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TWO-PERIODIC TERNARY RECURRENCES AND THEIR BINET-FORMULA

M. ALP, N. IRMAK AND L. SZALAY

Abstract. The properties of k-periodic binary recurrences have been discussed by several authors. In this paper, we define the notion of the two-periodic *ternary* linear recurrence. First we follow Cooper's approach to obtain the corresponding recurrence relation of order six. Then we provide explicit formulae linked to the three possible cases.

1. INTRODUCTION

Let a, b, c, d, and q_0, q_1 denote arbitrary complex numbers, and consider the the sequence $\{q_n\}$ $(n \in \mathbb{N})$ defined by

$$q_n = \begin{cases} aq_{n-1} + bq_{n-2} & \text{if } n \text{ is even} \\ cq_{n-1} + dq_{n-2} & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

The sequence $\{q_n\}$ is called two-periodic binary recurrence. It was first described by Edson and Yayenie in [**2**]. The authors discussed the specific case $q_0 = 0$, $q_1 = 1$ and $b = d = 1$, gave the generating function and Binet-type formula of $\{q_n\}$, further they proved several identities among the terms of $\{q_n\}$. In the same paper the sequence $\{q_n\}$ was investigated for arbitrary initial values q_0 and q_1, but $b = d = 1$ were presumed.

Later Yayenie [**6**] determined the Binet's formula for $\{q_n\}$, where b and d were arbitrary numbers, but held for the initial values $q_0 = 0$ and $q_1 = 1$.

The k-periodic binary recurrence

$$q_n = \begin{cases} a_0 q_{n-1} + b_0 q_{n-2} & \text{if } n \equiv 0 \pmod{k} \\ a_1 q_{n-1} + b_1 q_{n-2} & \text{if } n \equiv 1 \pmod{k} \\ \vdots & \vdots \\ a_{k-1} q_{n-1} + b_{k-1} q_{n-2} & \text{if } n \equiv k-1 \pmod{k} \end{cases} \tag{2}$$

was introduced by Cooper in [**1**], where mainly the combinatorial interpretation of the coefficients A_k and B_k appearing in the recurrence relation $q_n = A_k q_{n-k} + B_k q_{n-2k}$ was discussed. Edson, Lewis and Yayenie [**3**] also studied the k-periodic

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extension, again with $q_0 = 0$, $q_1 = 1$ and with the restrictions $b_0 = b_1 = \cdots = b_{k-1} = 1$.

The main tool in [**2**] and [**6**] is to work with the corresponding generating functions. Later we suggested a new approach (see [**4**]), namely to apply the fundamental theorem of homogeneous linear recurrences (Theorem 1). This powerful method made us possible to give the Binet's formula of $\{q_n\}$ for any b, d and for arbitrary initial values. Moreover, we were able to maintain the remaining case when the zeros of the polynomial

$$p_2(x) = x^2 - (ac + b + d)x + bd$$

coincide. We showed that the application of the fundamental theorem of linear recurrences was very effective and it could even be used at k-periodic sequences generally.

Now define the two-periodic ternary recurrence sequence by

$$\gamma_n = \begin{cases} a\gamma_{n-1} + b\gamma_{n-2} + c\gamma_{n-3} & \text{if } n \text{ is even} \\ d\gamma_{n-1} + e\gamma_{n-2} + f\gamma_{n-3} & \text{if } n \text{ is odd} \end{cases} \tag{3}$$

with arbitrary complex coefficients and initial conditions γ_0, γ_1.

In this paper, we provide a recurrence relation of order six for γ_n and then give the corresponding Binet-formulae (Theorem 2–4) by using the fundamental theorem of linear recurrences. According to the relation between the zeros of the characteristic polynomial of the recurrence we need to distinguish three principal cases (Case I, II and III).

At the end of the first section we recall a theorem for linear recurrences. A homogeneous linear recurrence $\{G_n\}_{n=0}^{\infty}$ of order k ($k \geq 1, k \in \mathbb{N}$) is defined by the recursion

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \qquad (n \geq k)\,, \tag{4}$$

where the initial values $G_0, \ldots, G_{k-1}$ and the coefficients $A_1, \ldots, A_k$ are complex numbers, $A_k \neq 0$ and $|G_0| + \cdots + |G_{k-1}| > 0$. The characteristic polynomial of the sequence $\{G_n\}$ is the polynomial

$$g(x) = x^k - A_1 x^{k-1} - \cdots - A_k\,.$$

Denote by $\alpha_1, \ldots, \alpha_t$ the distinct zeros of the characteristic polynomial $g(x)$ which can there be written in the form

$$g(x) = (x - \alpha_1)^{e_1} \cdots (x - \alpha_t)^{e_t}\,. \tag{5}$$

The following result (see e.g. [**5**]) plays a basic role in the theory of recurrence sequences, and here in our approach.

Theorem 1.1. *Let $\{G_n\}$ be a sequence satisfying the relation (4) with $A_k \neq 0$, and $g(x)$ its characteristic polynomial with distinct roots $\alpha_1, \dots, \alpha_t$. Let $K = \mathbb{Q}(\alpha_1, \dots, \alpha_t, A_1, \dots, A_k, G_0, \dots, G_{k-1})$ denote the extension of the field of rational numbers and let $g(x)$ be given in the form (5). Then there exist uniquely determined polynomials $g_i(x) \in K[x]$ of degree less than e_i $(i = 1, \dots, t)$ such that*

$$G_n = g_1(n)\alpha_1^n + \dots + g_t(n)\alpha_t^n \qquad (n \geq 0) \quad .$$

2. TWO-PERIODIC TERNARY RECURRENCE

Let a, b, c, d, e, f and γ_0, γ_1 denote complex numbers satisfying $cf \neq 0$ and $|\gamma_0| + |\gamma_1| \neq 0$. Recall the sequence $\{\gamma_n\}$ defined by (3).

Supposing that n is even, by Cooper's method (see [**1**]), we can built up the tree of γ_n (see Figure 1). For n odd we obtain a similar tree which leads to the same recurrence relation. Thus sequence $\{\gamma_n\}$ satisfies the recurrence relation

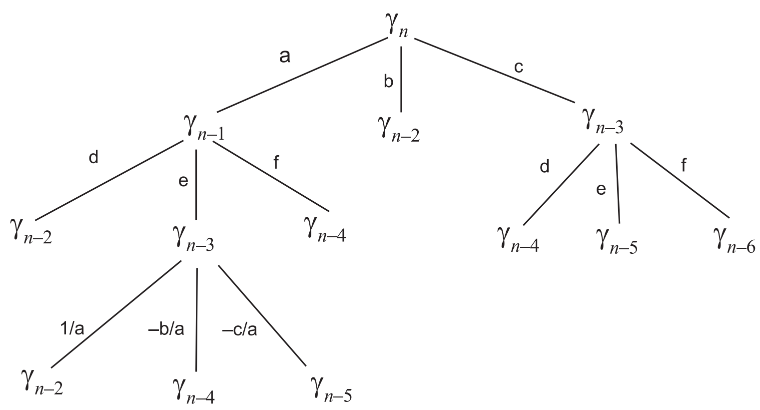

Figure 1

(6) $$\gamma_n = (ad + b + e)\,\gamma_{n-2} + (af - be + cd)\,\gamma_{n-4} + cf\gamma_{n-6}$$

of order six.

Let

(7) $$p\,(t) = t^3 - (ad + b + e)\,t^2 - (af - be + cd)\,t - cf$$

denote the polynomial determined by the characteristic polynomial

$$x^6 - (ad + b + e)\,x^4 - (af - be + cd)\,x^2 - cf$$

of the recurrence (6) by the substitution $t = x^2$. According to the coefficients of $p\,(t)$, we must distinguish the following cases: the polynomial $p(t)$ can possesses 3 or 2 or 1 different zeros (Case 1, 2 and 3, respectively). At this point, we apply Theorem 1 and obtain an appropriate Binet-formula.

2.1. Case 1

Let κ, τ and μ are three distinct zeros of (7). By Theorem 1, there are complex numbers κ_i, τ_i and μ_i $(i = 1, 2)$ such that

$$\gamma_n = \kappa_1 \left(\sqrt{\kappa}\right)^n + \kappa_2 \left(-\sqrt{\kappa}\right)^n + \mu_1 \left(\sqrt{\mu}\right)^n + \mu_2 \left(-\sqrt{\mu}\right)^n + \tau_1 \left(\sqrt{\tau}\right)^n + \tau_2 \left(-\sqrt{\tau}\right)^n.$$

Suppose first that n is even. Then we obtain

$$\gamma_n = (\kappa_1 + \kappa_2)(\sqrt{\kappa})^n + (\tau_1 + \tau_2)(\sqrt{\tau})^n + (\mu_1 + \mu_2)(\sqrt{\mu})^n,$$

which after considering the cases $n = 0,\ 2$ and 4 in order to determine κ_i, τ_i and μ_i $(i = 1,\ 2)$, leads to the explicit formula

$$\begin{aligned}\gamma_n = {} & \frac{\gamma_4 - (\mu+\tau)\,\gamma_2 + \mu\tau\gamma_0}{(\kappa-\mu)\,(\kappa-\tau)}\kappa^{\frac{n}{2}} + \frac{\gamma_4 - (\kappa+\tau)\,\gamma_2 + \kappa\tau\gamma_0}{(\mu-\tau)\,(\mu-\kappa)}\mu^{\frac{n}{2}} \\ & + \frac{\gamma_4 - (\mu+\kappa)\,\gamma_2 + \mu\kappa\gamma_0}{(\tau-\kappa)\,(\tau-\mu)}\tau^{\frac{n}{2}}.\end{aligned}$$

Contrary, suppose that n is odd. In similar manner it leads to

$$\begin{aligned}\gamma_n = {} & \frac{\gamma_5 - (\mu+\tau)\,\gamma_3 + \mu\tau\gamma_1}{\sqrt{\kappa}\,(\kappa-\mu)\,(\kappa-\tau)}\kappa^{\frac{n}{2}} + \frac{\gamma_5 - (\kappa+\tau)\,\gamma_3 + \kappa\tau\gamma_1}{\sqrt{\mu}\,(\mu-\tau)\,(\mu-\kappa)}\mu^{\frac{n}{2}} \\ & + \frac{\gamma_5 - (\mu+\kappa)\,\gamma_3 + \mu\kappa\gamma_1}{\sqrt{\tau}\,(\tau-\kappa)\,(\tau-\mu)}\tau^{\frac{n}{2}}.\end{aligned}$$

Comparing the two results above we proved the following theorem.

Theorem 2.1. *Let $\xi_n = n - 2\left\lfloor \frac{n}{2} \right\rfloor$. Suppose that the three different roots of (7) are κ, μ and τ. Then the terms of the sequence $\{\gamma_n\}$ satisfy*

$$\begin{aligned}\gamma_n = {} & \frac{\gamma_{4+\xi_n} - (\mu+\tau)\,\gamma_{2+\xi_n} + \mu\tau\gamma_{\xi_n}}{(\kappa-\mu)\,(\kappa-\tau)}\kappa^{\left\lfloor \frac{n}{2} \right\rfloor} + \frac{\gamma_{4+\xi_n} - (\kappa+\tau)\,\gamma_{2+\xi_n} + \kappa\tau\gamma_{\xi_n}}{(\mu-\tau)\,(\mu-\kappa)}\mu^{\left\lfloor \frac{n}{2} \right\rfloor} \\ & + \frac{\gamma_{4+\xi_n} - (\mu+\kappa)\,\gamma_{2+\xi_n} + \mu\kappa\gamma_{\xi_n}}{(\tau-\kappa)\,(\tau-\mu)}\tau^{\left\lfloor \frac{n}{2} \right\rfloor}.\end{aligned}$$

2.2. Case 2

In this case, we suppose that there are two distinct zeros of the polynomial (7). Say that $\kappa = \tau$ and $\mu \neq \tau$. Thus by Theorem 1, it results that γ_n can be written in the form

$$\gamma_n = (\kappa_1 n + \tau_1)\left(\sqrt{\kappa}\right)^n + (\kappa_2 n + \tau_2)\left(-\sqrt{\kappa}\right)^n + \mu_1\left(\sqrt{\mu}\right)^n + \mu_2\left(-\sqrt{\mu}\right)^n. \tag{8}$$

Firstly suppose again that n is even. Then we obtain

$$\gamma_n = \left((\kappa_1 + \kappa_2)\,n + (\tau_1 + \tau_2)\right)\left(\sqrt{\kappa}\right)^n + (\mu_1 + \mu_2)\left(\sqrt{\mu}\right)^n. \tag{9}$$

Observe that (9) at $n = 0,\ 2,\ 4$ is a system of three equations in $\kappa_1 + \kappa_2$, $\tau_1 + \tau_2$ and $\mu_1 + \mu_2$. One can easily get the solution

$$\kappa_1 + \kappa_2 = \frac{\gamma_4 - (\kappa+\mu)\,\gamma_2 + \kappa\mu\gamma_0}{2\kappa\,(\kappa-\mu)}, \quad \tau_1 + \tau_2 = -\frac{\gamma_4 - 2\kappa\gamma_2 + \left(2\kappa\mu - \mu^2\right)\gamma_0}{(\kappa-\mu)^2}$$

and

$$\mu_1 + \mu_2 = \frac{\gamma_4 - 2\kappa\gamma_2 + \kappa^2\gamma_0}{(\kappa-\mu)^2}.$$

Now, suppose that n is odd. Thus we obtain

$$\gamma_n = \left((\kappa_1 - \kappa_2)\, n + (\tau_1 - \tau_2) \right) \left(\sqrt{\kappa}\right)^n + (\mu_1 - \mu_2) \left(\sqrt{\mu}\right)^n,$$

where similarly to the previous case, one can determine

$$\kappa_1 - \kappa_2 = \frac{\gamma_5 - (\kappa+\mu)\,\gamma_3 + \kappa\mu\gamma_1}{2\kappa^{\frac{3}{2}}\,(\kappa-\mu)}, \qquad \mu_1 - \mu_2 = \frac{\gamma_5 - 2\kappa\gamma_3 + \kappa^2\gamma_1}{\mu^{\frac{1}{2}}\,(\kappa-\mu)^2}$$

and

$$\tau_1 - \tau_2 = \frac{(\mu - 3\kappa)\,\gamma_5 + \left(5\kappa^2 - \mu^2\right)\gamma_3 + \left(3\kappa\mu^2 - 5\kappa^2\mu\right)\gamma_1}{2\kappa^{\frac{3}{2}}\,(\kappa-\mu)^2}.$$

Hence we proved the following theorem.

Theorem 2.2. *If the polynomial $p(x)$ possesses two distinct zeros κ and μ, among them κ has the multiplicity 2, then the explicit formula*

$$\begin{aligned}\gamma_n = &\left\{ \frac{\gamma_{4+\xi_n} - (\kappa+\mu)\,\gamma_{2+\xi_n} + \kappa\mu\gamma_{\xi_n}}{2\kappa^{1+\frac{\xi_n}{2}}\,(\kappa-\mu)} n \right. \\ &\quad + (-1)^{\xi_{n+1}} \frac{(\mu-3\kappa)^{\xi_n}\gamma_{4+\xi_n} + (-2\kappa)^{\xi_{n+1}}\left(5\kappa^2-\mu^2\right)^{\xi_n}\gamma_{2+\xi_n}}{2\kappa^{1+\frac{\xi_n}{2}}\,(\kappa-\mu)^2} \\ &\quad \left. + (-1)^{\xi_{n+1}} \frac{\left(2\kappa\mu-\mu^2\right)^{\xi_{n+1}}\left(3\kappa\mu^2 - 5\kappa^2\mu\right)^{\xi_n}\gamma_{\xi_n}}{2\kappa^{1+\frac{\xi_n}{2}}\,(\kappa-\mu)^2} \right\} \kappa^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \\ &+ \frac{\gamma_{4+\xi_n} - 2\kappa\gamma_{2+\xi_n} + \kappa^2\gamma_{\xi_n}}{\mu^{\frac{\xi_n}{2}}\,(\kappa-\mu)^2} \mu^{\left\lfloor \frac{n}{2} \right\rfloor - 1}.\end{aligned}$$

describes the nth term of the sequence $\{\gamma\}$.

2.3. Case 3

In the last part, we suppose that the zeros of (7) coincide. Again by Theorem 1,

$$\gamma_n = \left(\kappa_1 n^2 + \tau_1 n + \mu_1\right)\left(\sqrt{\kappa}\right)^n + \left(\kappa_2 n^2 + \tau_2 n + \mu_2\right)\left(-\sqrt{\kappa}\right)^n \tag{10}$$

If n is even, then

$$\gamma_n = \left((\kappa_1 + \kappa_2)\, n^2 + (\tau_1 + \tau_2)\, n + (\mu_1 + \mu_2) \right) \left(\sqrt{\kappa}\right)^n$$

holds, where

$$\kappa_1 + \kappa_2 = \frac{\gamma_4 - 2\kappa\gamma_2 + \kappa^2\gamma_0}{8\kappa^2}, \quad \tau_1 + \tau_2 = -\frac{\gamma_4 - 4\kappa\gamma_2 + 3\kappa^2\gamma_0}{4\kappa^2} \quad \text{and} \quad \mu_1 + \mu_2 = \gamma_0.$$

Assuming odd n, (10) returns with

$$\gamma_n = \left((\kappa_1 - \kappa_2)\, n^2 + (\tau_1 - \tau_2)\, n + (\mu_1 - \mu_2) \right) \left(\sqrt{\kappa}\right)^n$$

where

$$\kappa_1-\kappa_2=\frac{\gamma_5-2\kappa\gamma_3+\kappa^2\gamma_1}{8\kappa^{\frac{5}{2}}},\ \tau_1-\tau_2=\frac{-\left(\gamma_5-3\kappa\gamma_3+2\kappa^2\gamma_1\right)}{2\kappa^{\frac{5}{2}}}$$

and

$$\mu_1-\mu_2=\frac{3\gamma_5-10\kappa\gamma_3+15\kappa^2\gamma_1}{8\kappa^{\frac{5}{2}}}.$$

Thus the proof of the forthcoming theorem is complete.

Theorem 2.3. *If $p(x)$ has only one zero with multiplicity 3, say κ, then*

$$\begin{aligned}\gamma_n = &\left\{\left(\frac{\gamma_{4+\xi_n}-2\kappa\gamma_{2+\xi_n}+\kappa^2\gamma_{\xi_n}}{8\kappa^{2+\frac{\xi_n}{2}}}\right)n^2\right.\\ &\quad-\left(\frac{\gamma_{4+\xi_n}-4^{\xi_{n+1}}3^{\xi_n}\kappa\gamma_{2+\xi_n}+3^{\xi_n}2^{\xi_{n+1}}\kappa^2\gamma_{\xi_n}}{2^{\xi_n}2\kappa^{2+\frac{\xi_n}{2}}}\right)n\\ &\quad\left.+\gamma_0^{\xi_{n+1}}\left(\frac{3\gamma_5-10\kappa\gamma_3+15\kappa^2\gamma_1}{8\kappa^{2+\frac{\xi_n}{2}}}\right)^{\xi_n}\right\}\kappa^{\left\lfloor\frac{n}{2}\right\rfloor-1}.\end{aligned}$$

References

1. Cooper C., *An identity for periodic k second order linear recurrence systems*, Cong. Numer., **200** (2010), 95–106.
2. Edson M. and Yayenie O., *A new generalization Fibonacci sequence and extended Binet's formula*, Integers **9** (2009), 639–654.
3. Edson M., Lewis S. and Yayenie O., *The k-periodic Fibonacci sequence and an extended Binet's formula*, Integers **11** (2011), 639–652.
4. Irmak N. and Szalay L., *On k-periodic binary recurrence*, submitted.
5. Shorey T. N. and Tijdeman R., *Exponential Diophantine Equations*, Cambridge University Press, 1986.
6. Yayenie O., *A note on generalized Fibonacci sequences*, Appl. Math. Comp. **217** (2011) 5603–5611.

M. Alp, Mathematics Department, Art and Science Faculty, Niğde University, Nigde, Turkey, 51240, *e-mail*: `muratalp@nigde.edu.tr`

N. Irmak, Mathematics Department, Art and Science Faculty, Niğde University, Nigde, Turkey, 51240, *e-mail*: `nirmak@nigde.edu.tr`

L. Szalay, Institute of Mathematics, University of West Hungary, Ady E. utca 5, H-9400 Sopron, Hungary, *e-mail*: `laszalay@emk.nyme.hu`