Omega_{\Pi(\rho)}$ where $(t,t') \in \Pi(\rho)$. We shall prove that

$$(t,t') \in \Pi(\rho) \Longrightarrow \theta(p \otimes tq)\ \rho\ \theta(p \otimes t'q). \tag{3}$$

The assumption means that $t \underset{\Pi_\rho}{\leq} t'$ and $t' \underset{\Pi_\rho}{\leq} t$. The first fact means that there exist $u_1, \ldots, u_n, u_1', \ldots, u_n' \in T$ such that

$$t \leq u_1 \Pi_\rho u_1' \leq u_2 \Pi_\rho u_2' \leq \ldots \leq u_n \Pi_\rho u_n' \leq t'.$$

Hence, for every $i \in \{1, \ldots, n\}$, there exist $p_i \in P$, $q_i \in Q$ and $(s_i, s_i') \in \rho$ such that $u_i = \phi(q_i \otimes s_i p_i)$ and $u_i' = \phi(q_i \otimes s_i' p_i)$. Using this, we have

$$\begin{aligned}
\theta(p \otimes tq) &\leq \theta(p \otimes u_1 q) = \theta(p \otimes \phi(q_1 \otimes s_1 p_1) q) \\
&= \theta(p \otimes q_1 s_1 \theta(p_1 \otimes q)) = \theta(p \otimes q_1) s_1 \theta(p_1 \otimes q) \\
&\ \rho\ \theta(p \otimes q_1) s_1' \theta(p_1 \otimes q) = \theta(p \otimes q_1 s_1' \theta(p_1 \otimes q)) \\
&= \theta(p \otimes \phi(q_1 \otimes s_1' p_1) q) = \theta(p \otimes u_1' q) \\
&\leq \theta(p \otimes u_2 q)\ \rho\ \theta(p \otimes u_2' q) \\
&\leq \ldots \leq \theta(p \otimes t'q),
\end{aligned}$$

i.e., $\theta(p \otimes tq) \underset{\rho}{\leq} \theta(p \otimes t'q)$. Similarly $t' \underset{\Pi_\rho}{\leq} t$ implies $\theta(p \otimes t'q) \underset{\rho}{\leq} \theta(p \otimes tq)$. Since ρ is a congruence, $(\theta(p \otimes tq), \theta(p \otimes t'q)) \in \rho$, and therefore $\Omega_{\Pi(\rho)} \subseteq \rho$. If now $(x,y) \in \Omega(\Pi(\rho))$, then $x \underset{\Omega_{\Pi(\rho)}}{\leq} y \underset{\Omega_{\Pi(\rho)}}{\leq} x$, which implies $x \underset{\rho}{\leq} y \underset{\rho}{\leq} x$, and since ρ is a congruence, $(x,y) \in \rho$. Consequently, $(\Omega\Pi)(\rho) \subseteq \rho$ and we have proven the equality $(\Omega\Pi)(\rho) = \rho$.

The proof of the equality $(\Pi\Omega)(\tau) = \tau$ is symmetric.

Now let us show that if $\rho \in \mathsf{Con}(S)$, then S/ρ and $T/\Pi(\rho)$ are strongly Morita equivalent. Let $\rho \in \mathsf{Con}(S)$ and denote $\tau := \Pi(\rho) \in \mathsf{Con}(T)$. We need to construct a Morita context containing S/ρ and T/τ. For this we define the sets

$$\begin{aligned}
H &:= \{(sp, s'p) \mid (s,s') \in \rho, p \in P\} \cup \{(pt, pt') \mid (t,t') \in \tau, p \in P\} \subseteq P \times P, \\
K &:= \{(qs, qs') \mid (s,s') \in \rho, q \in Q\} \cup \{(tq, t'q) \mid (t,t') \in \tau, q \in Q\} \subseteq Q \times Q.
\end{aligned}$$

Furthermore, let $\mu = \nu(H)$ and $\lambda = \nu(K)$ be the biposet congruences on ${}_SP_T$ and ${}_TQ_S$ induced by H and K, respectively. On the quotient sets P/μ and Q/λ we define the actions of the quotient posemigroups S/ρ and T/τ as follows:

$$\begin{aligned}
[s]_\rho \cdot [p]_\mu &:= [sp]_\mu, & [p]_\mu \cdot [t]_\tau &:= [pt]_\mu, \\
[q]_\lambda \cdot [s]_\rho &:= [qs]_\lambda, & [t]_\tau \cdot [q]_\lambda &:= [tq]_\lambda,
\end{aligned}$$

$p \in P$, $q \in Q$, $s \in S$, $t \in T$. Let $s\rho s'$ and $p\mu p'$, $s, s' \in S$, $p, p' \in P$. Since $H \subseteq \mu$ and μ is a left S-poset congruence, we obtain $sp\ \mu\ s'p\ \mu\ s'p'$, and hence $sp\ \mu\ s'p'$. Similarly one can show that all the other definitions are correct. Obviously we obtain biacts. To prove that the first action is monotone in the first argument, we suppose that $[s]_\rho \leq [s']_\rho$ for $s, s' \in S$. Then $s \underset{\rho}{\leq} s'$, i.e. $s \leq s_1 \rho s_1' \leq s_2 \rho \ldots \rho s_n' \leq s'$ for some $s_1, \ldots, s_n, s_1', \ldots, s_n' \in S$. This implies for each $p \in P$, $sp \leq s_1 p\ H\ s_1' p \leq$

$\ldots \leq s_n p \; H \; s_n' p \leq s' p$, hence $sp \underset{\alpha(H)}{\leq} s'p$ and $[sp]_\mu \leq [s'p]_\mu$. On the other hand, assuming that $s \in S$, $p, p' \in P$ and $[p]_\mu \leq [p']_\mu$, we have $p \underset{\alpha(H)}{\leq} p'$. The last inequality clearly implies $sp \underset{\alpha(H)}{\leq} sp'$, and so $[sp]_\mu \leq [sp']_\mu$. Thus we have obtained an $(S/\rho, T/\tau)$-biposet P/μ. Analogously, Q/λ is a $(T/\tau, S/\rho)$-biposet. Unitarity of P/μ and Q/λ follows from the unitarity of P and Q.

Define a mapping $\overline{\theta} : P/\mu \otimes Q/\lambda \to S/\rho$ by

$$\overline{\theta}([p]_\mu \otimes [q]_\lambda) := [\theta(p \otimes q)]_\rho,$$

$p \in P$, $q \in Q$. Let us prove that $\overline{\theta}$ preserves the order. First we notice that, for all $p \in P$, $q \in Q$, $s, s' \in S$, $u \in S^1$, $t, t' \in T$, $v \in T^1$,

$$(s, s') \in \rho \implies \theta(usp \otimes q) \; \rho \; \theta(us'p \otimes q), \tag{4}$$

$$(t, t') \in \tau \implies \theta(ptv \otimes q) \; \rho \; \theta(pt'v \otimes q). \tag{5}$$

The first implication holds because θ is a left S-poset homomorphism and ρ is compatible with multiplication. For the second implication we use that τ is compatible with multiplication and (3) holds.

Next we show that for all $x, x', p \in P$, $y, y', q \in Q$,

$$[x]_\mu \leq [x']_\mu \implies \theta(x \otimes q) \underset{\rho}{\leq} \theta(x' \otimes q), \tag{6}$$

$$[y]_\lambda \leq [y']_\lambda \implies \theta(p \otimes y) \underset{\rho}{\leq} \theta(p \otimes y'). \tag{7}$$

If $[x]_\mu \leq [x']_\mu$, then $x \underset{\alpha(H)}{\leq} x'$ and there exist $x_1, \ldots, x_n \in P$ such that

$$x \leq x_1 \alpha(H) x_1' \leq x_2 \alpha(H) x_2' \leq \ldots \leq x_n \alpha(H) x_n' \leq x',$$

where for each $j \in \{1, \ldots, n\}$ there exist a natural number n_j and $(s_i, s_i') \in \rho$, $(t_i, t_i') \in \tau$, $p_i \in P$, $u_i \in S^1$, $v_i \in T^1$, $i = 1, \ldots, n_j$, such that

$$\begin{array}{rcl} x_j & = & u_1 s_1 p_1 v_1 \qquad u_2 p_2 t_2' v_2 \;=\; u_3 s_3 p_3 v_3 \;\ldots\; u_{n_j} p_{n_j} t_{n_j}' v_{n_j} \;=\; x_j'. \\ & & u_1 s_1' p_1 v_1 \;=\; u_2 p_2 t_2 v_2. \end{array}$$

Using (4) and (5), we obtain

$$\begin{aligned} \theta(x_j \otimes q) \; &= \; \theta(u_1 s_1 p_1 v_1 \otimes q) \; \rho \; \theta(u_1 s_1' p_1 v_1 \otimes q) = \theta(u_2 p_2 t_2 v_2 \otimes q) \\ &\rho \;\; \theta(u_2 p_2 t_2' v_2 \otimes q) = \theta(u_3 s_3 p_3 v_3 \otimes q) \; \rho \ldots \rho \; \theta(x_j' \otimes q) \end{aligned}$$

which implies

$$\theta(x \otimes q) \leq \theta(x_1 \otimes q) \; \rho \; \theta(x_1' \otimes q) \leq \theta(x_2 \otimes q) \; \rho \; \theta(x_2' \otimes q) \leq \ldots \leq \theta(x' \otimes q).$$

Hence $\theta(x \otimes q) \underset{\rho}{\leq} \theta(x' \otimes q)$. The proof of the implication (7) is analogous.

Suppose now that $[p]_\mu \otimes [q]_\lambda \leq [p']_\mu \otimes [q']_\lambda$ in $P/\mu \otimes Q/\lambda$. From (1) we obtain $p_1, \dots, p_n \in P$, $q_2, \dots, q_n \in Q$, $t_1, \dots, t_n, w_1, \dots, w_n \in T^1$ such that

$$\begin{array}{rclcrcl} [p]_\mu & \leq & [p_1]_\mu [t_1]_\tau & & & & \\ [p_1]_\mu [w_1]_\tau & \leq & [p_2]_\mu [t_2]_\tau & & [t_1]_\tau [q]_\lambda & \leq & [w_1]_\tau [q_2]_\lambda \\ [p_2]_\mu [w_2]_\tau & \leq & [p_3]_\mu [t_3]_\tau & & [t_2]_\tau [q_2]_\lambda & \leq & [w_2]_\tau [q_3]_\lambda \\ & \dots & & & & \dots & \\ [p_n]_\mu [w_n]_\tau & \leq & [p']_\mu & & [t_n]_\tau [q_n]_\lambda & \leq & [w_n]_\tau [q']_\lambda. \end{array}$$

Using the implications (6) and (7), we obtain

$$\begin{aligned} \theta(p \otimes q) & \underset{\rho}{\leq} \theta(p_1 t_1 \otimes q) = \theta(p_1 \otimes t_1 q) \\ & \underset{\rho}{\leq} \theta(p_1 \otimes w_1 q_2) = \theta(p_1 w_1 \otimes q_2) \\ & \underset{\rho}{\leq} \dots \underset{\rho}{\leq} \theta(p' \otimes q'), \end{aligned}$$

and therefore $[\theta(p \otimes q)]_\rho \leq [\theta(p' \otimes q')]_\rho$. So $\overline{\theta}$ preserves the order. Let us show that $\overline{\theta}$ is a biposet morphism. For every $s, s' \in S$,

$$\begin{aligned} \overline{\theta}\left([s]_\rho \left([p]_\mu \otimes [q]_\lambda\right)\right) & = \overline{\theta}([sp]_\mu \otimes [q]_\lambda) = [\theta(sp \otimes q)]_\rho = [s\theta(p \otimes q)]_\rho \\ & = [s]_\rho [\theta\left(p \otimes q\right)]_\rho = [s]_\rho \overline{\theta}\left([p]_\mu \otimes [q]_\lambda\right). \end{aligned}$$

Similarly one can show that $\overline{\theta}$ preserves the right action. Surjectivity of $\overline{\theta}$ follows from the surjectivity of θ. Analogously one can construct a surjective morphism $\overline{\phi} : Q/\lambda \otimes P/\mu \to T/\tau$ of $(T/\tau, T/\tau)$-biposets. The equalities (2) are easy to check. $\square$

If a posemigroup S has an identity element 1 and ${}_S A$ is a left S-poset then $SA = A$ if and only if $1a = a$ for every $a \in A$. Thus the S-poset ${}_S A$ over a monoid S is unitary if and only if it is an S-poset in the sense of [**5**]. From Theorem 6 of [**5**] it follows that two posemigroups S and T with identity elements are strongly Morita equivalent if and only if they are Morita equivalent as pomonoids (in the sense of [**5**]). So we have the following corollary.

Corollary 1. *Congruence lattices of Morita equivalent pomonoids are isomorphic.*

In [**4**] one can find a list of non-isomorphic Morita equivalent monoids. These can be considered as Morita equivalent pomonoids with trivial order, and hence Corollary 1 applies to them. Moreover, an example of non-isomorphic Morita equivalent pomonoids with non-trivial order is given in [**5**].

Suppose that semigroups S and T with common joint weak local units are strongly Morita equivalent. We may consider S and T as posemigroups with trivial order and they will be strongly Morita equivalent as posemigroups. By Theorem 1 their lattices of posemigroup congruences are isomorphic. But for semigroups with trivial order the posemigroup congruences are precisely the semigroup congruences. Hence we have the following result.

Corollary 2 ([**6**]). *Congruence lattices of strongly Morita equivalent semigroups with common joint weak local units are isomorphic.*

In Theorem 1, we have proven that congruence lattices of strongly Morita equivalent posemigroups with common joint weak local units are isomorphic. As pointed out in [**10**], in general congruence lattices of strongly Morita equivalent posemigroups need not be isomorphic.

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