Acta Math. Univ. Comenianae
Vol. LXXXI, 2 (2012), pp. 143–149

A MEAN VALUE PROPERTY OF HARMONIC FUNCTIONS ON THE INTERIOR OF A HYPERBOLA

E. SYMEONIDIS

Abstract. We establish a mean value property for harmonic functions on the interior of a hyperbola. This property connects their boundary values with the interior ones on the axis of the hyperbola from the focus to infinity.

1. Introduction

Let Y denote the interior of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x > 0$ $(a, b > 0)$, that is,

$$Y := \left\{ (x, y) \in \mathbb{R}_+ \times \mathbb{R} : \frac{x^2}{a^2} - \frac{y^2}{b^2} > 1 \right\} .$$

Let ∂Y denote its boundary which consists of the hyperbola itself.

If h is a harmonic function on an open set containing $Y \cup \partial Y$ and if h decays at infinity, we are going to establish the following identity

$$\int\limits_{\sqrt{a^2+b^2}}^{\infty} \frac{h(x,0)}{\sqrt{x^2 - a^2 - b^2}} \, \mathrm{d}x = \frac{1}{2} \int\limits_{-\infty}^{\infty} h(a\cosh r, b\sinh r) \, \mathrm{d}r \tag{1}$$

that we regard as a mean value property.

The left side involves exactly the values of h on the axis of the hyperbola between the focus $(\sqrt{a^2+b^2}, 0)$ and infinity.

We avoid calling (1) a quadrature identity (and Y a quadrature domain) because the right side is not an integral with respect to the arc length. (For a complete discussion of quadrature domains and identities the reader is referred to [**3**].) This identity contributes to the few cases where explicit mean value properties are known for unbounded domains.

We would like to point out the striking analogy of (1) to the following mean value property for harmonic functions $\tilde{h}$ on (an open neighbourhood of) the elliptic

Received October 6, 2009.
2010 *Mathematics Subject Classification.* Primary 31A05, 31A10.
Key words and phrases. Mean value property; harmonic function; hyperbola.

disc $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ (here $a > b > 0$)

$$\int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} \frac{\tilde{h}(x,0)}{\sqrt{a^2-b^2-x^2}}\,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} \tilde{h}(a\cos s, b\sin s)\,\mathrm{d}s \tag{2}$$

(see [**5**], [**6**]). For the proof of (1) we do not use (2). Nevertheless, we believe that it is worth looking for a deeper connection between (1) and (2).

2. The mean value property

A simple dilatation transforms the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the "standard" form $\frac{x^2}{\cos^2 s_0} - \frac{y^2}{\sin^2 s_0} = 1$ with $s_0 \in]0, \frac{\pi}{2}[$ and (1) takes the form

$$\int_{1}^{\infty} \frac{h(x,0)}{\sqrt{x^2-1}}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} h(\cosh r\cos s_0, \sinh r\sin s_0)\,\mathrm{d}r. \tag{3}$$

For $s \in]0, s_0]$, we put

$$Y_s := \left\{(x,y) \in \mathbb{R}_+ \times \mathbb{R} : \frac{x^2}{\cos^2 s} - \frac{y^2}{\sin^2 s} > 1\right\}.$$

Every hyperbola ∂Y_s is parametrized by the map

$$\begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ r & \longmapsto & (\cosh r\cos s, \sinh r\sin s) \end{cases}.$$

A simple computation shows that disjoint hyperbolae ∂Y_s correspond to different $s \in]0, \frac{\pi}{2}[$ and $Y_{s_0} = \left[\bigcup_{0<s<s_0} \partial Y_s\right] \cup ([1,\infty[\times\{0\})$ holds. Thus, Y_{s_0} can be parametrized by

$$F(r,s) := (\cosh r\cos s, \sinh r\sin s)\,,$$

where $(r,s) \in \mathbb{R}\times]-s_0, s_0[$ ($F(r,s) = F(-r,-s)$). Since $F(r,s) = \cosh(r+\mathrm{i}s)$, F is a conformal mapping and the Laplacian in the coordinates r, s keeps its euclidean form $\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2}$.

The method we apply for obtaining (3) is the solving of the Dirichlet problem for Y_{s_0} by separation of variables in the Laplace equation. To this end, let $f\colon \partial Y_{s_0} \to \mathbb{R}$ be a continuous function. We assume that $f(F(r,s))$ decreases fast enough for $r \to \pm\infty$ so that all integrals in the sequel converge. A Dirichlet solution for (Y_{s_0}, f) is a harmonic function $H_f\colon Y_{s_0} \to \mathbb{R}$ which is continuously extendable to ∂Y_{s_0} by f. We require H_f to be bounded.

The method of separation of variables starts with the determination of all harmonic functions $u\colon Y_{s_0} \to \mathbb{R}$ of the form

$$u(F(r,s)) = U(r)V(s)\,.$$

The Laplace equation

$$\frac{\partial^2(u\circ F)}{\partial r^2} + \frac{\partial^2(u\circ F)}{\partial s^2} = 0$$

entails

$$\frac{U''(r)}{U(r)} + \frac{V''(s)}{V(s)} \equiv 0 \quad \text{so} \quad \frac{U''}{U} = -\frac{V''}{V} =: \gamma \in \mathbb{R}\,.$$

The following cases have to be examined separately.

Case 1. $\gamma < 0$: We write $\gamma = -\lambda^2$ with $\lambda > 0$ and have

$$V(s) = a\cosh(\lambda s) + b\sinh(\lambda s)\,, \quad U(r) = c\cos(\lambda r) + d\sin(\lambda r)$$

for $r \in \mathbb{R}$, $|s| < s_0$ with $a, b, c, d \in \mathbb{R}$. While combining U and V to obtain u, we have to pay attention to the condition $u\,(F(r,s)) = u\,(F(-r,-s))$ which results in the basic functions

$$\cos(\lambda r)\cosh(\lambda s) \quad \text{and} \quad \sin(\lambda r)\sinh(\lambda s) \tag{4}$$

for u ($\cos(\lambda r)\sinh(\lambda s)$ and $\sin(\lambda r)\cosh(\lambda s)$ have to be rejected). At this point it is still unsettled whether (4) represent harmonic functions in the cartesian coordinates x, y. The reason is that F is singular for $(r,s) = (0,0)$ which means that (4) may not be smooth at the focus $(x,y) = (1,0)$ of the hyperbola.

However, the harmonicity of $\sin(\lambda r)\sinh(\lambda s)$ is guaranteed by the reflection principle, since it is an odd function with respect to y (see, e. g. [**1**, Theorem 1.3.6]). Then, also $\cos(\lambda r)\cosh(\lambda s)$ is harmonic as a harmonic conjugate function (a conjugate with respect to (x,y), which must exist, has to be conjugate with respect to (r,s), too).

Case 2. $\gamma > 0$: Here we write $\gamma = \lambda^2$ with $\lambda > 0$ and have

$$V(s) = a\cos(\lambda s) + b\sin(\lambda s)\,, \quad U(r) = c\cosh(\lambda r) + d\sinh(\lambda r)$$

for $r \in \mathbb{R}$, $|s| < s_0$ with $a, b, c, d \in \mathbb{R}$. The condition $F(r,s) = F(-r,-s)$ results in the basic functions

$$\cosh(\lambda r)\cos(\lambda s) \quad \text{and} \quad \sinh(\lambda r)\sin(\lambda s)\,.$$

We are not going to further examine these functions because they are unbounded on Y_{s_0}.

Case 3. $\gamma = 0$: Here we have

$$V(s) = as + b\,, \quad U(r) = cr + d\,.$$

The only bounded functions u with $u\,(F(r,s)) = u\,(F(-r,-s))$ are the constant ones.

After the determination of the basic harmonic functions of separate variables we return to the Dirichlet setting and assume that the (bounded) solution H_f can be expressed in the form

$$H_f(F(r,s)) = c + \int_0^\infty a_\lambda \cosh(\lambda s)\cos(\lambda r)\,\mathrm{d}\lambda + \int_0^\infty b_\lambda \sinh(\lambda s)\sin(\lambda r)\,\mathrm{d}\lambda \tag{5}$$

with appropriate coefficient functions a_λ, b_λ. If the right side is continuously extendable to $s = s_0$, the boundary condition implies

$$\begin{aligned} &f(F(r,s_0)) \\ &\quad = c + \int_0^\infty a_\lambda \cosh(\lambda s_0)\cos(\lambda r)\,\mathrm{d}\lambda + \int_0^\infty b_\lambda \sinh(\lambda s_0)\sin(\lambda r)\,\mathrm{d}\lambda\,. \end{aligned} \tag{6}$$

Since by assumption the left side tends to zero for $r \to \pm\infty$, on the basis of the Riemann-Lebesgue lemma it should hold $c = 0$ provided that $a_\lambda \cosh(\lambda s_0)$ and $b_\lambda \sinh(\lambda s_0)$ are integrable in λ. (For this lemma and the following facts about the Fourier transform, see, e. g. [**2**].) Then, (6) is nothing but the Fourier inversion formula which at the same time forces $a_\lambda \cosh(\lambda s_0)$ and $b_\lambda \sinh(\lambda s_0)$ to constitute the Fourier transform of $r \mapsto f\,(F(r,s_0))$ as follows:

$$\begin{aligned} a_\lambda \cosh(\lambda s_0) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f\,(F(r,s_0))\cos(\lambda r)\,\mathrm{d}r\,, \\ b_\lambda \sinh(\lambda s_0) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f\,(F(r,s_0))\sin(\lambda r)\,\mathrm{d}r\,. \end{aligned} \tag{7}$$

At this point we have to clarify the conditions under which all previous plausible conclusions are allowed. By Fourier theory we know that if the function $r \mapsto f\,(F(r,s_0))$ is integrable and twice differentiable with integrable first and second derivative, then the left sides of (7) are integrable, (6) holds with $c = 0$, and (5) solves the Dirichlet problem for (Y_{s_0}, f). We remark that this H_f is a *bounded* function on Y_{s_0}.

Substituting a_λ and b_λ in (5) by their expressions from (7) and interchanging of integration lead to the following representation of the Dirichlet solution

$$\begin{aligned} &H_f(\cosh r\cos s, \sinh r\sin s) \\ &\quad = \int_{-\infty}^{\infty} f(\cosh\rho\cos s_0, \sinh\rho\sin s_0)\cdot\int_0^\infty \frac{\cosh(\lambda s)\cos(\lambda r)\cos(\lambda\rho)}{\pi\cosh(\lambda s_0)}\,\mathrm{d}\lambda\,\mathrm{d}\rho \\ &\qquad + \int_{-\infty}^{\infty} f(\cosh\rho\cos s_0, \sinh\rho\sin s_0)\cdot\int_0^\infty \frac{\sinh(\lambda s)\sin(\lambda r)\sin(\lambda\rho)}{\pi\sinh(\lambda s_0)}\,\mathrm{d}\lambda\,\mathrm{d}\rho \\ &\quad = \int_{-\infty}^{\infty} f(\cosh\rho\cos s_0, \sinh\rho\sin s_0) \\ &\qquad \cdot\int_0^\infty \left[\frac{\cosh(\lambda s)\cos(\lambda r)\cos(\lambda\rho)}{\pi\cosh(\lambda s_0)} + \frac{\sinh(\lambda s)\sin(\lambda r)\sin(\lambda\rho)}{\pi\sinh(\lambda s_0)}\right]\mathrm{d}\lambda\,\mathrm{d}\rho, \end{aligned}$$

where the inner integral can be regarded as a Poisson kernel.

In particular, for $s = 0$, we have

$$H_f(\cosh r, 0) = \int_{-\infty}^{\infty} f(\cosh\rho\cos s_0, \sinh\rho\sin s_0) \cdot \int_0^{\infty} \frac{\cos(\lambda r)\cos(\lambda\rho)}{\pi\cosh(\lambda s_0)}\,\mathrm{d}\lambda\,\mathrm{d}\rho\,. \tag{8}$$

At this point we need the following lemma.

Lemma. *Let $g\colon \partial Y_{s_0} \to \mathbb{R}$ be a bounded continuous function. Then there exists at most one bounded solution to the Dirichlet problem for (Y_{s_0}, g).*

Proof. It suffices to show that if $g = 0$ and the bounded function $h\colon Y_{s_0} \to \mathbb{R}$ solves the Dirichlet problem for (Y_{s_0}, g), then $h = 0$.

Let $\mathbb{R}^2 \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^2$, $\partial_\infty Y_{s_0} := \partial Y_{s_0} \cup \{\infty\}$ the boundary of Y_{s_0} in $\mathbb{R}^2 \cup \{\infty\}$. Since $\{\infty\}$ is a polar subset of $\mathbb{R}^2 \cup \{\infty\}$, under the above assumptions it holds

$$\lim_{\substack{z\to w\\ z\in Y_{s_0}}} h(z) = 0 \qquad \text{for every } w \in \partial Y_{s_0}$$

and therefore for quasi-every point $w \in \partial_\infty Y_{s_0}$. By a theorem of Bouligand (see [**4**]), this implies $h = 0$. □

Thus, if h is a bounded harmonic function on an open neighbourhood of $\overline{Y_{s_0}} = Y_{s_0} \cup \partial Y_{s_0}$, h is the unique solution of the Dirichlet problem for $\left(Y_{s_0}, h|_{\partial Y_{s_0}}\right)$, and if in addition the function $r \mapsto h\left(F(r, s_0)\right)$, $(r \in \mathbb{R})$ as well as its first two derivatives are integrable, (8) entails

$$h(\cosh r, 0) = \int_{-\infty}^{\infty} h(\cosh\rho\cos s_0, \sinh\rho\sin s_0) \cdot \int_0^{\infty} \frac{\cos(\lambda r)\cos(\lambda\rho)}{\pi\cosh(\lambda s_0)}\,\mathrm{d}\lambda\,\mathrm{d}\rho\,. \tag{9}$$

Due to the inner integral and the arbitrariness of r, this equation, though crucial in our investigation, is not yet "simple" enough to present a mean value property that deserves its name. However, it results in a whole class of such properties after being multiplied by suitable functions and integrated. Let $\alpha = \alpha(r)$ be a function for which the Fourier cosine transform and its inversion are applicable (see, e. g. [**2**])

$$\int_0^{\infty} \alpha(r)\cos(\lambda r)\,\mathrm{d}r =: \beta_\lambda \cosh(\lambda s_0)\,, \quad \alpha(r) = \frac{2}{\pi}\int_0^{\infty} \beta_\lambda \cosh(\lambda s_0)\cos(\lambda r)\,\mathrm{d}\lambda.$$

Then, from (9) we get

$$\int_0^\infty h(\cosh r, 0)\alpha(r)\,\mathrm{d}r = \int_{-\infty}^{\infty} h(\cosh\rho\cos s_0, \sinh\rho\sin s_0)$$

$$\cdot \int_0^\infty \frac{\cos(\lambda\rho)}{\pi\cosh(\lambda s_0)} \int_0^\infty \alpha(r)\cos(\lambda r)\,\mathrm{d}r\,\mathrm{d}\lambda\,\mathrm{d}\rho \tag{10}$$

$$= \int_{-\infty}^{\infty} h(\cosh\rho\cos s_0, \sinh\rho\sin s_0)\cdot \int_0^\infty \frac{\beta_\lambda}{\pi}\cos(\lambda\rho)\,\mathrm{d}\lambda\,\mathrm{d}\rho.$$

Indeed, relation (10) is a generator of many mean value properties. We shall look closer into the special case $\beta_\lambda = \mathrm{e}^{-c\lambda^2}$, $c > 0$; its Fourier cosine transform can be explicitly given

$$\int_0^\infty \mathrm{e}^{-c\lambda^2}\cos(\lambda\rho)\,\mathrm{d}\lambda = \frac{1}{2}\sqrt{\frac{\pi}{c}}\,\mathrm{e}^{-\frac{\rho^2}{4c}} \tag{11}$$

([**2**, p. 223]). In this case we have

$$\alpha(r) = \frac{1}{\pi}\int_{-\infty}^{\infty} \beta_\lambda\cosh(\lambda s_0)\cos(\lambda r)\,\mathrm{d}\lambda = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-c\lambda^2}\cdot\frac{\mathrm{e}^{\lambda s_0}+\mathrm{e}^{-\lambda s_0}}{2}\cos(\lambda r)\,\mathrm{d}\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-c\lambda^2+\lambda s_0}\cos(\lambda r)\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-c\lambda^2-\lambda s_0}\cos(\lambda r)\,\mathrm{d}\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-c\left[\left(\lambda-\frac{s_0}{2c}\right)^2-\frac{s_0^2}{4c^2}\right]}\cos(\lambda r)\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-c\left[\left(\lambda+\frac{s_0}{2c}\right)^2-\frac{s_0^2}{4c^2}\right]}\cos(\lambda r)\,\mathrm{d}\lambda$$

$$= \frac{1}{2\pi}\,\mathrm{e}^{\frac{s_0^2}{4c}}\int_{-\infty}^{\infty} \mathrm{e}^{-c\left(\lambda-\frac{s_0}{2c}\right)^2}\cos(\lambda r)\,\mathrm{d}\lambda + \frac{1}{2\pi}\,\mathrm{e}^{\frac{s_0^2}{4c}}\int_{-\infty}^{\infty} \mathrm{e}^{-c\left(\lambda+\frac{s_0}{2c}\right)^2}\cos(\lambda r)\,\mathrm{d}\lambda$$

$$= \frac{1}{2\pi}\,\mathrm{e}^{\frac{s_0^2}{4c}}\int_{-\infty}^{\infty} \mathrm{e}^{-cx^2}\cos\left(xr+\frac{s_0 r}{2c}\right)\mathrm{d}x + \frac{1}{2\pi}\,\mathrm{e}^{\frac{s_0^2}{4c}}\int_{-\infty}^{\infty} \mathrm{e}^{-cx^2}\cos\left(xr-\frac{s_0 r}{2c}\right)\mathrm{d}x$$

$$= \frac{1}{\pi}\,\mathrm{e}^{\frac{s_0^2}{4c}}\cos\frac{s_0 r}{2c}\cdot\int_{-\infty}^{\infty} \mathrm{e}^{-cx^2}\cos(xr)\,\mathrm{d}x = \frac{1}{\sqrt{c\pi}}\,\mathrm{e}^{\frac{s_0^2-r^2}{4c}}\cos\frac{s_0 r}{2c}$$

according to (11). Using (10) and (11), we arrive at the following theorem.

Theorem. *Let h be a bounded harmonic function on an open neighbourhood of $\overline{Y_{s_0}}$ for which $r \mapsto h(\cosh r \cos s_0, \sinh r \sin s_0)$ and its first two derivatives are integrable over $\mathbb{R}$. Then, for every $c > 0$, it holds*

$$\begin{aligned}(12)\qquad &\int_0^\infty h(\cosh r, 0) \cdot \mathrm{e}^{\frac{s_0^2 - r^2}{4c}} \cos \frac{s_0 r}{2c}\,\mathrm{d}r \\ &= \frac{1}{2} \int_{-\infty}^{\infty} h(\cosh \rho \cos s_0, \sinh \rho \sin s_0) \cdot \mathrm{e}^{-\frac{\rho^2}{4c}}\,\mathrm{d}\rho.\end{aligned}$$

If we pass to the limit $c \to \infty$ in (12), we obtain (since s_0 can be replaced by any smaller number) the following corollary.

Corollary. *Under the hypotheses of the theorem it holds*

$$\int_0^\infty h(\cosh r, 0)\,\mathrm{d}r = \frac{1}{2} \int_{-\infty}^{\infty} h(\cosh r \cos s, \sinh r \sin s)\,\mathrm{d}r$$

for every $s \in [0, s_0]$.

This establishes (3).

References

1. Armitage D. H. and Gardiner S. J., *Classical Potential Theory*, Springer, 2001.

2. Folland G. B., *Fourier Analysis and its Applications*, Wadsworth & Brooks/Cole, Belmont (California), 1992.

3. Gustafsson B. and Shapiro H. S., *What is a quadrature domain?*, in “Quadrature Domains and Applications — a Harold S. Shapiro Anniversary Volume” (eds. Ebenfelt P., Gustafsson B., Khavinson D., Putinar M.), Operator Theory: Advances and Applications, Vol. 156, Birkhäuser, 2005; as a pdf-file available under `www.math.kth.se/~gbjorn/qd.pdf`.

4. Helms L. L., *Introduction to Potential Theory*, John Wiley & Sons, Inc., 1963.

5. Royster W. C., *A Poisson Integral Formula for the Ellipse and some Applications*, Proc. Amer. Math. Soc. **15** (1964), 661–670.

6. Symeonidis E., *A Mean Value Property of Harmonic Functions on Prolate Ellipsoids of Revolution*, Acta Math. Univ. Comenianae Vol. **LXXVII(1)** (2008), 55–61.

E. Symeonidis, Mathematisch-Geographische Fakultät, Katholische Universität Eichstätt-Ingolstadt, Ostenstr. 26-28, 85072 Eichstätt, Germany, *e-mail*: `e.symeonidis@ku-eichstaett.de`