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NONUNIFORM INSTABILITY FOR EVOLUTION FAMILIES

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Abstract. The aim of this paper is to study the relationship between the notions of nonuniform exponential instability for evolutionary processes and admissibility of the pair of spaces $(\mathcal{C}_{00}(\mathbb{R}_+, X), \mathcal{C}(\mathbb{R}_+, X))$. The evolutionary processes considered are the most general in the sense that they do not require uniform or nonuniform growth. A necessary condition for expansiveness of the evolutionary processes is given.

1. Introduction

Consider the non-autonomous linear equation

$$(*) \qquad \frac{\mathrm{d}}{\mathrm{d}t}u(t) = A(t)u(t), \quad t \in \mathbb{R}_+ \text{ or } \mathbb{R},$$

on a Banach space X.

In 1930 O. Perron was the first who made the connection between the asymptotic properties of the solutions of $(*)$ and some specific properties of the operator

$$Lu(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t) - A(t)u(t), \quad t \in \mathbb{R}_+ \text{ or } \mathbb{R}$$

on a space of X valued functions. The solutions of $(*)$ generate an evolution family (evolutionary process) $(U(t,s))_{t\geq s}$ of bounded linear operators on X.

In fact, the differential equation is not the one investigated in most cases. Instead, the following integral equation is studied

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)\mathrm{d}\xi,$$

along with its connection with asymptotic properties of the evolutionary process $(U(t,s))_{t\geq s}$.

The researches on this subject were later developed by W. A. Coppel [**5**] and P. Hartman [**8**] for differential systems in finite dimensional spaces.

Further developments for differential systems in infinite dimensional spaces can be found in the monographs of J. L. Daleckij, M. G. Krein [**6**] and J. L. Massera, J. J. Schäffer [**11**]. The case of dynamical systems described by evolutionary

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processes was studied by C. Chicone, Y. Latushkin [**4**] and B. M. Levitan, V. V. Zhikov [**10**].

Other results concerning uniform exponential stability, exponential dichotomy and admissibility of exponentially bounded evolution families were obtained by N. van Minh [**13**], [**14**], [**15**], F. Räbiger [**15**], R. Schnaubelt [**15**], Y. Latushkin [**9**], T. Randolph [**9**], R. Nagel [**7**].

We also mention the results of L. Barreira and C. Valls [**1**], [**2**], [**3**], who analyzed evolutionary processes with nonuniform growth of the form $||\Phi(t,t_0)|| \leq M(t_0)\,\mathrm{e}^{\omega(t-t_0)}$ for all $t \geq t_0 \geq 0$, $M\colon \mathbb{R}_+ \to \mathbb{R}_+^*$, $\omega > 0$, and obtained nonuniform contraction conditions (i.e. nonuniform stability) and nonuniform dichotomy in terms of admissibility of some pairs of function spaces.

In the present paper we give a necessary condition for the nonuniform exponential instability of an evolutionary process. We take into consideration evolutionary processes in general, i.e. those which do not have any growth (uniform or nonuniform). Our technique is completely new and differs significantly from those used in the extended literature [**2**], [**3**], [**15**], [**14**], [**12**] devoted to this subject.

2. Preliminaries

Let X be a Banach space and $\mathcal{B}(X)$ the space of all linear and bounded operators acting on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $||\cdot||$.

Definition 2.1. A function $\Phi\colon \Delta = \{(t,t_0) \in \mathbb{R}^2\colon 0 \leq t_0 \leq t\} \to \mathcal{B}(X)$ is called an evolutionary process iff

1. $\Phi(t,t) = I$ for all $t \geq 0$,
2. $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$ for all $t \geq s \geq t_0 \geq 0$,
3. $t \mapsto \Phi(t,t_0)x : [t_0,\infty) \to X$ is continuous for every $x \in X$ and
   $s \mapsto \Phi(t,s)x : [0,t] \to X$ is continuous for every $x \in X$.

We consider the following function spaces (endowed with the sup-norm $|||\cdot|||$):

$$\begin{aligned}
\mathcal{C}(\mathbb{R}_+,X) &= \{f\colon \mathbb{R}_+ \to X : f \text{ is continuous and bounded}\},\\
\mathcal{C}_0(\mathbb{R}_+,X) &= \{f \in \mathcal{C}(\mathbb{R}_+,X) : \lim_{t\to\infty} f(t) = 0\},\\
\mathcal{C}_{00}(\mathbb{R}_+,X) &= \{f \in \mathcal{C}_0(\mathbb{R}_+,X) :\ f(0) = 0\},\\
\mathcal{C}^{t_0}(\mathbb{R}_+,X) &= \{f\colon [t_0,\infty) \to X : \exists u \in \mathcal{C}(\mathbb{R}_+,X) \text{ such that } u|_{[t_0,\infty)} = f\},\\
\mathcal{C}_{00}^{t_0}(\mathbb{R}_+,X) &= \{f\colon [t_0,\infty) \to X : \exists v \in \mathcal{C}_{00}(\mathbb{R}_+,X) \text{ such that } v|_{[t_0,\infty)} = f\}.
\end{aligned}$$

Definition 2.2. The evolutionary process Φ satisfies the Perron condition for instability (the pair of spaces $(\mathcal{C}_{00}(\mathbb{R}_+,X),\mathcal{C}(\mathbb{R}_+,X))$ is admissible to Φ) if for all $f \in \mathcal{C}_{00}(\mathbb{R}_+,X)$ there is a unique element $x \in X$ such that the function

$$x_f(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau$$

is in $\mathcal{C}(\mathbb{R}_+,X)$.

Proposition 2.1. *If the evolutionary process Φ satisfies the Perron condition for instability, then for all $f \in \mathcal{C}_{00}(\mathbb{R}_+, X)$, there is a unique function $u \in \mathcal{C}(\mathbb{R}_+, X)$ such that*

$$u(t) = \Phi(t,s)u(s) + \int_s^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau \quad \textit{for all } t \geq s \geq 0.$$

Proof. Existence. Let f be an element of $\mathcal{C}_{00}(\mathbb{R}_+, X)$ then, by Definition 2.2, there is a unique element x in X such that

$$x_f(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau$$

is in $\mathcal{C}(\mathbb{R}_+, X)$. If we put $u(t) = x_f(t)$ for all $t \geq 0$, then we have

$$x_f(t) = \Phi(t,s)\Phi(s,0)x + \int_0^s \Phi(t,s)\Phi(s,\tau)f(\tau)\mathrm{d}\tau + \int_s^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau.$$

Therefore

$$x_f = \Phi(t,s)x_f(s) + \int_s^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau \quad \text{for all } t \geq s \geq 0$$

and so $u = x_f$.

Uniqueness. We suppose there is another function $v \in \mathcal{C}(\mathbb{R}_+, X)$ such that

$$v(t) = \Phi(t,s)v(s) + \int_s^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau.$$

Let $w(t) = x_f(t) - v(t)$ for all $t \geq 0$, so w is an element of $\mathcal{C}(\mathbb{R}_+, X)$ and

$$w(t) = \Phi(t,s)w(s) + \int_s^t \Phi(t,\tau)0\mathrm{d}\tau \quad \text{for all } t \geq s \geq 0.$$

If we put $s = 0$, then

$$w(t) = \Phi(t,0)w(0) + \int_0^t \Phi(t,\tau)0\mathrm{d}\tau \quad \text{for all } t \geq 0.$$

But we also have

$$0 = \Phi(t,0)0 + \int_0^t \Phi(t,\tau)0\mathrm{d}\tau \quad \text{for all } t \geq 0,$$

therefore $w(0) = 0$, and this proves that $w(t) = 0$ for all $t \geq 0$ and $v = x_f$. □

Definition 2.3. The evolutionary process Φ is nonuniform exponential expansive if there is a function $N \colon \mathbb{R}_+ \to \mathbb{R}_+^*$ and a constant $\nu > 0$ such that

$$N(t)\|\Phi(t,t_0)x\| \geq \mathrm{e}^{\nu(t-t_0)}\,\|x\| \quad \text{for all } t \geq t_0 \geq 0 \text{ and } x \in X$$

and $\Phi(t,t_0)$ is surjective on X.

Remark 2.2. If $\Phi(t,t_0)$ is surjective on X, then for every $y \in X$ there is an element $x \in X$ such that $\Phi(t,t_0)x = y$. Then $x = \Phi^{-1}(t,t_0)y$.

If Φ is nonuniform exponential expansive, then we have

$$N(t)\|y\| \geq \mathrm{e}^{\nu(t-t_0)} \|\Phi^{-1}(t,t_0)y\| \quad \text{for all } t \geq t_0 \geq 0$$

and

$$\|\Phi^{-1}(t,t_0)y\| \leq N(t)\,\mathrm{e}^{-\nu(t-t_0)} \,||y|| \quad \text{for all } t \geq t_0 \geq 0 \text{ and } x \in X.$$

Proposition 2.3. *If $x \neq 0$, then $\Phi(t,0)x \neq 0$ for all $t \geq 0$.*

Proof. Indeed, if we assume that there is $t_0 > 0$ such that $\Phi(t_0,0)x = 0$, then $\Phi(t,0)x = 0$ for all $t \geq t_0 \geq 0$. This shows that $\Phi(\cdot,0)$ is an element of $\mathcal{C}(\mathbb{R}_+,X)$. However, $\Phi(t,0)x = \Phi(t,0)x + \int_0^t \Phi(t,\tau)0\mathrm{d}\tau$, so $0 = 0 + \int_0^t \Phi(t,\tau)0\mathrm{d}\tau$. Therefore $x = 0$, a contradiction. □

3. Results

Theorem 3.1. *If the evolutionary process Φ satisfies the Perron condition for instability, then there exists a constant $k > 0$ such that*

$$|||x_f||| \leq k|||f||| \quad \textit{for all } f \in \mathcal{C}_{00}(\mathbb{R}_+,X).$$

Proof. Let $\mathcal{U}\colon \mathcal{C}_{00}(\mathbb{R}_+,X) \to \mathcal{C}(\mathbb{R}_+,X)$, $\mathcal{U}f = x_f$. The operator $\mathcal{U}$ is well defined and we will show that it is closed.

In order to do that, we consider the sequence $(f_n)_n$ of functions in $\mathcal{C}_{00}(\mathbb{R}_+,X)$, $f_n \xrightarrow{\mathcal{C}_{00}(\mathbb{R}_+,X)} f$. We also assume that $\mathcal{U}f_n \xrightarrow{\mathcal{C}(\mathbb{R}_+,X)} g$ and we will prove that $\mathcal{U}f = g$.

We have that

$$\mathcal{U}f_n(t) = x_{f_n}(t) = \Phi(t,0)x_{f_n}(0) + \int_0^t \Phi(t,\tau)f_n(\tau)\mathrm{d}\tau.$$

Since $x_{f_n}(0) = \mathcal{U}_n f(0)$ and $\mathcal{U}f_n \xrightarrow{\mathcal{C}(\mathbb{R}_+,X)} g$, then $x_{f_n}(0) \xrightarrow[n\to\infty]{} g(0)$.

The application $\tau \mapsto \Phi(t,\tau)x\colon [0,t] \to X$ is continuous for all $x \in X$ and $t \geq 0$, so there exists $M_{t,x} > 0$ such that

$$\|\Phi(t,\tau)x\| \leq M_{t,x} \quad \text{for all } \tau \in [0,t].$$

By the Uniform Boundedness Principle, there exists a constant $M(t) > 0$ such that

$$\|\Phi(t,\tau)x\| \leq M(t)\|x\| \qquad \text{for all } \tau \in [0,t] \text{ and } x \in X,$$

so

$$\|\Phi(t,\tau)\| \leq M(t) \qquad \text{for all } \tau \in [0,t],\ x \in X,\ t \geq 0.$$

Then

$$\begin{aligned}\Big\|\int_0^t \Phi(t,\tau)f_n(\tau)\mathrm{d}\tau &- \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau\Big\| \\ &\le \int_0^t \|\Phi(t,\tau)(f_n(\tau)-f(\tau))\|\mathrm{d}\tau \le \int_0^t \|\Phi(t,\tau)\|\mathrm{d}\tau|||f_n-f||| \\ &\le tM(t)|||f_n-f||| \xrightarrow[n\to]{} 0 \quad \text{for all } t\ge 0.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} x_{f_n}(t) = \Phi(t,0)g(0) + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau \quad \text{for all } t\ge 0.$$

But $\mathcal{U}f_n \xrightarrow{\mathcal{C}(\mathbb{R}_+,X)} g$, so

$$\lim_{n\to\infty} \mathcal{U}f_n(t) = g(t), \quad \text{for all } t\ge 0.$$

So we proved that $\mathcal{U}f = g$, i.e. $\mathcal{U}$ is a closed operator and by the Closed Graph Theorem, there exists a constant $k>0$ such that

$$|||\mathcal{U}f||| \le k|||f||| \quad \text{for all } f\in\mathcal{C}_{00}(\mathbb{R}_+,X).$$

$\square$

Theorem 3.2. *If the evolutionary process Φ satisfies the Perron condition for instability (the pair of spaces $(\mathcal{C}_{00}(\mathbb{R}_+,X),\mathcal{C}(\mathbb{R}_+,X))$ is admissible to Φ), then there exists a function $N\colon\mathbb{R}_+\to\mathbb{R}_+^*$ and a constant $\nu>0$ such that*

$$(*) \qquad N(t)\|\Phi(t,0)x\| \ge \mathrm{e}^{\nu(t-t_0)}\|\Phi(t_0,0)x\| \quad \textit{for all } t\ge t_0 \textit{ and } x\in X.$$

Proof. Let $n\in\mathbb{N}^*$, $\delta>0$ and two functions χ_n^δ, $\chi^\delta\colon\mathbb{R}_+\to\mathbb{R}_+$

$$\chi_n^\delta(t) = \begin{cases} nt, & 0\le t<\frac{1}{n} \\ 1, & \frac{1}{n}\le t<\delta \\ 1+\delta-t, & \delta\le t<\delta+1 \\ 0, & t\ge\delta+1 \end{cases},$$

$$\chi^\delta(t) = \begin{cases} 1, & 0\le t<\delta \\ 1+\delta-t, & \delta\le t<\delta+1 \\ 0, & t\ge\delta+1 \end{cases}.$$

Let $x\in X\setminus\{0\}$ and $f_n\colon\mathbb{R}_+\to\mathbb{R}_+$, $f_n(t) = \chi_n^\delta\frac{\Phi(t,0)x}{\|\Phi(t,0)x\|}$ for all $n\in\mathbb{N}$, then obviously $f_n\in\mathcal{C}_{00}(\mathbb{R}_+,X)$ and $|||f_n|||=1$ for all $n\in\mathbb{N}$.

We consider the function

$$\begin{aligned} y_n(t) &= -\int_t^\infty \chi_n^\delta(\tau)\frac{\mathrm{d}\tau}{||\Phi(\tau,0)x||}\Phi(t,0)x \\ &= \Phi(t,0)\Big(-\int_0^\infty \chi_n^\delta(\tau)\frac{\mathrm{d}\tau}{||\Phi(\tau,0)x||}\cdot x\Big) + \int_0^t \Phi(t,\tau)f_n(\tau)\mathrm{d}\tau, \end{aligned}$$

then $y_n(t)=0$ for all $t\ge\delta+1$, so $y_n\in\mathcal{C}(\mathbb{R}_+,X)$.

By Theorem 3.1 we have that

$$\|y_n(t)\| \le |||y_n||| \le k|||f||| = k \quad \text{for all } t \ge 0.$$

Therefore

$$\int_t^\infty \chi_n^\delta(\tau)\frac{\mathrm{d}\tau}{||\Phi(\tau,0)x||}||\Phi(t,0)x|| \le k \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}^*.$$

But $\chi_n^\delta \xrightarrow[n\to\infty]{} \chi^\delta$ a.e., $\chi_n^\delta \le \chi^\delta$ and $\int_t^\infty \chi^\delta(\tau)\frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|} < \infty$, therefore $\lim_{n\to\infty}\int_t^\infty \chi_n^\delta(\tau)\frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|} = \int_t^\infty \chi^\delta(\tau)\frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}$. Then we have that

$$\int_t^\infty \chi^\delta(\tau)\frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\|\Phi(t,0)x\| \le k \quad \text{for all } t \ge 0 \text{ and } x \in X.$$

For $t < \delta$, we have that

$$\int_t^\delta \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\|\Phi(t,0)x\| \le k, \quad \text{for all } t \in [0,\delta) \text{ and all } \delta > 0.$$

This implies that if $\delta \to \infty$,

$$(**) \qquad \int_t^\infty \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\|\Phi(t,0)x\| \le k \quad \text{for all } t \ge 0.$$

Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \int_t^\infty \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}$, so $\dot\varphi(t) = -\|\Phi(t,0)x\|$. The inequality $(**)$ becomes $\varphi(t) \le -k\dot\varphi(t)$ for all $t \ge 0$, therefore

$$\varphi(t)\,\mathrm{e}^{\frac{1}{k}(t-t_0)} \le \varphi(t_0).$$

But $\varphi(t_0) \le -k\dot\varphi(t_0)$, so we have that

$$\int_t^\infty \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\cdot \mathrm{e}^{\frac{1}{k}(t-t_0)} \le \varphi(t_0) \le \frac{k}{\|\Phi(t_0,0)x\|}.$$

Then

$$\int_t^{t+1} \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\,\mathrm{e}^{\frac{1}{k}(t-t_0)} \le \varphi(t_0) \le \frac{k}{\|\Phi(t_0,0)x\|}.$$

Since

$$\begin{aligned}\|\Phi(\tau,0)x\| &\le ||\Phi(\tau,t)\Phi(t,0)x||\\ &\le \|\Phi(\tau,t)\|\|\Phi(t,0)x\| \le \sup_{\tau\in[t,t+1]}\|\Phi(\tau,t)\|\|\Phi(t,0)x\|,\end{aligned}$$

then

$$\frac{1}{N(t)\|\Phi(t,0)x\|}\,\mathrm{e}^{\frac{1}{k}(t-t_0)} \le \frac{k}{\|\Phi(t_0,0)x\|}.$$

We denote $N(t) = \sup_{\tau\in[t,t+1]}\|\Phi(\tau,t)\|$, $N\colon \mathbb{R}_+ \to \mathbb{R}_+^*$ and $\nu = \frac{1}{k} > 0$ and we have that

$$(*) \qquad N(t)\|\Phi(t,0)x\| \ge \mathrm{e}^{\nu(t-t_0)}\,\|\Phi(t_0,0)x\| \quad \text{for all } t \ge t_0 \text{ and } x \in X.$$

□

Theorem 3.3. *If for any* $t_0 \geq 0$ *and any function* $f \in \mathcal{C}_{00}^{t_0}(\mathbb{R}_+, X)$, *there exists a unique function* $u \in \mathcal{C}_{t_0}(\mathbb{R}_+, X)$ *such that*

$$u(t) = \Phi(t,s)u(s) + \int_s^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau \quad \text{for all } t \geq s \geq t_0 \geq 0,$$

then the evolutionary process Φ *is nonuniform exponential expansive.*

Proof. Let $t_0 > 0$ and the function $\chi_{t_0}^1 : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\chi_{t_0}^1 = \begin{cases} 0, & t < t_0 \\ 4(t - t_0), & t_0 \leq t < t_0 + \frac{1}{2} \\ 4(t_0 + 1 - t), & t_0 + \frac{1}{2} \leq t < t_0 + 1 \\ 0, & t \geq t_0 + 1 \end{cases}.$$

Let z be an element of X and $f(t) = \chi_{t_0}^1 \Phi(t, t_0)z$, $f \colon \mathbb{R}_+ \to X$. Obviously $f \in \mathcal{C}_{00}(\mathbb{R}_+, X)$.

We also consider the function $v \colon [t_0, \infty) \to X$,

$$v(t) = -\int_t^\infty \chi_{t_0}^1(\tau)\mathrm{d}\tau \Phi(t, t_0)z.$$

It is easy to see that

$$v(t) = \Phi(t,s)v(s) + \int_s^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau \quad \text{for all } t_0 \leq s \leq t.$$

In this case, $v = u$, so

$$v(t_0) = u(t_0) = z = \Phi(t_0, 0)u(0) + \int_0^{t_0} \Phi(t_0,\tau)f(\tau)\mathrm{d}\tau.$$

Therefore we found an element $t_0 \geq 0$ such that $z = \Phi(t_0, 0)u(0)$. So $\Phi(t_0, 0)\colon X \to X$ is a surjective function. □

Let y be an element in X, then by Theorem 3.3, there exists an element $x \in X$ such that $\Phi(t_0, 0)x = y$. In this case the inequality $(*)$ becomes

$$N(t)\|\Phi(t, t_0)y\| \geq \mathrm{e}^{\nu(t-t_0)}\|y\| \quad \text{for all } t \geq t_0 \geq 0 \text{ and } y \in X.$$

Remark 3.4. The converse of the theorem above is true.

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