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SUMS OF SEVENTH POWERS IN THE RING OF POLYNOMIALS OVER THE FINITE FIELD WITH FOUR ELEMENTS

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Abstract. We study representations of polynomials $P \in \mathbb{F}_4[T]$ as sums $P = X_1^7 + \dots + X_s^7$.

1. Introduction

Let F be a finite field of characteristic p with $q = p^m$ elements. Analogues of the Waring's problem for the polynomial ring $F[T]$ were investigated, ([**19**], [**12**], [**16**], [**6**], [**17**], [**8**], [**5**], [**13**], [**14**], [**10**], [**9**], [**2**], [**3**] [**4**]). Let $k > 1$ be an integer. Roughly speaking, Waring's problem over $F[T]$ consists in representing a polynomial $M \in F[T]$ as a sum

$$M = M_1^k + \dots + M_s^k \tag{1.1}$$

with $M_1, \dots, M_s \in F[T]$. Some obstructions to that may occur ([**15**]), and lead to consider Waring's problem over the subring $\mathcal{S}(F[T], k)$ formed by the polynomials of $F[T]$ which are sums of k-th powers. Some cancellations may occur in representations (1.1), so that it is possible to have a representation (1.1) with $\deg M$ small and $\deg(M_i^k)$ large. Without degree conditions in (1.1), the problem of representing M as sum (1.1) is close to the so called easy Waring's problem for $\mathbb{Z}$. In order to have a problem close to the non-easy Waring's problem, the degree conditions

$$k \deg M_i < \deg M + k \tag{1.2}$$

are required. Representations (1.1) satisfying degree conditions (1.2) are called *strict representations*, see [**6**, Definition 1.8] in opposition to representations without degree conditions. For the strict Waring's problem, analogue of the classical Waring numbers $g_{\mathbb{N}}(k)$ and $G_{\mathbb{N}}(k)$ have been defined as follows. Let $g(p^m, k)$ denote the least integer s (if it exists) such that every polynomial $M \in \mathcal{S}(F[T], k)$ may be written as a sum (1.1) satisfying the degree conditions (1.2); otherwise we put $g(p^m, k) = \infty$.

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Similarly, $G(p^m, k)$ denotes the least integer fulfilling the above condition for each polynomial $M \in \mathcal{S}(F[T], k)$ of sufficiently large degree. This notation is possible since these numbers depend only on p^m and k. The set $\mathcal{S}(F[T], k)$ and the parameters $G(p^m, k), g(p^m, k)$ are not sufficient to describle all possible cases, see [**1**, Proposition 4.4], so that in [**2**] and [**3**] we introduced new parameters defined as follows.

Let $\mathcal{S}^{\times}(F[T], k)$ denote the set of polynomials in $F[T]$ which are strict sums of k-th powers. Let $g^{\times}(p^m, k)$ denote the least integer s (if it exists) such that every polynomial $M \in \mathcal{S}^{\times}(F[T], k)$ may be written as a strict sum

$$M = M_1^k + \ldots + M_s^k.$$

Similarly, $G^{\times}(p^m, k)$ denotes the least integer s fulfilling the same condition for each polynomial $M \in \mathcal{S}^{\times}(F[T], k)$ of sufficiently large degree. Gallardo's method for cubes ([**8**] and [**5**]) was generalized in [**1**] and [**11**] where bounds for $g(p^m, k)$ and $G(p^m, k)$ were established when p^m and k satisfy some conditions. A bound for $g(p^m, k)$ was established in [**1**] in the case when $F = \mathcal{S}(F, k)$ if one of the two following conditions is satisfied:

i) $p > k$
ii) $p^n > k = hp^{\nu} - 1$ for some integers $\nu > 0$ and $0 < h \leq p$.

The smallest exponent k satisfying condition ii) is $k = 3$. It gave a matter for many articles, see [**8**], [**5**], [**9**], [**10**]. In the case of even characteristic, the second smallest exponent k satisfying condition ii) is $k = 7$. The case $k = 7, q = 2^m$ with $m > 3$ is covered by [**1**, Theorems 1.2 and 1.3] or by [**11**, Theorem 1.4]. For almost all $q = 2^m$, the upper bounds obtained in these articles for the numbers $G(2^m, 7)$ are comparable with the bound $G_{\mathbb{N}}(7) \leq 33$ known for the corresponding Waring's number for the integers ([**18**]). The case of the numbers $g(2^m, 7)$ is different. In the case when $m \notin \{1, 2, 3\}$ [**1**, Theorem 1.3] as well as [**11**, Theorem 1.4] gives $g(2^m, 7) \leq 239\ell(2^m, 7)$ when for the integers, it is known that $g_{\mathbb{N}}(7) = 143$ ([**7**]). In [**4**] we obtained better bounds for the numbers $g(2^m, 7)$ in the case when $m \notin \{1, 2, 3\}$, the method yielding also to better bounds for some numbers $G(2^m, 7)$. The aim of this paper is the study of one of the remaining cases, namely, the case $q = 4$. The case $q = 8$ will be the subject of a separate paper. When a finite field with 8 elements is not a 7-Waring field, every field with 4^h elements is a 7-Waring field, so that, from [**15**], $\mathcal{S}(\mathbb{F}_4[T], 7) = \mathbb{F}_4[T]$. We will see further that T is not a strict sum of seventh powers in the ring $\mathbb{F}_4[T]$, see Proposition 3.5 below, so that $\mathcal{S}(\mathbb{F}_4[T], 7) \neq \mathcal{S}^{\times}(\mathbb{F}_4[T], 7)$.

The main results proved in this work are summarized in the following theorems.

Theorem 1.1. *We have*

$$\mathcal{S}^{\times}(\mathbb{F}_4[T], 7) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_{\infty},$$

where

(i) $\mathcal{A}_1$ *is the set of polynomials* $A = \sum_{n=0}^{7} a_n T^n \in \mathbb{F}_4[T]$ *such that* $a_1 = a_4, a_2 = a_5, a_3 = a_6$*;*

(ii) $\mathcal{A}_2$ *is the set of polynomials* $A = \sum_{n=0}^{14} a_n T^n \in \mathbb{F}_4[T]$ *with* $7 < \deg A \leq 14$ *such that*

$$\begin{cases} a_1 + a_4 + a_{10} + a_{13} = 0, \\ a_2 + a_5 + a_8 + a_{11} = 0, \\ a_3 + a_6 + a_9 + a_{12} = 0; \end{cases}$$

(iii) $\mathcal{A}_3$ *is the set of polynomials* $A = \sum_{n=0}^{21} a_n T^n \in \mathbb{F}_4[T]$ *with* $14 < \deg A \leq 21$ *such that*

$$a_3 + a_6 + a_9 + a_{12} + a_{15} + a_{18} = 0;$$

(iv) $\mathcal{A}_\infty = \{A \in \mathbb{F}_4[T] \mid \deg A > 21\}$.

See Proposition 6.6 below.

Theorem 1.2. *Every polynomial* $P \in \mathbb{F}_4[T]$ *with degree* ≥ 435 *is a strict sum of* 33 *seventh powers, so that*

$$G(4,7) = G^\times(4,7) \leq 33,$$

and we have

$$g(4,7) = \infty,$$
$$g^\times(4,7) \leq 43.$$

This theorem is given by Corollaries 3.6, 6.4 and by Theorem 6.7

Proving that polynomials of small degree are sums or strict sums of seventh powers requires some results on the solvability of systems of algebraic equations over the finite field $\mathbb{F}_4$. This is done in Section 2. A characterization of polynomials of degree ≤ 21 that are strict sums of seventh powers is given in Section 3. In Section 4, using the general descent process described in [**1**], we obtain a first upper bound for $G(4,7)$. In Section 5 we describe other descent processes. They are used in Section 6 to get a better upper bound for $G(4,7)$ as well as a bound for $g(4,7)$. We denote by F the field $\mathbb{F}_4$ and by α a root of the equation $\alpha^2 = \alpha + 1$.

2. Equations

Proposition 2.1. *For every* $(a,b) \in F^2$*, the system*

$$(\mathcal{A}(a,b)) \qquad \begin{cases} x_1 + x_2 = a, \\ u_1 x_1 + u_2 x_2 = b, \end{cases}$$

has solutions $(u_1, u_2, x_1, x_2) \in F^4$ *satisfying the condition* $x_1 x_2 u_1 u_2 \neq 0$.

Proof. Suppose $a = b$. Choose $x_1 \in F - \{0, a\}$. Then, $(1, 1, x_1, a + x_1)$ is a solution of $(\mathcal{A}(a,b))$. Suppose $a \neq b$. There is $u_2 \in F - \mathbb{F}_2$ such that $au_2 + b \neq 0$. Then, $\left(1, u_2, \frac{au_2+b}{1+u_2}, a + \frac{au_2+b}{1+u_2}\right)$ is a solution of $(\mathcal{A}(a,b))$. Moreover, since $a \neq b$, we have $\frac{au_2+b}{1+u_2} \neq a$, so that

$$u_2 \times \frac{au_2+b}{1+u_2} \times \left(a + \frac{au_2+b}{1+u_2}\right) \neq 0.$$

□

Proposition 2.2. *For $(a,b,c) \in F^3$, let $(\mathcal{B}_s(a,b,c))$ denote the system of equations*

$$\begin{cases} x_1 + \ldots + x_s = a, \\ y_1 + \ldots + y_s = b, \\ x_1y_1 + \ldots + x_sy_s = c. \end{cases}$$

(I) For every $(a,b,c) \in F^\times \times F \times F$, the system $(\mathcal{B}_2(a,b,c))$ admits solutions $(x_1,x_2,y_1,y_2) \in F^4$ satisfying the condition $x_1x_2 \neq 0$.

(II) For every $(a,b,c) \in F^3$, the system $(\mathcal{B}_3(a,b,c))$ admits solutions $(x_1,x_2,x_3,y_1,y_2,y_3) \in F^6$ satisfying the condition $x_1x_2x_3y_1y_2y_3 \neq 0$.

(III) For every $(a,b,c) \in F^\times \times F \times F$, the system $(\mathcal{B}_3(a,b,c))$ admits solutions $(x_1,x_2,x_3,y_1,y_2,y_3) \in F^6$ satisfying the conditions

$$\begin{cases} x_1x_2x_3y_1y_2y_3 \neq 0, \\ x_1^2y_1 \neq x_2^2y_2. \end{cases}$$

Proof. (I) Suppose $a \neq 0$. Let $x_1 \in F - \{0,a\}$ and let $x_2 = a + x_1$. Then, $x_2 \neq 0$ and $x_2 \neq x_1$. The matrix

$$\begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix}$$

is invertible. Thus, for each $(b,c) \in F^2$, there exists $(y_1,y_2) \in F^2$ such that

$$\begin{cases} y_1 + y_2 = b, \\ x_1y_1 + x_2y_2 = c. \end{cases}$$

(II) Let $\mathbf{E}(a,b,c)$ denote the set of $(x_1,x_2,x_3,y_1,y_2,y_3) \in F^6$ solutions of $(\mathcal{B}_3(a,b,c))$ satisfying $x_1x_2x_3y_1y_2y_3 \neq 0$, and satisfying

$$\begin{cases} x_1x_2x_3y_1y_2y_3 \neq 0, \\ x_1^2y_1 \neq x_2^2y_2, \end{cases}$$

respectively. For $(x_1,x_2,x_3,y_1,y_2,y_3) \in F^6$, the three following statements are equivalent:

(i) $(x_1,x_2,x_3,y_1,y_2,y_3) \in \mathbf{E}(a,b,c)$,

(ii) $(y_1,y_2,y_3,x_1,x_2,x_3) \in \mathbf{E}(b,a,c)$,

(iii) $(x_1y_1, x_2y_2, x_3y_3, (y_1)^2, (y_2)^2, (y_3)^2) \in \mathbf{E}(c,b^2,a)$. Thus, it suffices to deal with the cases $(a,b,c) = (0,0,0)$, $(a,b,c) = (a,0,0)$ with $a \neq 0$, $(a,b,c) = (a,b,0)$ with $ab \neq 0$, and (a,b,c) with $abc \neq 0$. Firstly, we observe that if $x \in F - \mathbb{F}_2$, then $(1,x,x+1,1,x,x+1) \in \mathbf{E}(0,0,0)$. Now, we consider the systems with $a \neq 0$. Up to the automorphism $x \mapsto ax$, and the $\mathbb{F}_2$-automorphism $\alpha \mapsto \alpha + 1$, it suffices to consider the cases $(a,b,c) = (1,0,0)$, $(a,b,c) = (1,1,0)$,
$(a,b,c) = (1,1,1)$, $(a,b,c) = (1,1,\alpha)$. Observe that

$(1,1,1,1,\alpha,\alpha+1) \in \mathbf{E}(1,0,0)$,

$(\alpha+1,\alpha+1,1,1,\alpha,\alpha) \in \mathbf{E}(1,1,0)$,

$(1,\alpha,\alpha,1,1,1) \in \mathbf{E}(1,1,1)$,

$(1,\alpha+1,\alpha+1,\alpha+1,\alpha+1,1) \in \mathbf{E}(1,1,\alpha)$.

□

Proposition 2.3. *For every $(a,b,c)\in F^3$, the system*

$$(\mathcal{C}(a,b,c))\qquad \begin{cases} x_1+x_2=a,\\ y_1+y_2=b,\\ x_1y_1z_1^2+x_1^2y_1^2+x_2y_2z_2^2+x_2^2y_2^2=c \end{cases}$$

admits solutions $(x_1,x_2,y_1,y_2,z_1,z_2)\in F^6$ satisfying the condition $x_1x_2y_1y_2\neq 0$.

Proof. Let $x_1\in F$ be such that $x_1\neq 0,a$, let $y_1\in F$ be such that $y_1\neq 0,b$ and let $z_1\in F$. Let $x_2=a+x_1$ and $y_2=b+y_1$. Then, $x_1x_2y_1y_2\neq 0$. Let $z_2\in F$ be defined by the relation $x_2^2y_2^2z_2=c^2+x_1^2y_1^2z_1+x_1y_1+x_2y_2$. Then, $(x_1,x_2,y_1,y_2,z_1,z_2)$ is a solution of $(\mathcal{C}(a,b,c))$. □

Lemma 2.4. *For every $(a,b,c)\in F^\times\times F\times F$, the system of equations*

$$(\mathcal{S}_1(a,b,c))\qquad \begin{cases} z_1+\alpha z_2+(\alpha+1)z_3=b,\\ az_1+z_1^2+(\alpha+1)az_2+z_2^2+\alpha az_3+z_3^2=c, \end{cases}$$

admits solutions $(z_1,z_2,z_3)\in F^3$.

Proof. Let $\nu=\nu(a,b,c)$ denote the number of $(z_1,z_2,z_3)\in F^3$ solutions of $(\mathcal{S}_1(a,b,c))$. For $t\in F$ let

$$\Psi(t)=(-1)^{\mathrm{tr}(t)}$$

where $\mathrm{tr}\colon F\to\mathbb{F}_2$ is the absolute trace map. Then Ψ is a non-trivial character, so that by orthogonality,

$$\begin{aligned}\nu=&\sum_{(z_1,z_2,z_3)\in F^3}\frac{1}{4}\sum_{t\in F}\Psi(t(b+z_1+\alpha z_2+(\alpha+1)z_3))\\ &\times\frac{1}{4}\sum_{u\in F}\Psi(u(c+az_1+z_1^2+(\alpha+1)az_2+z_2^2+\alpha az_3+z_3^2)).\end{aligned}$$

Thus,

$$\begin{aligned}16\nu=&\sum_{(t,u)\in F^2}\Psi(bt+cu)\left(\sum_{z\in F}\Psi((t+au)z+uz^2)\right)\\ &\times\left(\sum_{z\in F}\Psi((\alpha t+a(\alpha+1)u)z+uz^2)\right)\\ &\times\left(\sum_{z\in F}\Psi(((\alpha+1)t+\alpha au)z+uz^2)\right).\end{aligned}$$

From [**2**, Proposition 2.3], for $(v,w)\in F^2$, we have

$$\sum_{z\in F}\Psi(vz+wz^2)=\begin{cases}4 & \text{if}\quad w=v^2,\\ 0 & \text{if}\quad w\neq v^2.\end{cases}$$

Therefore,

$$\nu=4\sum_{(t,u)\in\mathbf{E}}\Psi(bt+cu),$$

where $\mathbf{E}$ is the subset of F^2 formed by the pairs (t,u) satisfying the three conditions

$$\begin{cases} u = (t+au)^2, \\ u = (\alpha t + a(\alpha+1)u)^2, \\ u = ((\alpha+1)t + \alpha a u)^2. \end{cases}$$

Obviously, $(0,0) \in \mathbf{E}$. Conversely, let $(t,u) \in \mathbf{E}$. Then the first and second conditions give that $t + au = \alpha t + a(\alpha+1)u$ while the first and last conditions give that $t + au = (\alpha+1)t + \alpha a u$, so that $(\alpha+1)au = t = \alpha a u$ with $a \neq 0$. Thus, $t = u = 0$, so that $\mathbf{E} = \{(0,0)\}$ and $\nu = 4$. □

Proposition 2.5. *Let* $\mathbf{b} = (a,b,c,d) \in F^4$. *Then the system of equations*

$$(\mathcal{D}(\mathbf{b})) \qquad \begin{cases} \sum_{i=1}^{3} y_i = a, \\ \sum_{i=1}^{3} u_i y_i = b, \\ \sum_{i=1}^{3} u_i^2 z_i^2 y_i = c, \\ \sum_{i=1}^{3} (u_i^2 z_i + u_i z_i^2) y_i = d \end{cases}$$

admits solutions $(u_1,u_2,u_3,y_1,y_2,y_3,z_1,z_2,z_3) \in F^9$ *such that* $u_1u_2u_3y_1y_2y_3 \neq 0$.

Proof. (I) Suppose that there exists $(y_1,y_2,y_3,u) \in F^4$ satisfying the conditions:

$$\begin{cases} y_1 + y_2 + y_3 = a, \\ y_1 + y_2 + u y_3 = b, \\ y_1 y_2 y_3 u \neq 0, \\ y_1 \neq y_2, \end{cases}$$

and denote (H) this hypothesis. Then the matrix

$$\begin{pmatrix} y_1 & y_2 \\ y_1^2 & y_2^2 \end{pmatrix}$$

is invertible. Let $z_3 \in F$. There is $(z_1,z_2) \in F^2$ such that

$$\begin{aligned} y_1 z_1 + y_2 z_2 &= c + d + (u^2 z_3 + u^2 z_3^2 + u z_3^2) y_3, \\ y_1^2 z_1 + y_2^2 z_2 &= c^2 + u z_3 y_3^2. \end{aligned}$$

Then, we have

$$\begin{aligned} z_1^2 y_1 + z_2^2 y_2 + u^2 z_3^2 y_3 &= c, \\ (z_1 + z_1^2) y_1 + (z_2 + z_2^2) y_2 + (u^2 z_3 + u z_3^2) y_3 &= d, \end{aligned}$$

so that $(1,1,u,y_1,y_2,y_3,z_1,z_2,z_3)$ is a solution of $(\mathcal{D}(\mathbf{b}))$ such that $uy_1y_2y_3 \neq 0$.

(II) We prove that if one of the three following conditions:

(i) $a = b$,
(i) $a \notin \{0, b, (\alpha+1)b\}$,
(iii) $a = (\alpha+1)b \neq 0$, (so that $a \neq b$,)

is satisfied, then hypothesis (H) is satisfied, so that the conclusion of the proposition holds.

(i) Suppose $a = b$. If $a = 0$, then $(1, \alpha, \alpha+1)$ is a solution of (e_1). If $a \neq 0$, then (a, y, y) with $y \notin \{0, a\}$ is a solution of (e_1). Thus, in the two cases, (e_1) admits solutions $(y_1, y_2, y_3) \in F^3$ such that $y_1y_2y_3 \neq 0$ and $y_1 \neq y_2$. Hypothesis (H) is satisfied with $u = 1$.

(ii) Suppose $a \notin \{0, b, (\alpha+1)b\}$. Then $a+\alpha(a+b) \neq 0$. Let $u = \alpha$, $y_3 = \alpha(a+b)$. Choose $y_1 \in F - \{0, a + \alpha(a + b)\}$ and $y_2 = y_1 + a + \alpha(a + b)$. Then, $y_1 \neq y_2$ and $y_1y_2y_3 \neq 0$, so that (H) is satisfied.

(iii) Suppose $a = (\alpha+1)b \neq 0$. Let $u = (\alpha+1)$, $y_3 = b$. Choose $y_1 \in F - \{0, \alpha b\}$ and $y_2 = y_1 + \alpha b$. Then, $y_1 \neq y_2$, $y_1y_2y_3 \neq 0$ and $y_1 + y_2 + y_3 = (\alpha + 1)b = a$, $y_1 + y_2 + uy_3 = \alpha b + (\alpha + 1)b = b$, so that (H) is satisfied.

(III) We examine the remaining case, that is the case $a = 0$, $b \neq 0$. Lemma 2.4 gives the existence of $(z_1, z_2, z_3) \in F^3$, a solution of $(\mathcal{S}_1(b, c^2/b, d/b))$ such that

$$\begin{aligned} b^2z_1^2 + (\alpha + 1)b^2z_2^2 + \alpha b^2z_3^2 &= c,\\ b^2z_1 + bz_1^2 + (\alpha + 1)b^2z_2 + bz_2^2 + \alpha b^2z_3 + bz_3^2 &= d. \end{aligned}$$

Let

$$u_1 = b, u_2 = (\alpha + 1)b, u_3 = \alpha b, y_1 = 1, y_2 = \alpha, y_3 = \alpha + 1.$$

Then, $(u_1,\ u_2,\ u_3,\ y_1,\ y_2,\ y_3,\ z_1,\ z_2,\ z_3)$ is a solution of $(\mathcal{D}(\mathbf{b}))$ such that $u_1u_2u_3y_1y_2y_3 \neq 0$. □

Lemma 2.6. *Let $(a, b) \in F^2$. Then the system of equations*

$$(\mathcal{S}_2(a,b)) \qquad \begin{cases} u_1 + u_2 + u_3 = a,\\ x_1 + x_2 + x_3 = b,\end{cases}$$

admits solutions $(u_1, u_2, u_3, x_1, x_2, x_3) \in F^6$ satisfying the conditions

$$u_1u_2u_3 \neq 0, \tag{2.1}$$

$$\det \begin{pmatrix} 1 & 1 & 1\\ u_1 & u_2 & u_3\\ u_1x_1^2 & u_2x_2^2 & u_3x_3^2. \end{pmatrix} \neq 0. \tag{2.2}$$

Proof. If $(u_1, u_2, u_3, x_1, x_2, x_3) \in F^6$ is a solution of $(\mathcal{S}_2(0, 1))$ satisfying conditions (2.1) and (2.2), then for $b \in F, b \neq 0$, $(u_1, u_2, u_3, bx_1, bx_2, bx_3)$ is a solution of $(\mathcal{S}_2(0, b))$ satisfying conditions (2.1) and (2.2). If $(u_1, u_2, u_3, x_1, x_2, x_3) \in F^6$ is a solution of $(\mathcal{S}_2(1, 0))$ satisfying conditions (2.1) and (2.2), then for $a \in F, a \neq 0$, $(au_1, au_2, au_3, x_1, x_2, x_3)$ is a solution of $(\mathcal{S}_2(a, 0))$ satisfying conditions (2.1) and (2.2). If $(u_1, u_2, u_3, x_1, x_2, x_3) \in F^6$ is solution of $(\mathcal{S}_2(1, 1))$ satisfying conditions (2.1) and (2.2), then for $a, b \in F, ab \neq 0$, $(au_1, au_2, au_3, bx_1, bx_2, bx_3)$ is a solution of $(\mathcal{S}_2(a, b))$ satisfying conditions (2.1) and (2.2). It is sufficient to examine the cases $(a, b) = (0, 0), (a, b) = (0, 1), (a, b) = (1, 0), (a, b) = (1, 1)$. Observe that

$(1, \alpha, \alpha^2, \alpha, 1, \alpha^2)$ is a solution of $(\mathcal{S}_2(0, 0))$ satisfying conditions (2.1) and (2.2);
$(1, \alpha, \alpha^2, 0, 0, 1)$ is a solution of $(\mathcal{S}_2(0, 1))$ satisfying conditions (2.1) and (2.2);
$(1, \alpha, \alpha, 1, \alpha, \alpha^2)$ is a solution of $(\mathcal{S}_2(1, 0))$ satisfying conditions (2.1) and (2.2);
$(1, \alpha, \alpha, 0, 0, 1)$ is a solution of $(\mathcal{S}_2(1, 1))$ satisfying conditions (2.1) and (2.2). □

Lemma 2.7. *Let* $(u_1, u_2, u_3, x_1, x_2, x_3) \in F^6$ *be such that*

$$u_1 u_2 u_3 \neq 0, \tag{2.1}$$

$$\det \begin{pmatrix} 1 & 1 & 1 \\ u_1 & u_2 & u_3 \\ u_1 x_1^2 & u_2 x_2^2 & u_3 x_3^2. \end{pmatrix} \neq 0. \tag{2.2}$$

Then, for every $(c, d) \in F^2$*, there exists* $(y_1, y_2, y_3) \in F^3$ *such that*

$$(\mathcal{S}_3(c,d)) \qquad \begin{cases} y_1 + y_2 + y_3 = c, \\ u_1^2 y_1^2 + u_1 x_1^2 y_1 + \ldots + u_3^2 y_3^2 + u_3 x_3^2 y_3 = d. \end{cases}$$

Proof. Let N denote the number of $(y_1, y_2, y_3) \in F^3$ solutions of $(\mathcal{S}_3(c,d))$. With the notations used in the proof of Lemma 2.4, we have

$$\begin{aligned} N = \sum_{(y_1,y_2,y_3)\in F^3} & \frac{1}{4} \sum_{t \in F} \Psi(t(c + y_1 + y_2 + y_3)) \\ & \times \frac{1}{4} \sum_{u \in F} \Psi(u(d + u_1^2 y_1^2 + u_2^2 y_2^2 + u_3^2 y_3^2)). \end{aligned}$$

Thus,

$$16N = \sum_{(t,u)\in F^2} \Psi(ct + du) \prod_{i=1}^{3} \Theta_i(t, u),$$

where

$$\Theta_i(t, u) = \sum_{y \in F} \Psi(ty + u(u_i^2 y^2 + u_i x_i^2 y)).$$

From [**2**, Proposition 2.3], $\Theta_i(t, u) \in \{0, 4\}$ and $\Theta_i(t, u) = 4$ if and only if $uu_i^2 = (t + uu_i x_i^2)^2$. Thus,

$$N = 4 \sum_{(t,u)\in E} \Psi(ct + du),$$

where E is the set of pairs $(t, u) \in F^2$ such that

$$\begin{cases} t + uu_1 x_1^2 = u^2 u_1, \\ t + uu_2 x_2^2 = u^2 u_2, \\ t + uu_3 x_3^2 = u^2 u_3. \end{cases}$$

Observe that $(0, 0) \in E$. Moreover, if $(t, 0) \in E$, then $t = 0$. Suppose that $(t, u) \in E$ with $u \neq 0$. Then,

$$t = u(u_1 x_1^2 + uu_1) = u(u_2 x_2^2 + uu_2) = u(u_3 x_3^2 + uu_3),$$

so that

$$\begin{cases} u_1 x_1^2 + uu_1 = u_2 x_2^2 + uu_2, \\ u_1 x_1^2 + uu_1 = u_3 x_3^2 + uu_3. \end{cases}$$

Thus,

$$\begin{cases} u_1 x_1^2 + u_2 x_2^2 = u(u_1 + u_2), \\ u_1 x_1^2 + u_3 x_3^2 = u(u_1 + u_3), \end{cases}$$

so that

$$(u_1x_1^2 + u_2x_2^2)(u_1 + u_3) + (u_1x_1^2 + u_3x_3^2)(u_1 + u_2),$$

in contradiction with condition (2.2). $\square$

Proposition 2.8. *Let* $\mathbf{b} = (b_1, b_2, \ldots, b_7) \in F^7$. *Then the system of equations*

$$(\mathcal{E}(\mathbf{b})) \qquad \begin{cases} \sum_{i=1}^{3} u_i = b_1, \\ \sum_{i=1}^{3} x_i = b_2, \\ \sum_{i=1}^{3} (y_i + u_i^2x_i^2) = b_3, \\ \sum_{i=1}^{3} (z_i + u_ix_i^3) = b_4, \\ \sum_{i=1}^{3} (u_i^2y_i^2 + u_ix_i^2y_i) = b_5, \\ \sum_{i=1}^{3} (u_ix_i^2z_i + u_ix_iy_i^2 + u_i^2x_i^2) = b_6, \\ \sum_{i=1}^{3} (u_i^2z_i^2 + u_iy_i^3 + u_i^2x_iy_i + u_ix_i^3) = b_7 \end{cases}$$

admits solutions $(u_1, u_2, u_3, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) \in F^{12}$ *with* $u_1u_2u_3 \neq 0$.

Proof. Lemma 2.6 gives the existence of $(u_1, u_2, u_3, x_1, x_2, x_3) \in F^6$, a solution of $\mathcal{S}_2(b_1, b_2)$ satisfying (2.1) and (2.2). Lemma 2.7 gives the existence of $(y_1, y_2, y_3) \in F^3$, a solution of $\mathcal{S}_3(b_3 + \sum_{i=1}^{3} u_i^2x_i^2, b_5)$. Condition (2.2) insures the existence of $(z_1, z_2, z_3) \in F^3$ such that

$$\begin{cases} z_1 + z_2 + z_3 = b_4 + \sum_{i=1}^{3} u_ix_i^3, \\ u_1z_1 + u_2z_2 + u_3z_3 = b_7^2 + \sum_{i=1}^{3} (u_i^2x_i^3 + u_i^2y_i^3 + u_ix_i^2y_i^2, \\ u_1x_1^2z_1 + u_2x_2^2z_2 + u_3x_3^2z_3 = b_6 + \sum_{i=1}^{3} (u_ix_iy_i^2 + u_i^2x_i^2). \end{cases}$$

$\square$

Lemma 2.9. *Let* $(a, b, c) \in F^\times \times F^2$ *be such that* $ab + c \neq 0$. *Then the system*

$$(\mathcal{S}_4(a, b, c)) \qquad \begin{cases} u + v = a, \\ x + y = b, \\ ux + vy = c \end{cases}$$

admits a solution $(u, x, v, y) \in F^4$ *such that* $uv \neq 0$ *and* $u^2x + v^2y \neq 0$.

Proof. Let $u \in F - \{0, a\}$ and $v = u + a$. Then $uv(u + v) \neq 0$, so that with $x = (bu + c + ab)/a$ and $y = (bu + c)/a$, (u, x, v, y) is a solution of $(\mathcal{S}_4(a, b, c))$. Suppose that $u^2x + v^2y = 0$. Then, $u^2b + uab + ac = 0$. If $b = 0$, then $c = 0$, in contradiction with $ab + c \neq 0$. Thus $b \neq 0$, so that $u^2 + au + \frac{ac}{b} = 0$ and $\frac{c}{ab} \in \{0, 1\}$. Thus, $c = 0$. We have $u^2x + v^2y = 0$ and $ux + vy = 0$. Since $b \neq 0$, we have $(x, y) \neq (0, 0)$. If $x = 0$, then $vy = 0$, so that $y = 0$, a contradiction. Similarly, $y = 0$ is impossible. Thus, $xy \neq 0$. Therefore $u = u^2x/ux = v^2y/vy = v$ in contradiction with $u+v \neq 0$. Hence, (u, x, v, y) is a solution of $(\mathcal{S}_4(a, b, c))$ such that $uv \neq 0$ and $u^2x + v^2y \neq 0$. □

Proposition 2.10. *Let* $\mathbf{b} = (b_1, b_2, \dots, b_8) \in F^8$. *Then the system of equations*

$$(\mathcal{F}(\mathbf{b})) \qquad \begin{cases} \sum_{i=1}^{3} v_i = b_1, \\ \sum_{i=1}^{3} u_i = b_2, \\ \sum_{i=1}^{3} (x_i + v_i^2 u_i^2) = b_3, \\ \sum_{i=1}^{3} (y_i + v_i u_i^3) = b_4, \\ \sum_{i=1}^{3} (v_i^2 x_i^2 + v_i u_i^2 x_i + u_i + z_i) = b_5, \\ \sum_{i=1}^{3} (v_i u_i^2 y_i + v_i u_i x_i^2 + v_i^2 u_i^2) = b_6, \\ \sum_{i=1}^{3} (v_i u_i^3 + v_i x_i^3 + v_i^2 y_i^2 + v_i u_i^2 z_i + v_i^2 u_i x_i) = b_7, \\ \sum_{i=1}^{3} (v_i^2 u_i y_i + v_i u_i y_i^2 + v_i x_i^2 y_i) = b_8 \end{cases}$$

admits solutions

$$(v_1, v_2, v_3, u_1, u_2, u_3, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) \in F^{15}$$

satisfying the condition $v_1 v_2 v_3 \neq 0$.

Proof. Let $v_1 = 1$, $v_2 \in F - \{0, 1, b_1 + 1\}$ and let v_3 be defined by

$$v_1 + v_2 + v_3 = b_1.$$

Then we have

$$v_1 v_2 v_3 \neq 0, \quad v_1 \neq v_2.$$

Let $u_1 \in F - \{0, (v_1 v_2)^2\}$, $u_3 = 0$ and let u_2 be defined by

$$u_1 + u_2 + u_3 = b_2.$$

Then we have

$$(\dagger) \qquad v_1 u_1^2 \neq v_3 u_3^2.$$

(I) Suppose that $v_1 u_1 + v_2 u_2 = 0$. Let $y_1 \in F$, $y_2 \in F - \{u_2 y_1 / u_1\}$ and let y_3 be defined by

$$y_1 + y_2 + y_3 = b_4 + \sum_{i=1}^{3} v_i u_i^3.$$

Then we have

$$v_1v_2(u_1y_2 + u_2y_1) \neq 0,$$

so that

$$\det \begin{pmatrix} 1 & 1 & 1 \\ v_1u_1 & v_2u_2 & v_3u_3 \\ v_1y_1 & v_2y_2 & v_3y_3 \end{pmatrix} \neq 0.$$

(II) Suppose that $v_1u_1 + v_2u_2 \neq 0$. Let $y_1 \in F$. Since $u_1 \neq (v_1v_2)^2$, we have $1 + v_1v_2u_1 \neq 0$. Let $y_2 \in F$ be such that

$$(1 + v_1v_2u_1)y_2 + (1 + v_1v_2u_2)y_1 \neq v_3(v_1u_1 + v_2u_2)(b_4 + \sum_{i=1}^{3} v_iu_i^3),$$

and let y_3 be defined by

$$y_1 + y_2 + y_3 = b_4 + \sum_{i=1}^{3} v_iu_i^3.$$

Then we have

$$v_1v_2(u_1y_2 + u_2y_1) + v_3y_3(v_1u_1 + v_2u_2) \neq 0,$$

so that

$$\det \begin{pmatrix} 1 & 1 & 1 \\ v_1u_1 & v_2u_2 & v_3u_3 \\ v_1y_1 & v_2y_2 & v_3y_3 \end{pmatrix} \neq 0.$$

In both cases we get the existence of $(y_1, y_2, y_3) \in F^3$ satisfying

$$\det \begin{pmatrix} 1 & 1 & 1 \\ v_1^2u_1^2 & v_2^2u_2^2 & v_3^2u_3^2 \\ v_1^2y_1^2 & v_2^2y_2^2 & v_3^2y_3^2 \end{pmatrix} \neq 0$$

from which we deduce the existence of $(x_1, x_2, x_3) \in F^3$ such that

$$\begin{aligned} x_1 + x_2 + x_3 &= b_3 + \sum_{i=1}^{3} v_i^2u_i^2, \\ v_1^2u_1^2x_1 + v_2^2u_2^2x_2 + v_3^2u_3^2x_3 &= (b_1 + b_2 + b_6)^2 \sum_{i=1}^{3} v_iu_i^2y_i, \\ v_1^2y_1^2x_1 + v_2^2y_2^2x_2 + v_3^2y_3^2x_3 &= b_8^2 \sum_{i=1}^{3} (v_iu_i^2y_i^2 + v_i^2u_i^2y_i). \end{aligned}$$

From (†),

$$\det \begin{pmatrix} 1 & 1 \\ v_1u_1^2 & v_2u_2^2 \end{pmatrix} \neq 0.$$

Then there exists $(z_1, z_3) \in F^2$ such that

$$\begin{aligned} z_1 + z_3 &= b_2 + b_5 + \sum_{i=1}^{3} (v_i^2x_i^2 + v_iu_i^2x_i), \\ v_1u_1^2z_1 + v_3u_3^2z_3 &= b_7 + \sum_{i=1}^{3} (v_iu_i^3 + v_ix_i^3 + v_i^2y_i^2 + v_i^2u_ix_i). \end{aligned}$$

Let $z_2 = 0$. Then, $(v_1, v_2, v_3, u_1, u_2, u_3, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)$ is a solution of $(\mathcal{F}(\mathbf{b}))$ satisfying $v_1v_2v_3 \neq 0$. □

Proposition 2.11. *Let* $\mathbf{b} = (b_1, b_2, \ldots, b_9) \in F^9$. *Then the system of equations*

$$(\mathcal{G}(\mathbf{b})) \qquad \begin{cases} \sum_{i=1}^{8} u_i = b_1, \\ \sum_{i=1}^{8} x_i = b_2, \\ \sum_{i=1}^{8} (y_i + u_i^2 x_i^2) = b_3, \\ \sum_{i=1}^{8} (z_i + u_i x_i^3) = b_4, \\ \sum_{i=1}^{8} (u_i^2 y_i^2 + u_i x_i^2 y_i) = b_5, \\ \sum_{i=1}^{8} (u_i x_i^2 z_i + u_i x_i y_i^2 + u_i^2 x_i^2) = b_6, \\ \sum_{i=1}^{8} (u_i^2 z_i^2 + u_i y_i^3 + u_i^2 x_i y_i + u_i x_i^3) = b_7, \\ \sum_{i=1}^{8} (u_i^2 y_i z_i + u_i x_i^2 z_i + u_i y_i z_i^2) = b_8, \\ \sum_{i=1}^{8} (u_i^2 x_i z_i + u_i x_i z_i^2 + u_i y_i^2 z_i) = b_9 \end{cases}$$

admits solutions $(u_1, \ldots, u_8,\ x_1, \ldots, x_8, y_1, \ldots, y_8,\ z_1, \ldots, z_8) \in F^{32}$ *such that* $u_1 \ldots u_8 \neq 0$.

Proof. Proposition 2.1 insures the existence of a solution (x_1, x_2, u_1, u_2) of $(\mathcal{A}(b_2, b_6^2))$ such that $u_1u_2x_1x_2 \neq 0$. Thus, we have

$$\begin{aligned} x_1 + x_2 &= b_2, \\ u_1x_1^2z_1 + u_1x_1y_1^2 + u_1^2x_1^2 + u_2x_2^2z_2 + u_2x_2y_2^2 + u_2^2x_2^2 &= b_6. \end{aligned}$$

Let $y_1 = y_2 = z_1 = z_2 = 0$. Proposition 2.5 insures the existence of a solution $(u_3, u_4, u_5, y_3, y_4, y_5, z_3, z_4, z_5) \in F^9$ of $(\mathcal{D}((b_3 + b_6, b_5^2, b_9^2, b_8)))$ such that

$u_3u_4u_5y_3y_4y_5 \neq 0$. Let $x_3 = x_4 = x_5 = 0$. Then, we have

$$\begin{cases} \sum_{i=1}^{5} x_i = b_2, \\ \sum_{i=1}^{5} (y_i + u_i^2 x_i^2) = b_3, \\ \sum_{i=1}^{5} (u_i^2 y_i^2 + u_i x_i^2 y_i) = b_5, \\ \sum_{i=1}^{5} (u_i x_i^2 z_1 + u_i x_i y_i^2 + u_i^2 x_i^2) = b_6, \\ \sum_{i=1}^{5} (u_i^2 y_i z_i + u_i x_i^2 z_i + u_i y_i z_i^2) = b_8, \\ \sum_{i=1}^{5} (u_i^2 x_i z_i + u_i x_i z_i^2 + u_i y_i^2 z_i) = b_9. \end{cases}$$

Let

$$\beta_1 = b_1 + \sum_{i=1}^{5} u_i,$$

$$\beta_4 = b_4 + \sum_{i=1}^{5} (z_i + u_i x_i^3),$$

$$\beta_7 = b_7 + \sum_{i=1}^{5} (u_i^2 z_i^2 + u_i y_i^3 + u_i^2 x_i y_i + u_i x_i^3).$$

From Proposition 2.2, $(\mathcal{B}_3(\beta_1, \beta_4, \beta_7^2))$ admits a solution $(u_6, u_7, u_8, z_6, z_7, z_8) \in F^6$ such that $u_6u_7u_8z_6z_7z_8 \neq 0$. Let $x_6 = x_7 = x_8 = y_6 = y_7 = y_8 = 0$. Then, $(u_1, \dots, u_8, x_1, \dots, x_8, y_1, \dots, y_8, z_1, \dots, z_8)$ is a solution of $(\mathcal{G}(\mathbf{b}))$ such that $u_1 \dots u_8 \neq 0$. □

3. Strict sums of degree less than 21 in $F[T]$

The aim of this section is the proof of the three following theorems.

Theorem 3.1. *Let $A \in F[T]$ with degree ≤ 7, say*

$$A = \sum_{i=0}^{7} a_i T^i.$$

Then, A is a strict sum of seventh powers if and only if its coefficients a_i satisfy the conditions

$$\begin{cases} a_1 = a_4, \\ a_2 = a_5, \\ a_3 = a_6. \end{cases} \tag{3.1}$$

Moreover, if A is a strict sum of seventh powers, then A is a strict sum of 5 seventh powers.

Theorem 3.2. *Let $A \in F[T]$ with degree ≤ 14, say*

$$A = \sum_{i=0}^{14} a_i T^i.$$

Then, A is a strict sum of seventh powers if and only if its coefficients a_i satisfy the conditions

$$\begin{cases} a_1 + a_4 + a_{10} + a_{13} = 0, \\ a_2 + a_5 + a_8 + a_{11} = 0, \\ a_3 + a_6 + a_9 + a_{12} = 0. \end{cases} \tag{3.2}$$

Moreover, if A is a strict sum of seventh powers, then A is a sum of 11 seventh powers.

Theorem 3.3. *Let $A \in F[T]$ be such that $15 \leq \deg A \leq 21$, say*

$$A = \sum_{i=0}^{21} a_i T^i.$$

Then, A is a strict sum of seventh powers if and only if its coefficients $a_1, \dots, a_{21}$ satisfy the condition

$$a_3 + a_6 + a_9 + a_{12} + a_{15} + a_{18} = 0. \tag{3.3}$$

Moreover, if A satisfies condition (3.3), then A is a strict sum of 19 seventh powers.

Theorem 3.1 is a consequence of the two following propositions.

Proposition 3.4. *For $(a, b, c) \in F^3$,*

$$\begin{aligned} cT^7 &+ (aT^2 + bT + c)(T^4 + T) + a \\ &= ((a + b + c)(T + 1))^7 + ((\alpha^2 a + \alpha b + c)(T + \alpha))^7 \\ &\quad + ((\alpha a + \alpha^2 b + c)(T + \alpha^2))^7. \end{aligned} \tag{3.4}$$

Proof. A verification. □

Proposition 3.5.

(i) Let $A \in F[T]$ be such that $\deg A \leq 6$. If A is a strict sum of seventh powers, then its coefficients satisfy (3.1).

(ii) Let

$$A = \sum_{i=0}^{7} a_i T^i$$

in the polynomial ring $F[T]$ be such that conditions (3.1) are satisfied. Then, A is a strict sum of 5 seventh powers.

Proof. Let $A = a_0 + a_1 T + \ldots + a_6 T^6 \in F[T]$. Suppose that A is a strict sum of s seventh powers. Then,

$$A = \sum_{i=1}^{s} (x_i T + y_i)^7$$

with $x_i, y_i \in F$ for $i = 1, \dots, s$. Thus,

$$a_1 = a_4, a_2 = a_5, a_3 = a_6.$$

Now let $(a, b, c) \in F^3$ and let $A = a_7T^7 + (T^4 + T)(aT^2 + bT + c) + a_0$. From (3.4),

$$A + (a_7 + c)T^7 + a_0 + a = X_1^7 + X_2^7 + X_3^7,$$

where $X_1, X_2, X_3 \in F[T]$ have degree ≤ 1, so that

$$A = ((a_7 + c)T)^7 + (a_0 + a)^7 + X_1^7 + X_2^7 + X_3^7.$$

□

Corollary 3.6. *We have $\mathcal{S}^\times(F, 7) \neq \mathcal{S}(F, 7)$, so that $g(4, 7) = \infty$.*

Proof. Conditions (3.1) are not satisfied by T, so that $\mathcal{S}^\times(F[T], 7) \neq F[T]$. On the other hand, from Paley's theorem, [**15**], [**6**, Theorem 1.7], $\mathcal{S}(F[T], 7) = F[T]$. □

Theorem 3.2 is a consequence of the following proposition.

Proposition 3.7. *Let $A \in F[T]$ with degree ≤ 14, say $A = a_0 + a_1T + \dots + a_{14}T^{14}$.*

(i) If A is a sum

$$A = \sum_{i=1}^{s} (X_i)^7$$

with $X_i \in F[T]$ of degree ≤ 2, then the cofficients $a_1, \dots, a_{13}$ satisfy (3.2).

(ii) If $(a_1, \dots, a_{13}) \in F^{13}$ satisfies (3.2), then A is a sum

$$A = X_1^7 + \dots + X_{11}^7$$

of 11 seventh powers of polynomials X_i with $\deg X_i \leq 2$.

Proof. (i) Suppose that A is a sum

$$A = \sum_{i=1}^{s} \left(x_iT^2 + y_iT + z_i\right)^7$$

with $x_i, y_i, z_i \in F$ for $i = 1, \dots, s$. Then,

$$\begin{aligned} a_1 + a_4 + a_{10} + a_{13} = &\sum_{i=1}^{s} y_i(z_i)^3 + \sum_{i=1}^{s} \left((x_i)^2(z_i)^2 + x_i(y_i)^2z_i + y_i(z_i)^3\right) \\ &+ \sum_{i=1}^{s} \left((x_i)^2(z_i)^2 + x_i(y_i)^2z_i + (x_i)^3(y_i)\right) + \sum_{i=1}^{s} (x_i)^3y_i = 0. \end{aligned}$$

The proof of the other identities is similar.

(ii) Conversely, suppose that $(a_1, \dots, a_{13}) \in F^{13}$ satisfies (3.2). Proposition 2.3 insures the existence of $(x_1, x_2, y_1, y_2, z_1, z_2) \in F^6$ solution of $(\mathcal{C}(a_{11}, a_{13}, a_9))$ such that $x_1x_2y_1y_2 \neq 0$. For such a solution, we have

$$\begin{cases} a_{13} = \sum_{i=1}^{2} y_i = \sum_{i=1}^{2} x_i^3 y_i, \\ a_{11} = \sum_{i=1}^{2} x_i = \sum_{i=1}^{2} x_i y_i^3, \\ a_9 = \sum_{i=1}^{2} (x_i^2 y_i^2 + x_i y_i z_i^2). \end{cases}$$

Let

$$\begin{cases} a = a_8 + \sum_{i=1}^{2} (x_i z_i^3 + x_i^2 y_i z_i + x_i y_i^3), \\ b = a_{12} + \sum_{i=1}^{2} (x_i^3 z_i + x_i^2 y_i^2), \\ c = a_{10}^2 + \sum_{i=1}^{2} (x_i z_i + x_i^2 y_i z_i^2 + x_i^3 y_i^2). \end{cases}$$

Proposition 2.2 gives the existence of a solution $(x_3, x_4, x_5, z_3, z_4, z_5) \in F^6$ of $(\mathcal{B}_3(a, b, c))$ such that $x_3x_4x_5z_3z_4z_5 \neq 0$. For such a solution, we have

$$\begin{cases} a = \sum_{i=3}^{5} x_i = \sum_{i=3}^{5} x_i z_i^3, \\ b = \sum_{i=3}^{5} z_i = \sum_{i=3}^{5} x_i^3 z_i, \\ c^2 = \sum_{i=3}^{5} x_i z_i. \end{cases}$$

Let

$$x_6 = a_{14} + \sum_{i=1}^{5} x_i, \qquad y_3 = y_4 = y_5 = y_6 = z_6 = 0.$$

Thus, we have

$$\begin{cases} a_{12} = \sum_{i=1}^{6} (x_i^3 z_i + x_i^2 y_i^2), \\ a_{10} = \sum_{i=1}^{6} (x_i^2 z_i^2 + x_i y_i^2 z_i + x_i^3 y_i), \\ a_8 = \sum_{i=1}^{6} (x_i z_i^3 + x_i^2 y_i z_i + x_i y_i^3), \end{cases}$$

as well as

$$\begin{cases} a_{13} = \sum_{i=1}^{6} x_i^3 y_i, \\ a_{11} = \sum_{i=1}^{6} x_i y_i^3, \\ a_9 = \sum_{i=1}^{6} (x_i^2 y_i^2 + x_i y_i z_i^2). \end{cases}$$

Let

$$(\ddagger) \qquad B = A + \sum_{i=1}^{6}(x_iT^2 + y_iT + z_i)^7.$$

Then $\deg B \leq 7$. If

$$B = \sum_{i=0}^{7} b_iT^i,$$

then,

$$b_4 + b_1 = a_4 + a_1 + \sum_{i=1}^{6}(x_i^2z_i^2 + x_iy_i^2z_i),$$

$$b_5 + b_2 = a_5 + a_2 + \sum_{i=1}^{6}(x_iz_i^3 + x_i^2y_iz_i),$$

$$b_6 + b_3 = a_6 + a_3 = \sum_{i=1}^{6}(x_i^3z_i + x_iy_iz_i^2).$$

Condition (3.2) insures that

$$b_4 + b_1 = a_{13} + a_{10} + \sum_{i=1}^{6}(x_i^2z_i^2 + x_iy_i^2z_i) = 0,$$

$$b_5 + b_2 = a_{11} + a_8 + \sum_{i=1}^{6}(x_iz_i^3 + x_i^2y_iz_i) = 0,$$

$$b_6 + b_3 = a_{12} + a_9 = \sum_{i=1}^{6}(x_i^3z_i + x_iy_iz_i^2) = 0,$$

so that (3.1) is satisfied by $(b_1 \dots, b_6)$. Proposition 3.5 gives the existence of polynomials $X_1, \dots, X_5 \in F[T]$ of degree ≤ 1 such that

$$B = \sum_{i=0}^{5} X_i^7.$$

We conclude with ($\ddagger$). □

Theorem 3.3 is a consequence of the following proposition.

Proposition 3.8. *Let*

$$A = \sum_{i=0}^{21} a_iT^i$$

be a polynomial in $F[T]$ *with* $\deg A \leq 21$. *Then,* A *may be written as a sum*

$$A = \sum_{i}^{s}(X_i)^7$$

with $X_i \in F[T]$ of degree ≤ 3 if and only if its coefficients satisfy the condition

$$a_3 + a_6 + a_9 + a_{12} + a_{15} + a_{18} = 0. \tag{3.3}$$

Moreover, if A satisfies condition (3.3)*, then A is a sum of* 19 *seventh powers of polynomials $X_i \in F[T]$ of degree ≤ 3.*

Proof. (I) Let $A = \sum_{i=0}^{21} a_i T^i \in F[T]$. Suppose that A is a sum

$$A = \sum_{i=1}^{s} (u_i T^3 + x_i T^2 + y_i T + z_i)^7$$

with $u_i, x_i, y_i, z_i \in F$ for $i = 1, \ldots, s$. Then we have

$$a_3 + a_6 + a_9 + a_{12} + a_{15} + a_{18} = 0.$$

(II) Let $(a_0, a_1, \ldots, a_{20}, a_{21}) \in F^{22}$ satisfying (3.3). We construct a representation of A as a sum of seventh powers of polynomials of degree ≤ 3 in two steps.
(i) *First step* – From Proposition 2.11, there exists

$$(u_1, \ldots, u_8, x_1, \ldots, x_8, y_1, \ldots, y_8, z_1, \ldots, z_8) \in F^{32}$$

solution of $(\mathcal{G}(\mathbf{b}))$ with

$$\begin{aligned}\mathbf{b} = (&a_{21}, a_{20}, a_{19}, a_{18}, a_{17} + a_{20}, a_{16}, a_{15}, a_1 + a_4 + a_{10} + a_{13} + a_{16} + a_{19},\\ &a_2 + a_5 + a_8 + a_{11} + a_{17} + a_{20}),\end{aligned}$$

$u_1 \ldots u_8 \neq 0$. Therefore, we have

$$\begin{cases} \sum_{i=1}^{8} u_i = a_{21}, \\ \sum_{i=1}^{8} u_i^3 x_i = a_{20}, \\ \sum_{i=1}^{8} (u_i^3 y_i + u_i^2 x_i^2) = a_{19}, \\ \sum_{i=1}^{8} (u_i^3 z_i + u_i x_i^3) = a_{18}, \\ \sum_{i=1}^{8} (u_i^2 y_i^2 + u_i x_i^2 y_i) = a_{17} + a_{20}, \\ \sum_{i=1}^{8} (u_i x_i^2 z_i + u_i x_i y_i^2 + u_i^2 x_i^2) = a_{16}, \\ \sum_{i=1}^{8} (u_i^2 z_i^2 + u_i y_i^3 + u_i^2 x_i y_i + u_i x_i^3) = a_{15}, \\ \sum_{i=1}^{8} (u_i^2 y_i z_i + u_i x_i^2 z_i + u_i y_i z_i^2) = a_1 + a_4 + a_{10} + a_{13} + a_{16} + a_{19}, \\ \sum_{i=1}^{8} (u_i^2 x_i z_i + u_i x_i z_i^2 + u_i y_i^2 z_i) = a_2 + a_5 + a_8 + a_{11} + a_{17} + a_{20}, \end{cases}$$

so that

$$a_{17} = \sum_{i=1}^{8} (u_i^3 x_i + u_i^2 y_i^2 + u_i x_i^2 y_i).$$

(ii) *Second step* – Let

$$B = A + \sum_{i=1}^{8} (u_i T^3 + x_i T^2 + y_i T + z_i)^7. \tag{$\star$}$$

Then $\deg B \leq 14$. If

$$B = \sum_{i=0}^{14} b_i T^i,$$

then

$$b_{13} + b_{10} + b_4 + b_1 = a_{13} + a_{10} + a_4 + a_1 + \sum_{i=1}^{8} (u_i x_i y_i^2 + u_i^2 y_i z_i + u_i y_i z_i^2 + u_i^3 y_i),$$

$$b_{12} + b_9 + b_6 + b_3 = a_{12} + a_9 + a_6 + a_3 + \sum_{i=1}^{8} (u_i^2 x_i y_i + u_i^3 z_i + u_i^2 z_i^2 + u_i y_i^3),$$

$$b_{11} + b_8 + b_5 + b_2 = a_{11} + a_8 + a_5 + a_2 + \sum_{i=1}^{8} (u_i^2 x_i z_i + u_i x_i z_i^2 + u_i x_i^2 y_i + u_i y_i^2 z_i, + u_i^2 y_i^2)$$

We have

$$a_{19} + a_{16} + a_{13} + a_{10} + a_4 + a_1 = \sum_{i=1}^{8} (u_i^2 y_i z_i + u_i x_i^2 z_i + u_i y_i z_i^2),$$

and

$$a_{19} + a_{16} = \sum_{i=1}^{8} (u_i^3 y_i + u_i x_i^2 z_i + u_i x_i y_i^2),$$

so that

$$a_{13} + a_{10} + a_4 + a_1 = \sum_{i=1}^{8} (u_i^2 y_i z_i + u_i y_i z_i^2 + u_i x_i y_i^2 + u_i^3 y_i).$$

Thus,

$$b_{13} + b_{10} + b_4 + b_1 = 0.$$

Similarly, we prove that

$$b_{12} + b_9 + b_6 + b_3 = b_{11} + b_8 + b_5 + b_2 = 0.$$

Proposition 3.7 gives the existence of polynomials $X_1, \dots, X_{11} \in F[T]$ such that

$$B = \sum_{i=1}^{11} X_i^7, \qquad \deg X_i \le 2.$$

We conclude with $(\star)$. $\square$

Remarks. Proposition 3.7 proves that T is not a sum of seventh powers of polynomials of degree ≤ 2. From Proposition 3.8 we deduce that every $P \in F[T]$ of degree ≤ 2 may be written as a sum of 19 seventh powers of polynomials of degree ≤ 3, so that T is a sum of 19 seventh powers. This gives another proof of the equality $\mathcal{S}(F[T], k) = F[T]$. The following proposition gives a representation of T as a sum of 12 seventh powers of polynomials of degree ≤ 3.

Proposition 3.9. *We have*

$$\begin{aligned} T = {} & (T^3+T^2+1)^7 + (T^3+T^2+\alpha T)^7 + (T^3+T^2+(\alpha+1)T)^7 \\ & + (\alpha T^3 + \alpha T^2 + \alpha T + \alpha + 1)^7 + ((\alpha+1)T^3 + (\alpha+1)T^2 \\ & + (\alpha+1)T + \alpha)^7 + (T^2+T+1)^7 + (T^2+T)^7 + (T^2+\alpha)^7 \\ & + (T^2+\alpha+1)^7 + (T+\alpha)^7 + (T+\alpha+1)^7 + (T+1)^7. \end{aligned}$$

Proof. An easy verification. $\square$

4. The first descent

The process described in [**1**] or in [**11**] works when a representation of T as sum of k-th powers is known. In the case when $k = 7$ and $q = 4$, this process leads to the following.

Theorem 4.1.

(i) *Every polynomial $P \in F[T]$ with degree divisible by* 7 *and* ≥ 18599 *is a strict sum of* 32 *seventh powers.*

(ii) *Every polynomial $P \in F[T]$ with degree* ≥ 18593 *is a strict sum of* 33 *seventh powers.*

Proof. Let $P \in F[T]$ with $7(n-1) < \deg P \le 7n$. Let

$$\varepsilon(P) = \begin{cases} 0 & \text{if } \deg P = 7n, \\ 1 & \text{if } \deg P < 7n \end{cases}$$

and let

$$H = \varepsilon(P) T^{7n} + P.$$

Then, $\deg H = 7n$. From [**1**, Lemma 5.2], there is a sequence $H_0, H_1, \dots, H_i, \dots,$ of polynomials of $F[T]$ of degree $7n_0, 7n_1, \dots, 7n_i$, and a sequence $X_0, X_1, \dots, X_i$ of polynomials of degree $n_0, n_1, \dots, n_i$, such that $H = H_0$ and such that for each index i,

$$H_i = X_i^7 + H_{i+1}, \tag{4.1}$$

$$6n_i \leq 7n_{i+1} < 6n_i + 7. \tag{4.2}$$

Moreover, for each index i, there is a polynomial $Y_i \in F[T]$ of degree n_i such that

$$\deg(H_i + Y_i^7) < 6n_i. \tag{4.3}$$

We use (4.1) or (4.3) as long as the sequence (n_i) is decreasing. Let r, if it exists, be the least index such that $3(6n_r - 1) \leq n$. We use identity (1) r times, then we use identity (4.3) once. We get

$$H = X_0^7 + \cdots + X_{r-1}^7 + Y_r^7 + R,$$

with $3 \deg R \leq n$. From Proposition 3.9, there exist $R_1, \ldots R_{12} \in F[T]$ of degree $\leq 3 \deg R$ such that

$$R = R_1^7 + \ldots + R_{12}^7,$$

so that

$$H = X_0^7 + \cdots + X_{r-1}^7 + Y_r^7 + R_1^7 + \ldots + R_{12}^7 \tag{4.4}$$

with $\deg X_i = n_i \leq n_0 = n, \deg Y_r = n_r \leq n_0 = n, \deg R_j \leq 3 \deg R \leq n$. Thus, (4.4) is a strict sum of $r + 13$ seventh powers. From (4.2) we get that for $i \geq 1$,

$$7^i n_i \leq 6^i n + \sum_{j=0}^{i-1} 7^j 6^{i-j}.$$

Therefore, for any integer $r \geq 1$, we have

$$6n_r - 1 \leq 6 \left(\frac{6}{7}\right)^r n + 35 - 36 \left(\frac{6}{7}\right)^r.$$

For $r \geq 19$, we have $\left(\frac{6}{7}\right)^r < \frac{1}{18}$. Suppose $r = 19$. If $n \geq 2657$, then

$$6 \left(\frac{6}{7}\right)^{19} n + 35 - 36 \left(\frac{6}{7}\right)^{19} \leq \frac{n}{3}.$$

□

5. Others descents

The second descent process is based on very simple identities.

Proposition 5.1. *The following identity holds in the ring $F[X, Y]$,*

$$X^4Y^3 + XY^6 = X^7 + (X + Y)^7 + (X + \alpha Y)^7 + (X + (\alpha + 1)Y)^7. \tag{5.1}$$

Proof. A simple verification. □

Proposition 5.2. *For a non-negative integer i and $X \in F[T]$, let*

$$L_i(X) = X^4T^{3i} + XT^{6i}. \tag{5.2}$$

Then, the map L_i is $\mathbb{F}_2$-linear and we have

$$L_i(X) = X^7 + (X + T^i)^7 + (X + \alpha T^i)^7 + (X + (\alpha + 1)T^i)^7. \tag{5.3}$$

$$T^7 L_i(X) = L_{i+1}(TX). \tag{5.4}$$

Proof. Immediate. □

Corollary 5.3. *Let n be a non-negative integer and let $a \in F$. Then, we have*

$$aT^{4n} = L_0(aT^n) + aT^n, \tag{5.5}$$

$$aT^{4n+3} = L_1(aT^n) + aT^{n+6}. \tag{5.6}$$

If $n > 0$, then

$$aT^{4n+2} = L_2(aT^{n-1}) + aT^{n+11}. \tag{5.7}$$

If $n > 1$, then

$$aT^{4n+1} = L_3(aT^{n-2}) + aT^{n+16}. \tag{5.8}$$

Proof. (5.5) and (5.6) are immediate. We get (5.7) and (5.8) noting that $aT^{4n+2} = aT^{4(n-1)+6}$ and that $aT^{4n+1} = aT^{4(n-2)+9}$. □

Roughly speaking, the second descent process uses the following idea. Let $X = x_N T^N + x_{N-1}T^{N-1} + \ldots + x_1 T + x_0$ be a polynomial of $F[T]$. Making use of (5.5)–(5.8), we replace a monomial $x_k T^k$ by the sum of an appropriate $L_i(T^j)$ and a monomial of lower degree. We begin with $x_N T^N$ and we follow decreasing degrees as long as the process gives monomials of lower degree. For more details see [**4**, Proposition 5.4]. Mixing this process with the first descent process leads to the following proposition.

Proposition 5.4. *Let $H \in F[T]$ with degree $7n \geq 112$. Then, there exist $X_0, X_1, X_2, X_3, Y_0, Y_1, Y_2, Y_3, Z \in F[T]$ with $\deg X_i \leq n$, $\deg Y_j \leq n$ and $\deg Z \leq 21$ such that*

$$H = X_0^7 + X_1^7 + X_2^7 + X_3^7 + L_0(Y_0) + L_1(Y_1) + L_2(Y_2) + L_3(Y_3) + Z. \tag{5.9}$$

Proof. See [**4**, Proposition 5.5]. □

We continue with other descent processes.

Proposition 5.5. *Let $n \geq 3$ be an integer and let $A \in F[T]$ of degree $\leq 7n$. Then there exist $X_1, \ldots, X_4 \in F[T]$ of degree $\leq n$ such that*

$$\deg\left(A + \sum_{i=1}^{3} X_i^7\right) \leq 7(n-1),$$

so that there exist $X_1, \ldots, X_5 \in F[T]$ of degree $\leq n$ such that

$$\deg\left(A + \sum_{i=1}^{4} X_i^7\right) = 7(n-1).$$

Proof. Let

$$A = \sum_{i=0}^{21} a_i T^i$$

be a polynomial of degree ≤ 21. Proposition 2.8 gives the existence of

$$(u_1, u_2, u_3, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) \in F^{16},$$

a solution of $(\mathcal{E}(a_{21}, a_{20}, a_{19}, a_{18}, a_{17}+a_{20}, a_{16}, a_{15}))$ such that $u_1u_2u_3 \neq 0$. Therefore, we have

$$\begin{cases} \sum_{i=1}^{3} u_i = a_{21}, \\ \sum_{i=1}^{3} u_i^3 x_i = a_{20}, \\ \sum_{i=1}^{3} (u_i^3 y_i + u_i^2 x_i^2) = a_{19}, \\ \sum_{i=1}^{3} (u_i^3 z_i + u_i x_i^3) = a_{18}, \\ \sum_{i=1}^{3} (u_i^3 x_i + u_i^2 y_i^2 + u_i x_i^2 y_i) = a_{17}, \\ \sum_{i=1}^{3} (u_i x_i^2 z_i + u_i x_i y_i^2 + u_i^2 x_i^2) = a_{16}, \\ \sum_{i=1}^{3} (u_i^2 z_i^2 + u_i y_i^3 + u_i^2 x_i y_i + u_i x_i^3) = a_{15}. \end{cases}$$

Let

$$B = \left(A + \sum_{i=1}^{3} (u_i T^3 + x_i T^2 + y_i T + z_i)^7 \right),$$

so that

$$\deg B \leq 14.$$

This gives the first part of the proposition in the case when $n = 3$. We get the second part of the proposition in the case $n = 3$ taking

$$X_4 = \begin{cases} 0 & \text{if} \quad \deg B = 14, \\ T^2 & \text{if} \quad \deg B < 14. \end{cases}$$

Let $n \geq 3$ be an integer and let $A \in F[T]$ of degree $\leq 7n$. By euclidean division, there is a pair $(Q, R) \in F[T]$ such that, respectively,

$$A = T^{7(n-3)}Q + R, \qquad \deg Q \leq 21, \qquad \deg R < 7(n-3).$$

There exist $X_1, \ldots, X_3 \in F[T]$ and $X_1, \ldots, X_4 \in F[T]$, of degree ≤ 3 such that

$$\deg \left(Q + \sum_{i=1}^{3} X_i^7 \right) \leq 14,$$

and

$$\deg \left(Q + \sum_{i=1}^{4} X_i^7 \right) = 14,$$

respectively. Therefore,

$$\deg\left(T^{7(n-3)}(Q+\sum_{i=1}^{3}X_i^7)\right)\leq 7(n-1),$$

and

$$\deg\left(T^{7(n-3)}(Q+\sum_{i=1}^{4}X_i^7)\right)= 7(n-1),$$

respectively, so that,

$$\deg\left(R+T^{7(n-3)}Q+\sum_{i=1}^{3}X_i^7\right)\leq 7(n-1),$$

and

$$\deg\left(R+T^{7(n-3)}Q+\sum_{i=1}^{4}X_i^7\right)= 7(n-1),$$

respectively. □

6. End of the proof

Proposition 6.1. *Let*

$$A=\sum_{i=0}^{28}a_iT^i$$

be a polynomial in $F[T]$ *with* $\deg A\leq 28$. *Then* A *is a sum*

$$A=\sum_{i=1}^{3}X_i^7+\sum_{i=1}^{19}Y_i^7, \tag{6.1}$$

where $X_1,\dots,X_3,Y_1,\dots,Y_{19}$ *are polynomials of* $F[T]$ *such that* $\deg X_i\leq 4$ *and* $\deg Y_i\leq 3$.

Proof. Set

$$\sigma=a_{27}+a_{24}+a_{18}+a_{15}+a_{12}+a_9+a_6+a_3. \tag{6.2}$$

Proposition 2.10 gives the existence of

$$(v_1,\dots,v_3,u_1,\dots,u_3,x_1,\dots,x_3,y_1,\dots,y_3,z_1,\dots,z_3)\in F^{15},$$

a solution of $(\mathcal{F}(a_{28}, a_{27}, a_{26}, a_{25}, a_{24}, a_{23}, a_{22}, \sigma)$ such that $v_1 \dots v_4 \neq 0$. For such a solution, we have

$$\text{(6.3)} \qquad \begin{cases} a_{28} &= \sum_{i=1}^{3} v_i, \\ a_{27} &= \sum_{i=1}^{3} u_i = \sum_{i=1}^{3} v_i^3 u_i, \\ a_{26} &= \sum_{i=1}^{3} (x_i + v_i^2 u_i^2) = \sum_{i=1}^{3} (v_i^3 x_i + v_i^2 u_i^2), \\ a_{25} &= \sum_{i=1}^{3} (y_i + v_i u_i^3) = \sum_{i=1}^{3} (v_i^3 y_i + v_i u_i^3), \\ a_{24} &= \sum_{i=1}^{3} (v_i^2 x_i^2 + v_i u_i^2 x_i + u_i + z_i) \\ &= \sum_{i=1}^{3} (v_i^2 x_i^2 + v_i u_i^2 x_i + v_i^3 u_i + v_i^3 z_i), \\ a_{23} &= \sum_{i=1}^{3} (v_i u_i^2 y_i + v_i u_i x_i^2 + v_i^2 u_i^2), \\ a_{22} &= \sum_{i=1}^{3} (v_i u_i^3 + v_i x_i^3 + v_i^2 y_i^2 + v_i u_i^2 z_i + v_i^2 u_i x_i), \end{cases}$$

and

$$\text{(6.4)} \qquad \sigma = \sum_{i=1}^{3} (v_i^2 u_i y_i + v_i u_i y_i^2 + v_i x_i^2 y_i).$$

For $i = 1, 2, 3$, let

$$X_i = v_i T^4 + u_i T^3 + x_i T^2 + y_i T + z_i$$

and let

$$B = A + \sum_{i=1}^{3} X_i^7.$$

Identities (6.3) show that $\deg B \leq 21$. Set

$$B = \sum_{i=1}^{21} b_i T^i.$$

From (6.2), (6.3) and (6.4),

$$b_{18} + b_{15} + b_{12} + b_9 + b_6 + b_3 = 0,$$

so that from Theorem 3.3, there exist polynomials $Y_1, \dots, Y_{19} \in F[T]$ with degree ≤ 3 such that

$$B = \sum_{i=1}^{19} Y_i^7.$$

□

Corollary 6.2. *Let $A \in F[T]$ be such that $21 < \deg A \leq 28$. Then A is a strict sum of 22 seventh powers.*

Theorem 6.3.

(i) *Every polynomial $P \in F[T]$ whose degree ≥ 441 is divisible by 7 is a strict sum of 32 seventh powers.*
(ii) *Every polynomial $P \in F[T]$ with degree ≥ 435 is a strict sum of 33 seventh powers.*
(iii) *Every polynomial $P \in F[T]$ such that $\deg P \geq 112$ and $\deg P$ is divisible by 7 is a strict sum of 42 seventh powers.*
(iv) *Every polynomial $P \in F[T]$ with degree ≥ 106 is a strict sum of 43 seventh powers.*

Proof. As for the proof of Theorem 4.1, it is sufficient to prove (i) and (iii). Let $H \in F[T]$ of degree $7n$ with $n \geq 16$. From (5.9) and (5.3), we get that there exists $Z \in F[T]$ with $\deg Z \leq 21$ such that $H + Z$ is sum of 20 seventh powers of polynomials of degree $\leq n$. From Proposition 3.9, there exist $Z_1, \ldots, Z_{12}$ with $\deg Z_i \leq 63$ such that

$$Z = \sum_{i=1}^{12} Z_i^7.$$

If $n \geq 63$, then H is a strict sum of 32 seventh powers. This proves (i).

From Proposition 6.1, there exist $V_1, \ldots, V_{22} \in F[T]$ with $\deg V_i \leq 4 < n$ such that

$$Z = \sum_{i=1}^{22} V_i^7,$$

so that H is a strict sum of 42 seventh powers. This proves (iii). $\square$

Corollary 6.4. *We have*

$$G(4, 7) = G^{\times}(4, 7) \leq 33.$$

We end the study of the set $\mathcal{S}^{\times}(F, T)$ dealing with polynomials P such that $29 \leq \deg P \leq 105$.

Proposition 6.5. *Let $A \in F[T]$.*

(i) *If $29 \leq \deg A \leq 35$, then A is a strict sum of 25 seventh powers.*
(ii) *If $\deg A = 42$, then A is a strict sum of 26 seventh powers.*
(iii) *If $35 < \deg A < 42$, then A is a strict sum of 27 seventh powers.*
(iv) *If $\deg A = 7n$ with $7 \leq n < 14$, then A is a strict sum of $n + 20$ seventh powers.*
(v) *If $7n - 7 < \deg A < 7n$ with $7 \leq n < 14$, then A is a strict sum of $n + 21$ seventh powers.*
(vi) *If $\deg A = 7n$ with $14 \leq n < 21$, then A is a strict sum of $n + 19$ seventh powers.*
(vii) *If $7n - 7 < \deg A < 7n$ with $14 \leq n < 21$, then A is a strict sum of $n + 20$ seventh powers.*
(viii) *If $\deg A = 7n$ with $21 \leq n < 28$, then A is a strict sum of $n + 18$ seventh powers.*

(ix) If $7n-7<\deg A<7n$ with $21\le n<28$, then A is a strict sum of $n+19$ seventh powers.

Proof. As observed before, it suffices to prove (i), (ii), (iv), (vi) and (viii).

1. Suppose that $29\le \deg A\le 35$. From Proposition 5.5, there exist $X_1, X_2, X_3\in F[T]$ of degree ≤ 5 such that $\deg(A+\sum_{i=1}^{3} X_i^7)\le 28$. From Proposition 6.1, there exist $Y_1,\dots,Y_{22}\in F[T]$ of degree ≤ 4 such that

$$A+\sum_{i=1}^{3} X_i^7=\sum_{j=1}^{22} Y_j^7.$$

2. Suppose that $\deg A=42$. From [**1**, Lemma 5.2-(i)], there is a polynomial $X\in F[T]$ of degree 6 such that $\deg(A+X^7)\le 35$. From above, there exist $Y_1,\dots,Y_{25}\in F[T]$ of degree ≤ 5 such that

$$A+X^7=\sum_{j=1}^{25} Y_j^7.$$

3. We prove (iv), (vi) and (viii) by induction. Suppose that for $n\ge 7$, every polynomial of degree $7k$ with $k<n$ is a strict sum of $s(k)$ seventh powers. Let $A\in F[T]$ of degree $7n$. From [**1**, Lemma 5.2-(ii)], there is a polynomial $X\in F[T]$ of degree n such that $\deg(A+X^7)=7m(n)$ with $m(n)$ defined by the condition $6n\le 7m(n)<6n+7$. We have

$$m(n)=\begin{cases} n-1 & \text{if } \ 7\le n\le 13,\\ n-2 & \text{if } \ 14\le n\le 20\\ n-3 & \text{if } \ 21\le n\le 27.\end{cases}$$

The induction hypothesis gives that $A+X^7$ is a strict sum of $s(m(n))$ seventh powers, so that A is a strict sum of $s(m(n))+1$ seventh powers. We have $s(6)=26$. Thus,

$$s(n)=\begin{cases} n+20 & \text{if } \ 7\le n\le 13,\\ n+19 & \text{if } \ 14\le n\le 20,\\ n+18 & \text{if } \ 21\le n\le 27.\end{cases}$$

□

Proposition 6.6. *We have*

$$\mathcal{S}^{\times}(F[T],7)=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3\cup\mathcal{A}_\infty,$$

where

(i) $\mathcal{A}_1$ is the set of polynomials $A=\sum_{n=0}^{7} a_nT^n\in F[T]$ such that $a_1=a_4, a_2=a_5, a_3=a_6$,

(ii) $\mathcal{A}_2$ *is the set of polynomials* $A = \sum_{n=0}^{14} a_n T^n \in F[T]$ *with* $7 < \deg A \leq 14$ *such that*

$$\begin{cases} a_1 + a_4 + a_{10} + a_{13} = 0, \\ a_2 + a_5 + a_8 + a_{11} = 0, \\ a_3 + a_6 + a_9 + a_{12} = 0, \end{cases}$$

(iii) $\mathcal{A}_3$ *is the set of polynomials* $A = \sum_{n=0}^{21} a_n T^n \in F[T]$ *with* $14 < \deg A \leq 21$ *such that*

$$a_3 + a_6 + a_9 + a_{12} + a_{15} + a_{18} = 0,$$

(iv) $\mathcal{A}_\infty = \{A \in F[T] \mid \deg A > 21\}$.

Proof. With Theorems 3.1, 3.2, 3.3, Corollary 6.2 and Theorem 6.3. □

Theorem 6.7. *We have*

$$g^\times(4, 7) \leq 43.$$

Proof. From Theorems 3.1, 3.2, 3.3, every polynomial $A \in \mathcal{S}^\times(F[T], 7)$ of degree ≤ 21 is a strict sum of 19 seventh powers. From Corollary 6.2 and Proposition 6.5, every polynomial $A \in \mathcal{S}^\times(F[T], 7)$ such that $21 < \deg A \leq 175$ is a strict sum of 43 seventh powers. From Theorem 6.3, every polynomial $A \in \mathcal{S}^\times(F[T], 7)$ such that $\deg A \geq 106$ is a strict sum of 43 seventh powers. □

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