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ASSOCIATED PRIMES OF TOP LOCAL HOMOLOGY MODULES WITH RESPECT TO AN IDEAL

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Abstract. Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ be an ideal of R and M be a non-zero Artinian R-module with $\operatorname{Ndim}_R M = n$. In this paper we determine the associated primes of the top local homology module $\mathrm{H}^{\mathfrak{a}}_n(M)$.

1. Introduction

Throughout this paper assume that $(R,\mathfrak{m})$ is a commutative Noetherian local ring, $\mathfrak{a}$ is an ideal of R and M is an R-module. In [**2**] Cuong and Nam defined the local homology modules $\mathrm{H}^{\mathfrak{a}}_i(M)$ with respect to $\mathfrak{a}$ by

$$\mathrm{H}^{\mathfrak{a}}_i(M) = \varprojlim_n \operatorname{Tor}^R_i(R/\mathfrak{a}^n, M).$$

This definition is dual to Grothendieck's definition of local cohomology modules and coincides with the definition of Greenless and May in [**6**] for an Artinian R-module M. For basic results about local homology we refer the reader to [**2, 3**] and [**13**]; for local cohomology see [**1**].

In [**8**] Macdonald and Sharp studied the top local cohomology module with respect to the maximal ideal and showed that $\operatorname{Att}(\mathrm{H}^n_{\mathfrak{m}}(N)) = \{\mathfrak{p} \in \operatorname{Ass} N : \dim R/\mathfrak{p} = n\}$, where N is a finitely generated R-module of dimension n. Cuong and Nam proved in [**2**] a dual result stating that

$$\operatorname{Ass}_{\hat{R}}(\mathrm{H}^{\mathfrak{m}}_d(M)) = \{\mathfrak{p} \in \operatorname{Att}_{\hat{R}}(M) : \dim \hat{R}/\mathfrak{p} = d\}$$

for a non-zero Artinian R-module M of Noetherian dimension d. In this paper we study the top local homology module $\mathrm{H}^{\mathfrak{a}}_n(M)$, where M is a non-zero Artinian R-module of Noetherian dimension n and $\mathfrak{a}$ is an arbitrary ideal of R. The module $\mathrm{H}^{\mathfrak{a}}_n(M)$ is called a top local homology module because $\max\{i : \mathrm{H}^{\mathfrak{a}}_i(M) \neq 0\} \leq n$ by [**2**, Proposition 4.8].

A non-zero R-module M is called secondary if the multiplication map by any element a of R is either surjective or nilpotent. A secondary representation of the R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable. A prime ideal $\mathfrak{p}$ of R

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is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation $M = S_1 + S_2 + \ldots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of M is equal to $\{\sqrt{0 :_R S_i}$ for $i = 1, \ldots, n\}$. Note that every Artinian R-module M is representable and minimal elements of the set $\mathrm{V}(\mathrm{Ann}(M))$, the set of prime ideals of R containing ideal $\mathrm{Ann}(M)$, belong to $\mathrm{Att}(M)$. It is well known that if N is a submodule of Artinian R-module M, then $\mathrm{Att}(M/N) \subseteq \mathrm{Att}(M) \subseteq \mathrm{Att}(N) \cup \mathrm{Att}(M/N)$ (See [**9**, Section 6]).

We now recall the concept of Noetherian dimension $\mathrm{Ndim}_R(M)$ of an R-module M. For $M = 0$ we define $\mathrm{Ndim}_R(M) = -1$. Then by induction, for any integer $t \geq 0$, we define $\mathrm{Ndim}_R(M) = t$ when

i) $\mathrm{Ndim}_R(M) < t$ is false, and
ii) for every ascending chain $M_1 \subseteq M_2 \subseteq \ldots$ of submodules of M there exists an integer m_0 such that $\mathrm{Ndim}_R(M_{m+1}/M_m) < t$ for all $m \geq m_0$.

Thus M is non-zero and finitely generated if and only if $\mathrm{Ndim}_R(M) = 0$. If M is Artinian module, then $\mathrm{Ndim}_R(M) < \infty$. (For more details see [**7**] and [**11**]).

Following [**5**], for any R-module M, we define the cohomological dimension of M with respect to $\mathfrak{a}$ as

$$\mathrm{cd}(\mathfrak{a}, M) = \max\{i : \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}.$$

By [**1**, Theorem 6.1.2 and Theorem 6.1.4], we have $\mathrm{cd}(\mathfrak{a}, M) \leq \dim M$ and $\mathrm{cd}(\mathfrak{m}, M) = \dim M$. We will call

$$\mathrm{hd}(\mathfrak{a}, M) := \max\{i : \mathrm{H}^{\mathfrak{a}}_i(M) \neq 0\}$$

the homological dimension of M with respect to $\mathfrak{a}$. It follows from [**2**, Propositions 4.8 and 4.10] that if M is an Artinian R-module, then $\mathrm{hd}(\mathfrak{a}, M) \leq \mathrm{Ndim}_R(M)$ and $\mathrm{hd}(\mathfrak{m}, M) = \mathrm{Ndim}_R(M)$.

Throughout the paper, for an R-module M, $\mathrm{E}(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$ and $\mathrm{D}(.)$ denotes the Matlis duality functor $\mathrm{Hom}_R(., \mathrm{E}(R/\mathfrak{m}))$. It is well known that $\dim \mathrm{D}(M) = \dim M$. Also, if M is an Artinian R-module, then $M \simeq \mathrm{D}\,\mathrm{D}(M)$ and $\mathrm{D}(M)$ is a Noetherian $\hat{R}$-module. (See [**1**, Theorem 10.2.19] and [**10**, Theorem 1.6(5)]).

Note that if M is an Artinian R-module, then $\mathrm{H}^{\mathfrak{a}}_i(M) \simeq \mathrm{D}(\mathrm{H}^i_{\mathfrak{a}}(\mathrm{D}(M)))$ for all i (See [**2**, Proposition 3.3(ii)]), and therefore $\mathrm{hd}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, \mathrm{D}(M))$. Thus $\mathrm{hd}(\mathfrak{a}, M) \leq \dim \mathrm{D}(M) = \dim M$.

The main result of this paper shows that if M is a non-zero Artinian R-module such that $\mathrm{Ndim}_R M = n$, then

$$\mathrm{Ass}_R(\mathrm{H}^{\mathfrak{a}}_n(M)) = \{\mathfrak{P} \cap R : \mathfrak{P} \in \mathrm{Att}_{\hat{R}} M \text{ and } \mathrm{cd}(\mathfrak{a}\hat{R}, \hat{R}/\mathfrak{P}) = n\}.$$

2. THE RESULTS

To prove our main result, we need the following lemmas.

Lemma 2.1. *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathfrak{a}$ *be an ideal of* R *and* $0 \to L \to M \to N \to 0$ *be an exact sequence of Artinian* R*-modules. Then* $\mathrm{hd}(\mathfrak{a}, M) = \mathrm{Max}\{\mathrm{hd}(\mathfrak{a}, L), \mathrm{hd}(\mathfrak{a}, N)\}$.

Proof. Since $\mathrm{D}(M)$ is Noetherian $\hat{R}$-module, by [**5**, Corollary 2.3(i)], $\mathrm{cd}(a\hat{R}, \mathrm{D}(N)) \leq \mathrm{cd}(a\hat{R}, \mathrm{D}(M))$. Hence by the Independence Theorem ([**1**, Theorem 4.2.1]), $\mathrm{cd}(\mathfrak{a}, \mathrm{D}(N)) \leq \mathrm{cd}(\mathfrak{a}, \mathrm{D}(M))$. Therefore $\mathrm{hd}(a, N) \leq \mathrm{hd}(a, M)$. From the long exact sequence

$$\mathrm{H}^{\mathfrak{a}}_{i+1}(L) \to \mathrm{H}^{\mathfrak{a}}_{i+1}(M) \to \mathrm{H}^{\mathfrak{a}}_{i+1}(N) \to \mathrm{H}^{\mathfrak{a}}_{i}(L) \to \mathrm{H}^{\mathfrak{a}}_{i}(M) \to \dots$$

we deduce that $\mathrm{hd}(\mathfrak{a}, L) \leq \mathrm{hd}(\mathfrak{a}, M)$. Hence $\mathrm{Max}\{\mathrm{hd}(\mathfrak{a}, L), \mathrm{hd}(\mathfrak{a}, N)\} \leq \mathrm{hd}(\mathfrak{a}, M)$. From the above long exact sequence we also infer that $\mathrm{hd}(\mathfrak{a}, M) \leq \mathrm{Max}\{\mathrm{hd}(\mathfrak{a}, L), \mathrm{hd}(\mathfrak{a}, N)\}$ and the proof is complete. $\square$

Lemma 2.2. *Let* $(R, \mathfrak{m})$ *be a complete local ring,* $\mathfrak{a}$ *be an ideal of* R *and* M *be a non-zero Artinian module. Then* $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \mathrm{hd}(\mathfrak{a}, M)$ *for all* $\mathfrak{p} \in \mathrm{Att}(M)$.

Proof. Since $\mathrm{D}(M)$ is a Noetherian R-module and $\mathrm{Supp}(R/\mathfrak{p}) \subseteq \mathrm{Supp}(\mathrm{D}(M))$ for all $\mathfrak{p} \in \mathrm{Ass}\,\mathrm{D}(M)$, by [**5**, Theorem 2.2] we infer that $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \mathrm{cd}(\mathfrak{a}, \mathrm{D}(M))$ for all $\mathfrak{p} \in \mathrm{Ass}\,\mathrm{D}(M)$. Since $\mathrm{Att}(M) = \mathrm{Ass}\,\mathrm{D}(M)$ and $\mathrm{cd}(\mathfrak{a}, \mathrm{D}(M)) = \mathrm{hd}(\mathfrak{a}, M)$, we obtain $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \mathrm{hd}(\mathfrak{a}, M)$ for all $\mathfrak{p} \in \mathrm{Att}(M)$. $\square$

Lemma 2.3. *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathfrak{a}$ *be an ideal of* R *and* M *be an Artinian* R*-module. Then* $\mathrm{hd}(\mathfrak{a}, M) \leq \mathrm{cd}(\mathfrak{a}, R/\operatorname{Ann} M)$.

Proof. Let $R^{'} := R/\operatorname{Ann} M$. By [**12**, Theorem 3.3], $\mathrm{H}^{\mathfrak{a}}_{i}(M) \simeq \mathrm{H}^{\mathfrak{a}R^{'}}_{i}(M)$ for all i. Thus $\mathrm{hd}(\mathfrak{a}, M) = \mathrm{hd}(\mathfrak{a}R^{'}, M)$. Since $\mathrm{hd}(\mathfrak{a}R^{'}, M) \leq \mathrm{cd}(\mathfrak{a}R^{'}, R^{'})$ (see [**6**, Corollary 3.2]) and $\mathrm{cd}(\mathfrak{a}R^{'}, R^{'}) = \mathrm{cd}(\mathfrak{a}, R^{'})$ (see [**5**, Lemma 2.1]), we conclude that $\mathrm{hd}(\mathfrak{a}, M) \leq \mathrm{cd}(\mathfrak{a}, R^{'})$. $\square$

Lemma 2.4. *Let* $(R, \mathfrak{m})$ *be a complete local ring,* $\mathfrak{a}$ *be an ideal of* R *and* M *be a non-zero Artinian module of dimension* n *with* $\mathrm{hd}(\mathfrak{a}, M) = n$. *Then the set*

$$\Sigma := \{N^{'} : N^{'} \textit{is a submodule of } M \textit{ and } \mathrm{hd}(\mathfrak{a}, M/N^{'}) < n\}$$

has a smallest element N. *The module* N *has the following properties:*

i) $\mathrm{hd}(\mathfrak{a}, N) = \dim N = n$.
ii) N *has no proper submodule* L *such that* $\mathrm{hd}(\mathfrak{a}, N/L) < n$.
iii) $\mathrm{Att}(N) = \{\mathfrak{p} \in \mathrm{Att}(M) : \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n\}$.
iv) $\mathrm{H}^{\mathfrak{a}}_{n}(N) \simeq \mathrm{H}^{\mathfrak{a}}_{n}(M)$.

Proof. It is clear that $M \in \Sigma$ and thus Σ is not empty. Since M is an Artinian R-module, the set Σ has a minimal member N. By Lemma 2.1, if $N_1, N_2 \in \Sigma$, then $\mathrm{hd}(\mathfrak{a}, M/N_1 \cap N_2) < n$. Since the intersection of any two members of Σ is again in Σ, it follows that N is contained in every member of Σ implying that N is the smallest element of Σ.

i) Since $\mathrm{hd}(\mathfrak{a}, M/N) < n$, from the exact sequence $0 \to N \to M \to M/N \to 0$ and Lemma 2.1 we obtain $\mathrm{hd}(\mathfrak{a}, N) = n$. From $n = \mathrm{hd}(\mathfrak{a}, N) \leq \dim N \leq \dim M = n$ we derive $\dim N = n$.

ii) Suppose that L is a submodule of N such that $\mathrm{hd}(\mathfrak{a}, N/L) < n$. From the exact sequence

$$0 \to N/L \to M/L \to M/N \to 0$$

and Lemma 2.1 we infer $\mathrm{hd}(\mathfrak{a}, M/L) < n$. Hence $L \in \Sigma$ and $L = N$.

iii) If $\mathfrak{p} \in \mathrm{Att}(N)$, then $\mathfrak{p} = \mathrm{Ann}(N/L)$, where L is a submodule of N. By (ii), $\mathrm{hd}(\mathfrak{a}, N/L) = n$. Hence $n = \mathrm{hd}(\mathfrak{a}, N/L) \leq \dim R/\mathfrak{p} \leq \dim(M) = n$. Thus $\dim(R/\mathfrak{p}) = \dim(M)$. Since $\dim(M) = \dim(R/\mathrm{Ann}(M))$, we conclude that $\mathfrak{p}$ is a minimal element of the set $\mathrm{V}(\mathrm{Ann}(M))$. Thus $\mathfrak{p} \in \mathrm{Att}(M)$.

On the other hand, using Lemma 2.3, we derive $n = \mathrm{hd}(\mathfrak{a}, N/L) \leq \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \dim(R/\mathfrak{p}) \leq \dim(M) = n$. Therefore $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n$.

Now suppose that $\mathfrak{p} \in \mathrm{Att}(M)$ and $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n$. Since $\mathrm{hd}(\mathfrak{a}, M/N) < n$ and $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n$, Lemma 2.2 implies that $\mathfrak{p} \notin \mathrm{Att}(M/N)$. Therefore $\mathfrak{p} \in \mathrm{Att}(N)$.

iv) The exact sequence $0 \to N \to M \to M/N \to 0$ induces the exact sequence

$$\mathrm{H}^{\mathfrak{a}}_{n+1}(M/N) \to \mathrm{H}^{\mathfrak{a}}_{n}(N) \to \mathrm{H}^{\mathfrak{a}}_{n}(M) \to \mathrm{H}^{\mathfrak{a}}_{n}(M/N) \to .$$

Since $\mathrm{hd}(\mathfrak{a}, M/N) < n$, $\mathrm{H}^{\mathfrak{a}}_{n+1}(M/N) = \mathrm{H}^{\mathfrak{a}}_{n}(M/N) = 0$. Therefore $\mathrm{H}^{\mathfrak{a}}_{n}(N) \simeq \mathrm{H}^{\mathfrak{a}}_{n}(M)$. ☐

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ be an ideal of R and M be a non-zero Artinian module of dimension n. Then*

$$\mathrm{Ass}(\mathrm{H}^{\mathfrak{a}}_{n}(M)) = \{\mathfrak{p} \in \mathrm{Att}(M) : \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n\}.$$

Proof. If $n = 0$, then M has a finite length and therefore $\mathfrak{a}^k M = 0$ for some $k \in \mathbb{N}$. Hence

$$\mathrm{Ass}(\mathrm{H}^{\mathfrak{a}}_{n}(M)) = \mathrm{Ass}(M) = \{\mathfrak{m}\} = \mathrm{Att}(M) = \{\mathfrak{p} \in \mathrm{Att}(M) : \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = 0\}.$$

Thus we can assume that $n > 0$. If $\mathrm{H}^{\mathfrak{a}}_{n}(M) = 0$, then $\mathrm{hd}(\mathfrak{a}, M) < n$. Hence by Lemma 2.2 $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) < n$ for all $\mathfrak{p} \in \mathrm{Att}(M)$. This implies $\{\mathfrak{p} \in \mathrm{Att}(M) : \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n\} = \emptyset = \mathrm{Ass}(\mathrm{H}^{\mathfrak{a}}_{n}(M))$ and the result has been proved in this case. Now assume that $n > 0$ and $\mathrm{H}^{\mathfrak{a}}_{n}(M) \neq 0$. Then $\mathrm{hd}(\mathfrak{a}, M) = \dim M = n$. By Lemma 2.4, we can assume that M has no proper submodule L with $\mathrm{hd}(\mathfrak{a}, M/L) < n$ and we must show that $\mathrm{Ass}(\mathrm{H}^{\mathfrak{a}}_{n}(M)) = \mathrm{Att}(M)$.

If $r \notin \cup_{\mathfrak{p} \in \mathrm{Att}\, M} \mathfrak{p}$, then the exact sequence $0 \to (0 :_M r) \to M \xrightarrow{r} M \to 0$ induces the exact sequence $\mathrm{H}^{\mathfrak{a}}_{n}(0 :_M r) \to \mathrm{H}^{\mathfrak{a}}_{n}(M) \xrightarrow{r} \mathrm{H}^{\mathfrak{a}}_{n}(M)$. Using [**3**, Lemma 4.7], we obtain $\mathrm{Ndim}_R(0 :_M r) \leq n-1$, and therefore $\mathrm{H}^{\mathfrak{a}}_{n}(0 :_M r) = 0$. Since $0 \to \mathrm{H}^{\mathfrak{a}}_{n}(M) \xrightarrow{r} \mathrm{H}^{\mathfrak{a}}_{n}(M)$ is exact, we infer $r \notin \cup_{\mathfrak{p} \in \mathrm{Ass}\, \mathrm{H}^{\mathfrak{a}}_{n}(M)} \mathfrak{p}$ and $\cup_{\mathfrak{p} \in \mathrm{Ass}\, \mathrm{H}^{\mathfrak{a}}_{n}(M)} \mathfrak{p} \subseteq \cup_{\mathfrak{p} \in \mathrm{Att}\, M} \mathfrak{p}$. Since $\mathrm{Att}\, M$ is a finite set, every $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}^{\mathfrak{a}}_{n}(M))$ is included in some $\mathfrak{q} \in \mathrm{Att}\, M$. For such $\mathfrak{q}$ there exists a submodule L of M satisfying $\mathfrak{q} = \mathrm{Ann}(M/L)$. Hence $n = \mathrm{hd}(\mathfrak{a}, M/L) \leq \dim M/L \leq \dim R/\mathfrak{q} \leq \dim R/\mathfrak{p} \leq n$. This shows $\mathfrak{p} = \mathfrak{q}$ and $\mathrm{Ass}\, \mathrm{H}^{\mathfrak{a}}_{n}(M) \subseteq \mathrm{Att}(M)$.

To prove the reverse inclusion, assume $\mathfrak{p} \in \mathrm{Att}(M)$. There exists a submodule L of M such that $\mathrm{Att}(L) = \{\mathfrak{p}\}$. Since we have assumed that M has no proper submodule U with $\mathrm{hd}(\mathfrak{a}, M/U) < n$, Lemma 2.4 implies that $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n$. Hence by Lemma 2.2, we have $\mathrm{hd}(\mathfrak{a}, L) = n$ and $\mathrm{H}^{\mathfrak{a}}_{n}(L) \neq 0$. Since $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n$ and $\mathrm{Att}(L/U) \subseteq \mathrm{Att}\, L = \{\mathfrak{p}\}$ for all submodules U, Lemma 2.2 shows that L cannot have any proper submodule U such that $\mathrm{hd}(\mathfrak{a}, L/U) < n$. Analogously as above, we obtain $\mathrm{Ass}\, \mathrm{H}^{\mathfrak{a}}_{n}(L) \subseteq \mathrm{Att}(L) = \{\mathfrak{p}\}$. Since $\mathrm{H}^{\mathfrak{a}}_{n}(L) \neq 0$, we establish that $\mathrm{Ass}\, \mathrm{H}^{\mathfrak{a}}_{n}(L) = \{\mathfrak{p}\}$. However, from the exact sequence $0 \to \mathrm{H}^{\mathfrak{a}}_{n}(L) \to \mathrm{H}^{\mathfrak{a}}_{n}(M) \to$

$\mathrm{H}_n^{\mathfrak{a}}(M/L)$ we see that $\{\mathfrak{p}\} = \operatorname{Ass} \mathrm{H}_n^{\mathfrak{a}}(L) \subseteq \operatorname{Ass} \mathrm{H}_n^{\mathfrak{a}}(M)$. Therefore $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}_n^{\mathfrak{a}}(M)$, that completes the proof. □

Corollary 2.6. *Let* $(R, \mathfrak{m})$ *be a complete local ring,* $\mathfrak{a}$ *be an ideal of* R *and* M *be a non-zero Artinian module of dimension* n*. Then*

$$\operatorname{Ass}(\mathrm{H}_n^{\mathfrak{m}}(M)) = \{\mathfrak{p} \in \operatorname{Att}(M) : \dim(R/\mathfrak{p}) = n\}.$$

Proof. Since $\operatorname{cd}(\mathfrak{m}, R/\mathfrak{p}) = \dim R/\mathfrak{p}$, it follows from Theorem 2.5. □

The following Theorem is the main result of this paper.

Theorem 2.7. *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathfrak{a}$ *be an ideal of* R *and* M *be a non-zero Artinian* R*-module with* $\operatorname{Ndim}_R M = n$*. Then*

$$\operatorname{Ass}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \{\mathfrak{P} \cap R : \mathfrak{P} \in \operatorname{Att}_{\hat{R}} M \ \textit{and} \ \operatorname{cd}(\mathfrak{a}\hat{R}, \hat{R}/\mathfrak{P}) = n\}.$$

Proof. Since $\dim_{\hat{R}} \mathrm{D}(M) = \dim_{\hat{R}} M = \operatorname{Ndim}_R M = n$ (for details consult [**4**]), by [**1**, Theorem 7.1.6], $\mathrm{H}^n_{\mathfrak{a}\hat{R}}(\mathrm{D}(M))$ is an Artinian local cohomology module and $\mathrm{D}(\mathrm{H}^n_{\mathfrak{a}\hat{R}}(\mathrm{D}(M))) \simeq \mathrm{H}_n^{\mathfrak{a}\hat{R}}(M)$ is a Noetherian $\hat{R}$-module. It is well known that $\operatorname{Ass}_R(L) = \{\mathfrak{P} \cap R : \mathfrak{P} \in \operatorname{Ass}_{\hat{R}} L\}$ for each finitely generated $\hat{R}$-module L (See [**9**, Exercise 6.7]). Thus $\operatorname{Ass}_R(\mathrm{H}_n^{\mathfrak{a}\hat{R}}(M)) = \{\mathfrak{P} \cap R : \mathfrak{P} \in \operatorname{Ass}_{\hat{R}}(\mathrm{H}_n^{\mathfrak{a}\hat{R}}(M))\}$. Since by [**13**, Proposition 4.3], $\mathrm{H}_n^{\mathfrak{a}}(M) \simeq \mathrm{H}_n^{\mathfrak{a}\hat{R}}(M)$ as R-modules, we conclude that $\operatorname{Ass}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \{\mathfrak{P} \cap R : \mathfrak{P} \in \operatorname{Ass}_{\hat{R}}(\mathrm{H}_n^{\mathfrak{a}\hat{R}}(M))\}$. According to Theorem 2.5, $\operatorname{Ass}_{\hat{R}}(\mathrm{H}_n^{\mathfrak{a}\hat{R}}(M)) = \{\mathfrak{P} : \mathfrak{P} \in \operatorname{Att}_{\hat{R}} M \text{ and } \operatorname{cd}(\mathfrak{a}\hat{R}, \hat{R}/\mathfrak{P}) = n\}$. Therefore $\operatorname{Ass}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \{\mathfrak{P} \cap R : \mathfrak{P} \in \operatorname{Att}_{\hat{R}} M \text{ and } \operatorname{cd}(\mathfrak{a}\hat{R}, \hat{R}/\mathfrak{P}) = n\}$. □

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