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BLOWUP FOR A TIME-OSCILLATING NONLINEAR HEAT EQUATION

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Abstract. In this paper, we study a nonlinear heat equation with a periodic time-oscillating term in factor of the nonlinearity. In particular, we give examples showing how the behavior of the solution can drastically change according to both the frequency of the oscillating factor and the size of the initial value.

1. Introduction

Let Ω be a smooth, bounded domain of $\mathbb{R}^N$ and fix $\alpha > 0$. Let $\tau > 0$ and let $\theta \in C(\mathbb{R}, \mathbb{R})$ be a τ-periodic function. Given $\omega \in \mathbb{R}$ and $\phi \in C_0(\Omega)$ (the space of continuous functions on $\overline{\Omega}$ that vanish on $\partial\Omega$), we consider the nonlinear heat equation

$$\begin{cases} u_t = \Delta u + \theta(\omega t)|u|^\alpha u, \\ u_{|\partial\Omega} = 0, \\ u(0, \cdot) = \phi(\cdot), \end{cases} \tag{1.1}$$

and the (formally) limiting equation

$$\begin{cases} U_t = \Delta U + A(\theta)|U|^\alpha U, \\ U_{|\partial\Omega} = 0, \\ U(0, \cdot) = \phi(\cdot), \end{cases} \tag{1.2}$$

where

$$A(\theta) = \frac{1}{\tau} \int_0^\tau \theta(s)\, \mathrm{d}s, \tag{1.3}$$

i.e., $A(\theta)$ is the average of θ.

In Ref. [**3, 9**], the authors study a similar problem, but for Schrödinger's equation on $\mathbb{R}^N$ instead of the heat equation on Ω. Under appropriate assumptions, the solution of the time-oscillating Schrödinger equation converges as $|\omega| \to \infty$ to the solution of the limiting Schrödinger equation with the same initial value.

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Moreover, if the solution of the limiting equation is global and decays (in an appropriate sense) as $t \to \infty$, then the solution of the time-oscillating equation is also global for $|\omega|$ large. It is natural to expect that if the solution of the limiting equation blows up in finite time, then so does the solution of the time-oscillating equation for $|\omega|$ large, but this question seems to be open. (See [**3**, Question 1.7].)

We note that the proofs in [**3, 9**] are based on Strichartz estimates for the Schrödinger group. Since the heat equation satisfies the same Strichartz estimates as Schrödinger's equation, results similar to the results in [**3, 9**] hold for the equation (1.1). However, the heat equation enjoys specific properties, such as the maximum principle, so that much more can be said. This is our main motivation for studying the equation (1.1).

It is not difficult to prove by standard contraction arguments that the initial value problem (1.1) is locally well-posed in $C_0(\Omega)$. (Apply Proposition 2.1 below with $f(t) \equiv \theta(\omega t)$.) As $|\omega| \to \infty$, the solution u of (1.1) converges in an appropriate sense to the solution U of the limiting equation (1.2), as shows the following result.

Proposition 1.1. Let $\tau > 0$ and let $\theta \in C(\mathbb{R}, \mathbb{R})$ be τ-periodic. Given $\phi \in C_0(\Omega)$, let U be the corresponding solution of (1.2), defined on the maximal existence interval $[0, T_{\max})$. For every $\omega \in \mathbb{R}$, let u_ω be the (maximal) solution of (1.1). If $0 < T < T_{\max}$, then u_ω exists on $[0, T]$ provided $|\omega|$ is sufficiently large. Moreover, $\|u_\omega - U\|_{L^\infty((0,T)\times\Omega)} \to 0$ as $|\omega| \to \infty$.

Note that 0 is an exponentially stable stationary solution of (1.2). (See Remark 2.3 (i) below.) It follows in particular that any global solution of (1.2) either converges exponentially to 0 as $t \to \infty$ or else is bounded away from 0. If the limiting solution as given by Proposition 1.1 is global and exponentially decaying, then the solution of (1.1) is global (and exponentially decaying) for all large $|\omega|$, as the next result shows. (Note that this property is classical in the framework of ordinary equations, see e.g. [**14, 15**].)

Proposition 1.2. Let $\tau > 0$ and let $\theta \in C(\mathbb{R}, \mathbb{R})$ be τ-periodic. Let $\phi \in C_0(\Omega)$ and suppose the corresponding solution U of (1.2) is global and $U(t) \to 0$ as $t \to \infty$. For every $\omega \in \mathbb{R}$, let u_ω be the (maximal) solution of (1.1). It follows that u_ω is global provided $|\omega|$ is sufficiently large. Moreover, there exist constants $C, \lambda > 0$ such that $\|u_\omega(t)\|_{L^\infty} + \|U(t)\|_{L^\infty} \le C\,\mathrm{e}^{-\lambda t}$ for all $t \ge 0$ and all sufficiently large $|\omega|$. In particular, $\|u_\omega - U\|_{L^\infty((0,\infty)\times\Omega)} \to 0$ as $|\omega| \to \infty$.

Let $A(\theta)$ be defined by (1.3). If $A(\theta) \le 0$, then all solutions of (1.2) are global and exponentially decaying, so that, by Proposition 1.2, all solutions of (1.1) are global (and exponentially decaying) for large $|\omega|$.

If $A(\theta) > 0$, then the set of initial values ϕ for which the solution of (1.2) is global and converges to 0 is an open neighborhood of 0. For such ϕ, the solution of (1.1) is also global (and exponentially decaying) for large $|\omega|$.

On the other hand (still assuming $A(\theta) > 0$), there exist initial values ϕ for which the solution of (1.2) blows up in finite time. For such ϕ, we may wonder if the solution of (1.1) also blows up in finite time for large $|\omega|$. In this regard, it is instructive to consider the ODE associated with the heat equation (1.1), i.e.,

$$v' + av = \theta(\omega t)|v|^\alpha v, \tag{1.4}$$

where $a > 0$ and the limiting ODE

$$V' + aV = A(\theta)|V|^\alpha V. \tag{1.5}$$

The solutions v of (1.4) and V of (1.5) with the initial conditions $v(0) = V(0) = x > 0$ are given by

$$v(t) = \mathrm{e}^{-at}(x^{-\alpha} - h(t,\omega))^{-\frac{1}{\alpha}}, \tag{1.6}$$

where

$$h(t,\omega) = \alpha \int_0^t \mathrm{e}^{-a\alpha s}\, \theta(\omega s)\, \mathrm{d}s, \tag{1.7}$$

and

$$V(t) = \mathrm{e}^{-at}(x^{-\alpha} - a^{-1}A(\theta)[1 - \mathrm{e}^{-a\alpha t}])^{-\frac{1}{\alpha}}. \tag{1.8}$$

The solution V blows up in finite time if and only if $x^{-\alpha} < a^{-1}A(\theta)$. For such x, there exists $T_1 > 0$ such that $x^{-\alpha} < a^{-1}A(\theta)[1 - \mathrm{e}^{-a\alpha T_1}]$. Since $h(T_1,\omega) \to a^{-1}A(\theta)[1 - \mathrm{e}^{-a\alpha T_1}]$ as $|\omega| \to \infty$, it follows that $x^{-\alpha} < h(T_1,\omega)$ for $|\omega|$ large; and so by formula (1.6), v blows up in a finite time $T_2 < T_1$. Thus we see that if the solution of the limiting equation (1.5) blows up in finite time, then so does the solution of (1.4) if $|\omega|$ is sufficiently large.

The above calculations can be adapted to a nonlinear heat equation with a nonlocal nonlinearity. More precisely, consider the nonlinear heat equation

$$\begin{cases} u_t = \Delta u + \theta(\omega t)\|u\|_{L^2}^\alpha u, \\ u_{|\partial\Omega} = 0, \\ u(0,\cdot) = \phi(\cdot), \end{cases} \tag{1.9}$$

and the (formally) limiting equation

$$\begin{cases} U_t = \Delta U + A(\theta)\|U\|_{L^2}^\alpha U, \\ U_{|\partial\Omega} = 0, \\ U(0,\cdot) = \phi(\cdot). \end{cases} \tag{1.10}$$

It is easy to show that both problems are locally well posed in $L^2(\Omega)$, and that analogues of Propositions 1.1 and 1.2 hold. Moreover, we have the following result.

Theorem 1.3. Let $\tau > 0$ and let $\theta \in C(\mathbb{R},\mathbb{R})$ be τ-periodic. Given $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, let U be the corresponding solution of (1.10) and, for every $\omega \in \mathbb{R}$, let u_ω be the (maximal) solution of (1.9). If U blows up in finite time, then u_ω blows up in finite time provided $|\omega|$ is sufficiently large.

Our proof of Theorem 1.3 makes use of the very particular structure of the equations (1.9) and (1.10). It is based on an abstract result (see Section A), relying on an explicit calculation of the solution.

We are not aware of any result similar to Theorem 1.3 for the heat equation (1.1), so we emphasize the following open problem.

Open problem 1.4. Let $\tau > 0$, let $\theta \in C(\mathbb{R}, \mathbb{R})$ be τ-periodic and let $A(\theta)$ be defined by (1.3). Assume $A(\theta) > 0$ and let $\phi \in C_0(\Omega)$ be such that the corresponding solution of (1.2) blows up in finite time. Does the solution of (1.1) blow up in finite time if $|\omega|$ is sufficiently large?

Note that the answer to the open problem 1.4 might depend on whether or not the exponent α is Sobolev subcritical (i.e. $\alpha < 4/(N-2)$ if $N \geq 3$). Indeed, if α is subcritical, then the set of initial values producing blowup in the limiting problem (1.2) is an open subset of $C_0(\Omega)$. (This follows easily from [**16**].)) On the other hand, if α is supercritical, then the set of initial values producing blowup in the limiting problem (1.2) is not an open subset of $C_0(\Omega)$ (see [**5**, Theorem B]). In other words, blowup is stable with respect to small perturbations of the initial value if α is subcritical, but not if α is supercritical. It is possible that a similar phenomenon occurs for the stability of blowup with respect to perturbations of the equation.

The difficulty in proving a general blowup result for (1.1) (when θ is not constant) comes from the fact that the standard techniques that are used for the autonomous equation (1.2) seem to fail. Levine's energy method [**13**] (see also Ball [**1**, Theorem 3.2] for a slightly different argument) uses the decay of the energy associated with (1.2). There is an energy identity for (1.1), but it contains the time derivative of the function $\theta(\omega t)$, which is difficult to control (especially when $|\omega| \to \infty$). On the other hand, Kaplan's argument [**12**, Theorem 8] (see also [**10**, Theorem 2.6]) and Weissler's argument [**19**, Theorem 1] only apply to positive solutions and when $\theta(\omega t) \geq 0$ on the time interval on which the argument is performed. Therefore, Kaplan's argument can be applied to prove blowup for positive initial values when $\theta(0) > 0$ and $|\omega|$ is small; or when θ is bounded from below and the initial value is sufficiently large, in which case blowup occurs for all ω. However, it does not seem to be applicable on a time interval where θ takes negative values. Thus we mention the following open problem.

Open problem 1.5. Let $\tau > 0$, let $\theta \in C(\mathbb{R}, \mathbb{R})$ be τ-periodic and let $A(\theta)$ be defined by (1.3). Suppose $A(\theta) > 0$ and $\theta(0) < 0$. Does there exist ϕ and ω for which the solution of (1.1) blows up in finite time?

Note that the problems 1.4 and 1.5 seem to be open even in the apparently simple situation when $N = 1$, $\Omega = (-1, 1)$ and ϕ is positive and even.

Of course, a positive answer to the problem 1.4 would yield a positive answer to the problem 1.5. We are not aware of any general result of the type suggested in Open Problem 1.5. However, it is easy to construct an initial value ϕ and a function θ as in Problem 1.5 such that the solution of (1.1) with $\omega = 1$ blows up in finite time after picking up negative values of θ. (See Remark 2.7 below.) On the other hand, it is also easy to construct a function θ as in Problem 1.5 such that for all $\phi \in C_0(\Omega)$, the solution of (1.1) with $\omega = 1$ is global. (See Remark 2.8 below.)

In the following result, we describe an interesting situation where, for a given, nonnegative function θ, the behavior of the solution of (1.1) changes drastically according to both the frequency ω and the size of the initial value.

Theorem 1.6. There exist $\tau > 0$, a τ-periodic, positive $\theta \in C^\infty(\mathbb{R})$, a positive $\psi \in C_0(\Omega)$ and $0 < k_0 < k_1 < k_2 < k_3 < \infty$ with the following properties. Let $k > 0$, $\phi = k\psi$ and, given $\omega > 0$, let $u_{k,\omega}$ be the solution of (1.1).

(i) If $0 \le k \le k_0$, then $u_{k,\omega}$ is global (and exponentially decaying) for all $\omega > 0$.
(ii) If $k = k_1$, then $u_{k,\omega}$ blows up in finite time if $0 < \omega \le 1$ and is global (and exponentially decaying) if ω is large.
(iii) If $k = k_2$, then $u_{k,\omega}$ blows up in finite time if $0 < \omega \le 1$ and if ω is large, and it is global (and exponentially decaying) for some $\omega_0 > 1$.
(iv) If $k \ge k_3$, then $u_{k,\omega}$ blows up in finite time for all $\omega > 0$.

The solution $u_{k,\omega}$ of Theorem 1.6 is global (and exponentially decaying) if k is small ($k \le k_0$) and blows up in finite time if k is large ($k \ge k_3$). This is certainly not surprising. The interesting features of Theorem 1.6 appear for intermediate values of k, for which the behavior of $u_{k,\omega}$ (blowup or global) changes in terms of ω. As k increases from k_0, $u_{k,\omega}$ blows up for small values of $\omega \ge 0$ while it remains global for larger values of ω. As one keeps increasing k (below k_3), we see that $u_{k,\omega}$ blows up for both small and large values of ω, while it remains global for intermediate values of ω. (See Figure 1.)

We prove Theorem 1.6 by constructing an appropriate function θ. If θ is bounded from below and above by positive constants, the existence of k_0 and k_3 is straightforward. Furthermore, if $\theta(t) \equiv 1$ for t in a neighborhood of 0 and $A(\theta) < 1$, then it is not difficult to prove the existence of k_1. The existence of k_2 is more involved. Showing that for some $k_2 \in (k_1, k_3)$ the solution $u_{k_2,\omega}$ blows up for both small and large ω is easy, but the fact that $u_{k_2,\omega}$ is global for an intermediate value ω_0 relies on a delicate balance in the various parameters introduced in the construction of θ. The idea is to make θ small on a long interval (a, b). The parameters are adjusted in such a way that u_{k_2,ω_0} exists on $[0, a/\omega_0]$. On $[a/\omega_0, b/\omega_0]$, $\theta(\omega_0 t)$ is very small, so the equation (1.1) is close to the linear heat equation. Therefore, u_{k_2,ω_0} decays exponentially on $[a/\omega_0, b/\omega_0]$. Thus if b is sufficiently large $u_{k_2,\omega_0}(b/\omega_0)$ will be so small as to ensure global existence.

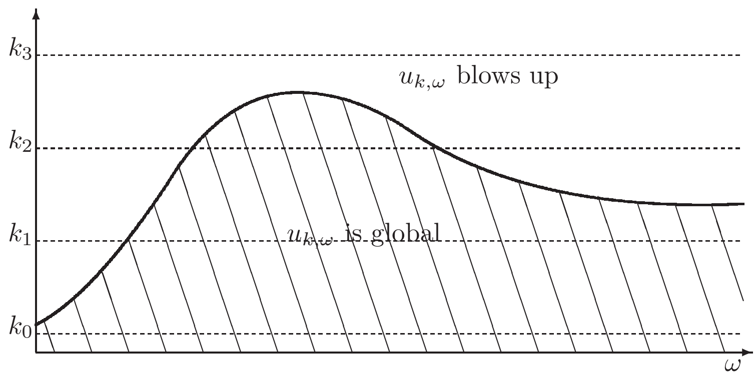

Figure 1. The ω, k picture of Theorem 1.6.

In the following result, we describe a situation in which the behavior of $u_{k,\omega}$ for intermediate values of k is in some sense opposite to the behavior described in Theorem 1.6.

Theorem 1.7. There exist $\tau > 0$, a τ-periodic, positive $\theta \in C^\infty(\mathbb{R})$, a positive $\psi \in C_0(\Omega)$ and $0 < k_0 < k_1 < k_2 < k_3 < \infty$ with the following properties. Let $k > 0$, $\phi = k\psi$ and, given $\omega > 0$, let $u_{k,\omega}$ be the solution of (1.1).

(i) If $0 \le k \le k_0$, then $u_{k,\omega}$ is global (and exponentially decaying) for all $\omega > 0$.
(ii) If $k = k_1$, then $u_{k,\omega}$ is global (and exponentially decaying) if ω is small and if ω is large, and it blows up in finite time for $\omega = 1$.
(iii) If $k = k_2$, then $u_{k,\omega}$ is global (and exponentially decaying) if ω is small, and it blows up in finite time if ω is large.
(iv) If $k \ge k_3$, then $u_{k,\omega}$ blows up in finite time for all $\omega > 0$.

The solution $u_{k,\omega}$ of Theorem 1.7 is global (and exponentially decaying) if k is small ($k \le k_0$) and blows up in finite time if k is large ($k \ge k_3$). As k increases from k_0, $u_{k,\omega}$ blows up for intermediate values of $\omega > 0$ while it remains global for both small and large values of ω. As one keeps increasing k (below k_3), we see that $u_{k,\omega}$ blows up for small values of ω, while it remains global for large values of ω. (See Figure 2.) In fact, while the behavior of $u_{k,\omega}$ for intermediate values of k is very different in Theorems 1.6 and 1.7, the function θ which we use in the proof of Theorem 1.7 is simply deduced by reflection and translation from the function θ of the proof of of Theorem 1.6.

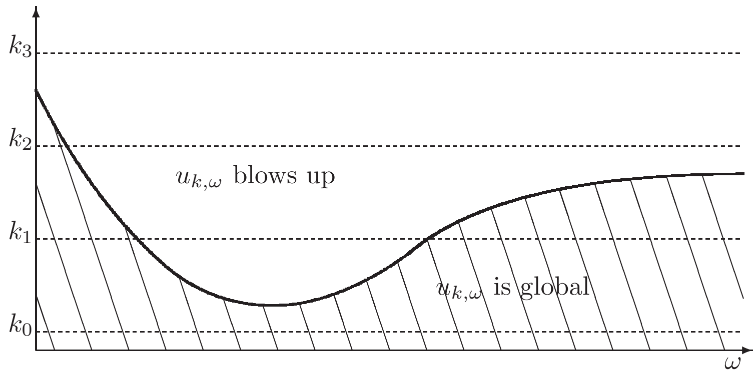

Figure 2. The ω, k picture of Theorem 1.7.

We note that equations of the form (1.1) were studied by Esteban [**7, 8**], Quittner [**17**] and Húska [**11**], where positive, time-periodic solutions are constructed under certain assumptions. More precisely, let θ be as above and suppose further that $\theta \in W^{1,\infty}(\mathbb{R})$ and $\min \theta > 0$. If $\alpha < 2/(N-2)$, or if $\alpha < 4/(N-2)$ and $|\omega|$ is sufficiently small, then there exists a positive, τ/ω-periodic solution of (1.1).

The rest of this paper is organized as follows. In Section 2, we recall some properties of the initial value problem (1.1). Section 3 is devoted to the proofs of the convergence results (Propositions 1.1 and 1.2), while Theorems 1.6, 1.7 and 1.3 are proved in Sections 4, 5, and 6 respectively. The last section of the paper is an appendix devoted to an abstract result which we use in the proof ot Theorem 1.3.

Notation

We denote by $\lambda_1 > 0$ the first eigenvalue of $-\Delta$ in $L^2(\Omega)$ with Dirichlet boundary condition and we let φ_1 be the eigenvector of $-\Delta$ corresponding to the first eigenvalue λ_1 and normalized by the condition $\max \varphi_1 = 1$. We denote by $(\mathrm{e}^{t\Delta})_{t\geq 0}$ the heat semigroup in Ω with Dirichlet boundary condition, so that $\mathrm{e}^{t\Delta}\varphi_1 = \mathrm{e}^{-\lambda_1 t}\varphi_1$.

2. Local properties

We recall below some properties concerning local well-posedness for the equations (1.1) and (1.2). Although these are well-known results, we state them explicitly because we use the precise values of some of the constants. For further reference, we consider the slightly more general problem

$$\begin{cases} v_t = \Delta v + f(t)|v|^\alpha v, \\ v_{|\partial\Omega} = 0, \\ v(0,\cdot) = v_0(\cdot), \end{cases} \tag{2.1}$$

where $f \in L^\infty(0,\infty)$, which we study in the equivalent form

$$v(t) = \mathrm{e}^{t\Delta} v_0 + \int_0^t f(s)\,\mathrm{e}^{(t-s)\Delta}\,|v|^\alpha v(s)\,\mathrm{d}s. \tag{2.2}$$

Recall that

$$\|\,\mathrm{e}^{t\Delta} w\|_{L^\infty} \leq t^{-\frac{N}{2p}}\|w\|_{L^p}, \tag{2.3}$$

for all $t > 0$ and $1 \leq p \leq \infty$, and that there exists a constant $C_\Omega \geq 1$ such that

$$\|\,\mathrm{e}^{t\Delta} w\|_{L^\infty} \leq C_\Omega\,\mathrm{e}^{-\lambda_1 t}\,\|w\|_{L^\infty}, \tag{2.4}$$

for all $t \geq 0$. (See e.g. [**2**, Corollary 3.5.10] or [**18**, Proposition 48.5].) It follows from (2.3) and (2.4) that, with $C_\Omega \geq 1$ possibly larger,

$$\|\,\mathrm{e}^{t\Delta} w\|_{L^\infty} \leq C_\Omega t^{-\frac{N}{2p}}\,\mathrm{e}^{-\frac{\lambda_1}{2}t}\,\|w\|_{L^p}, \tag{2.5}$$

for all $t > 0$ and $1 \leq p \leq \infty$.

The following result is a consequence of a standard contraction argument.

Proposition 2.1. Let C_Ω be given by (2.4). There exists $\delta > 0$ such that if $f \in L^\infty(0,\infty)$, $v_0 \in C_0(\Omega)$ and $0 < T \leq \infty$ satisfy

$$(1 - \mathrm{e}^{-\alpha\lambda_1 T})\|f\|_{L^\infty(0,T)}\|v_0\|^\alpha_{L^\infty(\Omega)} \leq \delta, \tag{2.6}$$

then there exists a unique solution $v \in C([0,T), C_0(\Omega))$ of (2.2). Moreover,

$$\|v(t)\|_{L^\infty} \leq 2C_\Omega\,\mathrm{e}^{-\lambda_1 t}\,\|v_0\|_{L^\infty}, \tag{2.7}$$

for all $0 \le t < T$. In addition, if v_0, w_0 both satisfy (2.6) and v, w are the corresponding solutions of (2.1), then

$$\|v(t) - w(t)\|_{L^\infty} \le 2C_\Omega \,\mathrm{e}^{-\lambda_1 t}\, \|v_0 - w_0\|_{L^\infty}, \tag{2.8}$$

for all $0 \le t < T$. Moreover, the solution v can be extended to a maximal existence interval $[0, T_{\max})$, and if $T_{\max} < \infty$ then $\|v(t)\|_{L^\infty} \to \infty$ as $t \uparrow T_{\max}$.

Proof. Existence follows by applying Banach's fixed point theorem to the map $v \mapsto \Phi_{v_0}(v)$, where

$$\Phi_{v_0}(v)(t) = \mathrm{e}^{t\Delta}\, v_0 + \int_0^t f(s)\, \mathrm{e}^{(t-s)\Delta}\, |v|^\alpha v(s)\, \mathrm{d}s,$$

in the ball of radius $2C_\Omega \|v_0\|_{L^\infty}$ of the Banach space

$$\mathcal{X}_T = \begin{cases} C([0,T], C_0(\Omega)) & \text{if } T < \infty, \\ \{v \in C([0,\infty), C_0(\Omega));\ \sup_{t\ge 0} \mathrm{e}^{\lambda_1 t}\, \|v(t)\|_{L^\infty} < \infty\} & \text{if } T = \infty, \end{cases}$$

equipped with the norm $\|v\|_{\mathcal{X}_T} = \|\,\mathrm{e}^{\lambda_1 \cdot}\, v\|_{L^\infty((0,T),L^\infty)}$. Indeed, using (2.4) and setting

$$\delta = \frac{\alpha\lambda_1}{(\alpha+1)2^{\alpha+1} C_\Omega^{\alpha+1}},$$

one obtains by straightforward calculations that $\|\Phi_{v_0}(v)\|_{\mathcal{X}_T} \le 2C_\Omega \|v_0\|_{L^\infty}$ and

$$\|\Phi_{v_0}(v) - \Phi_{v_0}(w)\|_{\mathcal{X}_T} \le \frac{1}{2} \|v - w\|_{\mathcal{X}_T} \tag{2.9}$$

provided (2.6) holds. This proves the existence statement. Uniqueness easily follows from Gronwall's inequality, while the continuous dependence statement (2.8) follows from (2.9). Finally, by uniqueness, one can extend the solution to a maximal interval $[0, T_{\max})$ by the standard procedure. The fact that $\|v(t)\|_{L^\infty}$ blows up at $T_{\max}$ if $T_{\max} < \infty$ follows from the local existence property applied to an appropriate translation of f. □

Remark 2.2. It follows from the smoothing properties of the heat semigroup that the solution v of (2.1) given by Proposition 2.1 is smooth at positive times, as much as the regularity of f and that of the map $v \mapsto |v|^\alpha v$ allow. In any case, $v \in C((0, T_{\max}), C^2(\overline{\Omega}))$ and $v_t \in L^\infty((0, T_{\max}), C_0(\Omega))$.

Remark 2.3. Here are some immediate consequences of Proposition 2.1.

(i) Letting $T = \infty$ in (2.6), we see that if $\|f\|_{L^\infty(0,\infty)} \|v_0\|_{L^\infty(\Omega)}^\alpha \le \delta$, then (2.2) has a global solution $v \in C([0,\infty), C_0(\Omega))$ which satisfies (2.7) for all $t \ge 0$.

(ii) Since $1 - \mathrm{e}^{-r} \le r$, we deduce from (2.6) that if $\alpha\lambda_1 T \|f\|_{L^\infty(0,T)} \|v_0\|_{L^\infty(\Omega)}^\alpha \le \delta$, then there exists a solution $v \in C([0,T), C_0(\Omega))$ of (2.2) which satisfies (2.7) for all $0 \le t < T$.

Here is a result based on Kaplan's argument [**12**].

Lemma 2.4. Let $f \in L^\infty(0,\infty)$, $f \ge 0$ and let $v \in C([0,T_{\max}), C_0(\Omega))$ be the maximal solution of (2.2) with $v_0 \ge 0$, $v_0 \not\equiv 0$. If

$$\alpha \int_0^1 f(s)\,\mathrm{d}s > \mathrm{e}^{\alpha\lambda_1} \|\varphi_1\|_{L^1}^{\alpha} \Big(\int_\Omega v_0\varphi_1 \Big)^{-\alpha}, \tag{2.10}$$

then $T_{\max} < 1$.

Proof. Note that, by the strong maximum principle, $v(t) > 0$ for all $0 < t < T_{\max}$. Next, multiplying the equation (2.1) by φ_1 and integrating by parts, we obtain

$$\frac{d}{dt}\int_\Omega v(t)\varphi_1 + \lambda_1 \int_\Omega v(t)\varphi_1 = f(t)\int_\Omega v^{\alpha+1}\varphi_1 \ge f(t)\|\varphi_1\|_{L^1}^{-\alpha}\Big(\int_\Omega v\varphi_1\Big)^{\alpha+1},$$

where we used Hölder in the last inequality. Setting

$$h(t) = \mathrm{e}^{\lambda_1 t}\int_\Omega v(t)\varphi_1,$$

we deduce that $h'(t) \ge \mathrm{e}^{-\alpha\lambda_1 t} f(t)\|\varphi_1\|_{L^1}^{-\alpha} h(t)^{\alpha+1}$, so that

$$h(s)^{-\alpha} \ge \alpha\|\varphi_1\|_{L^1}^{-\alpha}\int_s^t \mathrm{e}^{-\alpha\lambda_1\sigma} f(\sigma)\,d\sigma,$$

for all $0 < s < t < T_{\max}$. Letting $s \downarrow 0$ and $t \uparrow T_{\max}$, we conclude that

$$(h(0))^{-\alpha} \ge \alpha\|\varphi_1\|_{L^1}^{-\alpha}\int_0^{T_{\max}} \mathrm{e}^{-\alpha\lambda_1\sigma} f(\sigma)\,d\sigma.$$

In particular, if

$$(h(0))^{-\alpha} < \alpha\|\varphi_1\|_{L^1}^{-\alpha}\int_0^1 \mathrm{e}^{-\alpha\lambda_1\sigma} f(\sigma)\,d\sigma,$$

then necessarily $T_{\max} < 1$. Since $\mathrm{e}^{-\alpha\lambda_1\sigma} > \mathrm{e}^{-\alpha\lambda_1}$ for $\sigma < 1$, the result follows. $\square$

Remark 2.5. Set

$$K = \mathrm{e}^{\lambda_1}\frac{\|\varphi_1\|_{L^1}}{\|\varphi_1\|_{L^2}^2} \ge \mathrm{e}^{\lambda_1} > 1. \tag{2.11}$$

(Note that $\|\varphi_1\|_{L^\infty} = 1$, so that $\|\varphi_1\|_{L^2}^2 \le \|\varphi_1\|_{L^1}$ by Hölder.) If $v_0 = k\varphi_1$ with $k > 0$ and

$$\alpha\int_0^1 f(s)\,\mathrm{d}s > K^\alpha k^{-\alpha}, \tag{2.12}$$

then it follows from Lemma 2.4 that $T_{\max} < 1$.

Remark 2.6. Set $\eta = \alpha^{-1}K^\alpha$, where K is defined by (2.11). It follows from Remark 2.5 that if $f(t) \equiv 1$ and if $v_0 = k\varphi_1$ with $k^\alpha > \eta$, then $T_{\max} < 1$.

Remark 2.7. We claim that there exist an initial value ϕ and a function θ as in Problem 1.5 such that the solution of (1.1) with $\omega = 1$ blows up in finite time after picking up negative values of θ. To see this, let $\xi \in C_c^\infty(\Omega)$, $\xi \geq 0$, $\xi \not\equiv 0$ and let ζ be the solution of

$$\begin{cases} -\Delta\zeta = \xi & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows from the strong maximum principle that $\zeta \geq a\varphi_1$ for some $a > 0$. Moreover, since ξ has compact support we see that $\zeta^{\alpha+1} \geq \nu\xi$ for some $\nu > 0$. We now let $\phi = A\zeta$ with

$$A \geq \max\Big\{\nu^{-\frac{1}{\alpha}}, 2a^{-1}\alpha^{-\frac{1}{\alpha}}K\Big\}, \tag{2.13}$$

where K is defined by (2.11), and we let u be the corresponding solution of (2.2) with $f(t) \equiv 1$ defined on the maximal interval $[0, T_{\max})$. Note that

$$\Delta\phi + |\phi|^\alpha\phi = A(-\xi + A^\alpha\zeta^{\alpha+1}) \geq A(-\xi + A^\alpha\nu\xi) \geq 0,$$

by the first inequality in (2.13). It follows in particular that $u_t \geq 0$ on $(0, T_{\max}) \times \Omega$. Next, we deduce from the second inequality in (2.13) that

$$\alpha > \mathrm{e}^{\alpha\lambda_1}\, \|\varphi_1\|_{L^1}^\alpha \Big(\int_\Omega \phi\varphi_1\Big)^{-\alpha}, \tag{2.14}$$

so that $T_{\max} < 1$ by Lemma 2.4. We fix $0 < T < T_{\max}$. Since $u(T) \geq \phi$, we deduce from (2.14) that

$$\alpha > \mathrm{e}^{\alpha\lambda_1}\, \|\varphi_1\|_{L^1}^\alpha \Big(\int_\Omega u(T)\varphi_1\Big)^{-\alpha}. \tag{2.15}$$

We now consider a function $f \in C(\mathbb{R})$ such that $f(t) = 1$ for $T \leq t \leq T + 1$ and we let v be the corresponding solution of (2.2) with the initial value ϕ. It follows from a standard argument that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|f - 1\|_{L^1(0,T)} \leq \delta$ then v is defined on $[0, T]$ and $\|u(T) - v(T)\|_{L^\infty} \leq \varepsilon$. In particular, if $\delta > 0$ is sufficiently small and $\|f - 1\|_{L^1(0,T)} \leq \delta$ then v is defined on $[0, T]$ and, by (2.15),

$$\alpha > \mathrm{e}^{\alpha\lambda_1}\, \|\varphi_1\|_{L^1}^\alpha \Big(\int_\Omega v(T)\varphi_1\Big)^{-\alpha}. \tag{2.16}$$

Since $f(t) = 1$ for $T \leq t \leq T+1$, it follows in particular from (2.16) and Lemma 2.4 (applied with $f(t) \equiv 1$) that v blows up before the time $T + 1$. Note that, once ϕ is fixed, the only restriction is that $\|f - 1\|_{L^1(0,T)}$ is small. This allows f to be negative at $t = 0$ (and even to be highly oscillatory on the interval $(0, T)$). The claim now follows by choosing $\tau > T + 1$ and a τ-periodic function $\theta \in C(\mathbb{R})$ such that $\theta(t) = f(t)$ for $0 \leq t \leq T + 1$ and $A(\theta) > 0$.

Remark 2.8. We claim that there exists a function θ as in Problem 1.5 such that the solution of (1.1) with $\omega = 1$ is global for all $\phi \in C_0(\Omega)$. Indeed, let $\delta > 0$ be as in Proposition 2.1 and set $T = 1/(\alpha\delta)$. Fix $\tau > 2T$ and a τ-periodic function $\theta \in C(\mathbb{R})$ such that $\theta(t) = -1$ for $0 \leq t \leq T$, $\|\theta\|_{L^\infty} \leq 1$ and $A(\theta) > 0$.

Given any $\phi \in C_0(\Omega)$, let u be the corresponding solution of (1.1) with $\omega = 1$. It follows easily by comparison with the solution $(\alpha t)^{-\frac{1}{\alpha}}$ of the ODE $u' = -|u|^\alpha u$ that u exists up to the time T and that $\|u(T)\|_{L^\infty} \le (\alpha T)^{-\frac{1}{\alpha}} = \delta^{-\frac{1}{\alpha}}$. Applying Remark 2.3 (i) with $f(t) \equiv \theta(t+1)$, we conclude that u is global.

3. Proofs of Propositions 1.1 and 1.2

Proposition 1.1 could be proved (with convergence in L^p, $p < \infty$, rather than in L^∞) by the "periodic unfolding method", see [**6**]. We give here a direct proof which relies on the following elementary lemma.

Lemma 3.1. Given $0 < T < \infty$ and $h \in L^\infty((0,T) \times \Omega)$, it follows that

$$\int_0^t \theta(\omega(s+t_0))\, \mathrm{e}^{(t-s)\Delta}\, h(s)\, \mathrm{d}s \underset{|\omega|\to\infty}{\longrightarrow} A(\theta) \int_0^t \mathrm{e}^{(t-s)\Delta}\, h(s)\, \mathrm{d}s, \tag{3.1}$$

in $L^\infty((0,T) \times \Omega)$, uniformly in $t_0 \in \mathbb{R}$.

Proof. Set

$$\psi(t) = \theta(t) - A(\theta), \quad \Psi(t) = \int_0^t \psi(s)\, \mathrm{d}s,$$

so that Ψ is τ-periodic, hence bounded. It follows from (2.5) that

$$\begin{aligned}
\Big\| \int_0^t \psi(\omega(s+t_0))\, \mathrm{e}^{(t-s)\Delta}\, h(s)\, \mathrm{d}s \Big\|_{L^\infty} &\le C \int_0^t \mathrm{e}^{-\frac{\lambda_1}{2}(t-s)} (t-s)^{-\frac{N}{2(N+1)}} \|h(s)\|_{L^{N+1}} \\
&\le C \Big(\int_0^t \mathrm{e}^{-\frac{\lambda_1 (N+1)}{2N} s}\, s^{-\frac{1}{2}} \Big)^{\frac{N}{N+1}} \|h\|_{L^{N+1}((0,T)\times\Omega)} \\
&\le C \|h\|_{L^{N+1}((0,T)\times\Omega)},
\end{aligned}$$

for every $0 \le t \le T$. Therefore, by density, we need only prove (3.1) for $h \in C_{\mathrm{c}}^\infty((0,T) \times \Omega)$. Since $\psi(\omega(s+t_0)) = \frac{1}{\omega} \frac{d}{\mathrm{d}s} \Psi(\omega(s+t_0))$, an integration by parts yields

$$\begin{aligned}
\int_0^t \psi(\omega(s+t_0))\, \mathrm{e}^{(t-s)\Delta}\, h(s)\, \mathrm{d}s &= \frac{1}{\omega} \Psi(\omega(t+t_0)) h(t) \\
&\quad - \frac{1}{\omega} \int_0^t \Psi(\omega(s+t_0))\, \mathrm{e}^{(t-s)\Delta} [h_t(s) - \Delta h(s)]\, \mathrm{d}s.
\end{aligned}$$

Thus, by (2.4),

$$\begin{aligned}
&\Big\| \int_0^t \psi(\omega(s+t_0))\, \mathrm{e}^{(t-s)\Delta}\, h(s)\, \mathrm{d}s \Big\|_{L^\infty} \\
&\qquad \le \frac{C}{|\omega|} \|\Psi\|_{L^\infty} [\|h\|_{L^\infty((0,\infty)\times\Omega)} + \|h_t - \Delta h\|_{L^\infty((0,\infty)\times\Omega)}] \underset{|\omega|\to\infty}{\longrightarrow} 0,
\end{aligned} \tag{3.2}$$

uniformly in $t \in [0,T]$ and $t_0 \in \mathbb{R}$. This completes the proof. $\square$

Proof of Proposition 1.1. Fix $0 < T < T_{\max}$ and set

$$M = 2C_\Omega \sup_{0\le t\le T} \|U(t)\|_{L^\infty},$$

where $C_\Omega \ge 1$ is given by (2.4). In particular, $\|\phi\|_{L^\infty} \le M/2$, so we may define $T_\omega > 0$, for every $\omega \in \mathbb{R}$, by

$$T_\omega = \min\{T, \sup\{t > 0;\ u_\omega \text{ exists on } (0,t) \text{ and } \|u_\omega\|_{L^\infty((0,t)\times\Omega)} \le M\}\}.$$

For $0 \le t < T_\omega$, we have

$$\begin{aligned} U(t) - u_\omega(t) &= \int_0^t \theta(\omega s)\, \mathrm{e}^{(t-s)\Delta}[|U|^\alpha U - |u_\omega|^\alpha u_\omega]\, \mathrm{d}s \\ &\quad + \int_0^t [A(\theta) - \theta(\omega s)]\, \mathrm{e}^{(t-s)\Delta}\, |U|^\alpha U\, \mathrm{d}s \stackrel{\mathrm{def}}{=} a_\omega(t) + b_\omega(t). \end{aligned} \tag{3.3}$$

It follows from Lemma 3.1 that

$$\|b_\omega\|_{L^\infty((0,T)\times\Omega)} \underset{|\omega|\to\infty}{\longrightarrow} 0. \tag{3.4}$$

Moreover,

$$\|a_\omega(t)\|_{L^\infty} \le 2(\alpha+1)\|\theta\|_{L^\infty} M^\alpha \int_0^t \|U(s) - u_\omega(s)\|_{L^\infty}\, \mathrm{d}s,$$

so we deduce from (3.3) and Gronwall's inequality that

$$\|U - u_\omega\|_{L^\infty((0,T_\omega)\times\Omega)} \le \|b_\omega\|_{L^\infty((0,T)\times\Omega)}\, \mathrm{e}^{2(\alpha+1)\|\theta\|_{L^\infty} M^\alpha T}. \tag{3.5}$$

Applying (3.5) and (3.4), we may now assume that $|\omega|$ is sufficiently large so that $\|U - u_\omega\|_{L^\infty((0,T_\omega)\times\Omega)} \le M/4$. Since $\|U\|_{L^\infty((0,T_\omega)\times\Omega)} \le M/2$ by definition of M, we conclude that $\|u_\omega\|_{L^\infty((0,T_\omega)\times\Omega)} \le 3M/4 < M$. Thus we see that $T_\omega = T$. This proves the first statement of Proposition 1.1. Moreover, we may now apply (3.5) with T_ω replaced by T and the second statement of Proposition 1.1 follows from (3.4). □

Proof of Proposition 1.2. Let $\delta > 0$ be given by Proposition 2.1 and fix S large enough so that

$$\|\theta\|_{L^\infty(\mathbb{R})} \|U(S)\|^\alpha_{L^\infty(\Omega)} \le \frac{\delta}{2}. \tag{3.6}$$

Applying Proposition 2.1 with $f(t) \equiv A(\theta)$, $T = \infty$ and $v_0 = U(S)$, we deduce that

$$\|U(t)\|_{L^\infty} \le 2C_\Omega \|U(S)\|_{L^\infty}\, \mathrm{e}^{-\lambda_1(t-S)}, \tag{3.7}$$

for all $t \ge S$. Moreover, it follows from Proposition 1.1 that if $|\omega|$ is sufficiently large, then u_ω exists on $[0,S]$ and

$$\|U(S) - u_\omega(S)\|_{L^\infty} \underset{|\omega|\to\infty}{\longrightarrow} 0. \tag{3.8}$$

Applying (3.8) and (3.6), we conclude that

$$\|\theta\|_{L^\infty(\mathbb{R})} \|u_\omega(S)\|^\alpha_{L^\infty(\Omega)} \le \delta,$$

if $|\omega|$ is sufficiently large. We may now apply Proposition 2.1 with $f(t) \equiv \theta(\omega(S + t))$, $T = \infty$ and $v_0 = u_\omega(S)$, and we deduce that u_ω is globally defined and $\|u_\omega(S+t)\|_{L^\infty} \le C\,\mathrm{e}^{-\lambda_1 t}$ for all $t \ge 0$, where C is independent of ω. This proves the first and second statements of Proposition 1.2 (with $\lambda = \lambda_1$). The last statement follows from Proposition 1.1. $\square$

4. Proof of Theorem 1.6

Let C_Ω be given by (2.4), let the constants δ, K and η be as defined in Proposition 2.1 and Remarks 2.5 and 2.6, respectively. Set

$$k_0 = \delta^{\frac{1}{\alpha}}, \tag{4.1}$$

$$k_1 = \mathrm{e}^{2\lambda_1}\,\eta^{\frac{1}{\alpha}}, \tag{4.2}$$

$$k_2 = K\Big(1 + \frac{4}{\alpha\delta}\Big)^{\frac{1}{\alpha}} k_1, \tag{4.3}$$

$$c = \frac{\delta}{2k_1^\alpha}, \tag{4.4}$$

$$\ell = 2 + \frac{2\alpha k_2^\alpha}{\delta}\log\frac{4C_\Omega^2 k_2}{\delta^{\frac{1}{\alpha}}}, \tag{4.5}$$

$$\tau = \max\Big\{2(\ell+2), \frac{8}{c}\Big\}. \tag{4.6}$$

Fix

$$0 < \varepsilon \le \min\Big\{\frac{c}{2}, \frac{1}{(\ell-2)(2C_\Omega)^\alpha}, \frac{\delta}{k_2^\alpha}\Big\}, \tag{4.7}$$

and set

$$k_3 = 2K(\alpha\varepsilon)^{-\frac{1}{\alpha}}. \tag{4.8}$$

Note that (4.3) and (4.4) imply that

$$\frac{c}{2} = \frac{\delta K^\alpha}{4k_2^\alpha}\Big(1 + \frac{4}{\alpha\delta}\Big) > \frac{K^\alpha}{\alpha k_2^\alpha}. \tag{4.9}$$

Note also that by Remark 2.3 (i) (applied with $f(t) \equiv 1$) and Remark 2.6, $\eta \ge \delta$. Therefore, and since $K > 1$ by (2.11), we deduce from (4.3) and (4.2) that $k_2 > k_1 > \delta^{\frac{1}{\alpha}}$. Since $C_\Omega \ge 1$, it follows from (4.5) that

$$\ell > 2. \tag{4.10}$$

Let $\Phi \in C^\infty(\mathbb{R})$ be τ-periodic and satisfy $\varepsilon \le \Phi \le 1$ and

$$\Phi(t) = \begin{cases} 1 & 0 \le t \le 1 \\ \varepsilon & 2 \le t \le \ell \\ c & \ell + 1 \le t \le \tau - 1. \end{cases} \tag{4.11}$$

(See Figure 3.) Note that this makes sense by (4.10) and (4.6). Note also that

$$\|\Phi\|_{L^\infty(\mathbb{R})} = 1. \tag{4.12}$$

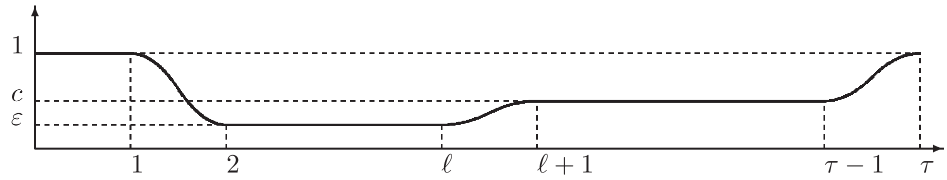

Figure 3. The function $\theta = \Phi$ of Theorem 1.6.

Furthermore,

$$A(\Phi) \le \frac{1}{\tau}[4 + c(\tau - \ell - 2) + \varepsilon(\ell - 2)] \le \frac{1}{\tau}[4 + c\tau + \varepsilon\tau] = c + \varepsilon + \frac{4}{\tau}.$$

Since $\varepsilon \le c/2$ by (4.7) and $4/\tau \le c/2$ by (4.6), we deduce that $A(\Phi) \le 2c$. Applying (4.4), we conclude that

$$A(\Phi)k_1^\alpha \le \delta. \tag{4.13}$$

Next, we observe that

$$A(\Phi) \ge c\frac{\tau - \ell - 2}{\tau} \ge \frac{c}{2},$$

where we used (4.6) in the last inequality. Applying (4.9), we deduce that

$$A(\Phi) > \frac{K^\alpha}{\alpha k_2^\alpha}. \tag{4.14}$$

We let

$$\theta = \Phi,$$

and, given $k > 0$ and $\omega \in \mathbb{R}$, we consider the solution $u_{k,\omega}$ of (1.1) with $\phi = k\varphi_1$. We proceed in several steps.

Step 1. Since k_0 is defined by (4.1), it follows from (4.12) and Remark 2.3 (i) (applied with $f(t) \equiv \theta(\omega t)$) that if $k \le k_0$, then $u_{k,\omega}$ is global and exponentially decaying for all $\omega \in \mathbb{R}$.

Step 2. Let $k = k_1$ defined by (4.2). It follows from (4.13) and Remark 2.3 (i) (with $f(t) \equiv A(\theta)$) that the solution U of (1.2) with $\phi = k\varphi_1$ is global and exponentially decaying. Applying Proposition 1.2, we deduce that if $|\omega|$ is sufficiently large, then $u_{k,\omega}$ is global and exponentially decaying.

Step 3. Let $k = k_1$ defined by (4.2) and let $0 < \omega \le 1$, so that $\theta(\omega t) = 1$ for $0 \le t \le 1$. It follows from (4.2) and Remark 2.6 that $u_{k,\omega}$ blows up before $t = 1$.

Step 4. Let $k = k_2$ defined by (4.3). Since

$$\int_0^1 \theta(\omega s)\,\mathrm{d}s \underset{|\omega|\to\infty}{\longrightarrow} A(\theta),$$

we deduce from (4.14) that if $|\omega|$ is large, then

$$\alpha\int_0^1 \theta(\omega s)\,\mathrm{d}s > \frac{K^\alpha}{k_2^\alpha}. \tag{4.15}$$

Applying Remark 2.5 (with $f(t) \equiv \theta(\omega t)$), we conclude by using (2.12) that $u_{k,\omega}$ blows up before the time $t = 1$. Thus we see that if $|\omega|$ is sufficiently large, then $u_{k,\omega}$ blows up in finite time.

Step 5. Let $k = k_2$ defined by (4.3). If $0 < \omega \le 1$, then $\theta(\omega t) = 1$ for $0 \le t \le 1$. Since $k_2 \ge k_1 \ge \eta^{\frac{1}{\alpha}}$ by (4.3) and (4.2), it follows from Remark 2.6 that $u_{k,\omega}$ blows up before $t = 1$.

Step 6. Let $k = k_2$ defined by (4.3) and $\omega = \omega_0$ where

$$\omega_0 = \frac{2\alpha\lambda_1 k_2^\alpha}{\delta}. \tag{4.16}$$

It follows from (4.16) and Remark 2.3 (ii) (with $T = 2/\omega_0$ and $f(\cdot) = \theta(\omega_0 \cdot)$) that $u_{k,\omega}$ exists up to the time $2/\omega_0$ and

$$\|u_{k,\omega}(2/\omega_0)\|_{L^\infty} \le 2C_\Omega k_2. \tag{4.17}$$

On the other hand, we deduce from (4.7) and (4.16) that

$$\varepsilon \le \frac{1}{(\ell-2)(2C_\Omega)^\alpha} = \frac{\omega_0 \delta}{2\alpha\lambda_1(\ell-2)k_2^\alpha(2C_\Omega)^\alpha}. \tag{4.18}$$

Inequalities (4.17) and (4.18) imply

$$\alpha\lambda_1\Big(\frac{\ell-2}{\omega_0}\Big)\varepsilon\|u_{k,\omega}(2/\omega_0)\|_{L^\infty}^\alpha \le \delta. \tag{4.19}$$

Since $\theta(\omega_0 t) = \varepsilon$ for $t \in (2/\omega_0, \ell/\omega_0)$, it follows from (4.19) and Remark 2.3 (ii) (with $T = (\ell-2)/\omega_0$, $f(\cdot) \equiv \theta(2 + \omega_0 \cdot)$ and $\phi = u_{k,\omega}(2/\omega_0)$) that $u_{k,\omega}$ exists up to the time ℓ/ω_0 and that

$$\|u_{k,\omega}(\ell/\omega_0)\|_{L^\infty} \le 2C_\Omega \|u_{k,\omega}(2/\omega_0)\|_{L^\infty}\, \mathrm{e}^{-\lambda_1 \frac{\ell-2}{\omega_0}} \le 4C_\Omega^2 k_2\, \mathrm{e}^{-\lambda_1 \frac{\ell-2}{\omega_0}}, \tag{4.20}$$

where we used (4.17) in the last inequality. Note that, by (4.5) and (4.16),

$$\lambda_1 \frac{\ell-2}{\omega_0} = \log \frac{4C_\Omega^2 k_2}{\delta^{\frac{1}{\alpha}}},$$

so that (4.20) implies $\|u_{k,\omega}(\ell/\omega_0)\|_{L^\infty}^\alpha \le \delta$. Applying Remark 2.3 (i) (with $f(t) \equiv \theta(\ell + \omega_0 t)$) and $\phi = u_{k,\omega}(\ell/\omega_0)$), we conclude that $u_{k,\omega}$ is global and exponentially decaying.

Step 7. Let k_3 be defined by (4.8). Since $\theta \ge \varepsilon$, we see that for every $\omega \in \mathbb{R}$,

$$\alpha \int_0^1 \theta(\omega s)\, \mathrm{d}s \ge \alpha\varepsilon > \frac{K^\alpha}{k_3^\alpha},$$

where we used (4.8) in the last inequality. It follows that if $k \ge k_3$ then

$$\alpha \int_0^1 \theta(\omega s)\, \mathrm{d}s > \frac{K^\alpha}{k^\alpha}.$$

Applying Remark 2.5 (with $f(t) \equiv \theta(\omega t)$), we conclude by using (2.12) that $u_{k,\omega}$ blows up before the time $t = 1$ for all $\omega \in \mathbb{R}$.

Step 8. Conclusion. Property (i) follows from Step 1. Property (ii) follows from Steps 2 and 3. Property (iii) follows from Steps 4, 5 and 6. Property (iv) follows from Step 7.

5. Proof of Theorem 1.7

We consider Φ, k_0, k_1, k_2, k_3 as in the preceding section and we let

$$\theta(t) \equiv \Phi(3-t).$$

(See Figure 4.) Given $k > 0$ and $\omega > 0$, we consider the solution $u_{k,\omega}$ of (1.1) with

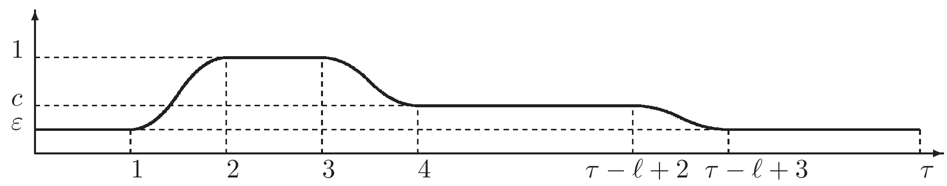

Figure 4. The function θ of Theorem 1.7.

$\phi = k\varphi_1$. Property (i) (respectively, Property (iv)) follows from the argument of Step 1 (respectively, Step 7) in the preceding section. It remains to prove Properties (ii) and (iii), and we proceed in several steps.

Step 1. Let $k \le k_2$ defined by (4.3). Given $\omega > 0$, note that $\theta(\omega t) = \varepsilon$ for $0 \le t \le 1/\omega$. It follows from (4.7) that $\varepsilon \|\phi\|_{L^\infty}^\alpha \le \delta$, so that by Remark 2.3 (i) (applied with $f(t) \equiv \theta(\omega t)$) $u_{k,\omega}$ exists up to the time $1/\omega$ and

$$\|u_{k,\omega}(1/\omega)\|_{L^\infty} \le 2C_\Omega k_2 \, e^{-\frac{\lambda_1}{\omega}} \underset{\omega \downarrow 0}{\longrightarrow} 0.$$

Thus we see that if $\omega > 0$ is sufficiently small, then $\|u_{k,\omega}(1/\omega)\|_{L^\infty}^\alpha \le \delta$. Applying again Remark 2.3 (i) (with $f(t) \equiv \theta(1+\omega t)$ and $\phi = u(1/\omega)$), we conclude that if $\omega > 0$ is sufficiently small, then $u_{k,\omega}$ is globally defined and exponentially decaying.

Step 2. Let $k = k_1$ defined by (4.2). Since $A(\theta)k_1^\alpha \le \delta$ by (4.13), we conclude with the argument of Step 2 of the preceding section that if $|\omega|$ is sufficiently large, then $u_{k,\omega}$ is global and exponentially decaying.

Step 3. Let $k = k_1$ defined by (4.2) and $\omega = 1$. We claim that $u_{k,\omega}$ blows up in finite time. Indeed, assume by contradiction that $u_{k,\omega}$ is global. Since $\theta \ge 0$, we observe that $u_{k,\omega}(t) \ge e^{t\Delta}\phi = k_1 e^{-\lambda_1 t} \varphi_1$. In particular,

$$u_{k,\omega}(2) \ge k_1 e^{-2\lambda_1} \varphi_1 \ge \eta^{\frac{1}{\alpha}} \varphi_1, \tag{5.1}$$

where we used (4.2) in the last inequality. We note that for $t \in [2,3]$, $u_{k,\omega}$ solves the equation (2.1) with $f(t) \equiv 1$. Thus it follows from (5.1) and Remark 2.6 that u blows up before the time $T = 3$, which is a contradiction.

Step 4. Let $k = k_2$ defined by (4.3). The argument of Step 4 in the preceding section shows that if $|\omega|$ is sufficiently large, then $u_{k,\omega}$ blows up in finite time.

Step 5. Conclusion. Property (ii) follows from Steps 1, 2 and 3. Property (iii) follows from Steps 1 and 4.

6. Proof of Theorem 1.3

Given $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, let U be the corresponding solution of (1.10) and, for every $\omega \in \mathbb{R}$, let u_ω be the (maximal) solution of (1.9). Suppose U blows up at the time $T < \infty$. We first apply Theorem A.1 with $H = L^2(\Omega)$, $L = \Delta$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$, $F(t,u) = A(\theta)\|u\|_{L^2}^\alpha$, $\varphi = \phi$ and $\kappa = 1$. (Note that the map $u \mapsto f(u) \overset{\text{def}}{=} \|u\|_{L^2}^\alpha u$ is locally Lipschitz $L^2(\Omega) \to L^2(\Omega)$. Indeed, it is not difficult to show that, even if $\alpha < 1$, $\|f(u) - f(v)\|_{L^2} \le (\alpha+1)\max\{\|u\|_{L^2}^\alpha, \|v\|_{L^2}^\alpha\}\|u-v\|_{L^2}$.) It is easy to show that, with the notation of Theorem A.1, $0 < \Lambda < \infty$ and the supremum in (A.7) is not achieved, i.e.

$$\Lambda = \int_0^\infty \mu(s)\rho(s)^{-\frac{\alpha}{2}}\,\mathrm{d}s. \tag{6.1}$$

Thus we deduce from property (iv) of Theorem A.1 that

$$\alpha\Lambda\|\phi\|_{L^2}^\alpha > 1. \tag{6.2}$$

Since $\mu(t) > 0$ for all $t \ge 0$, we deduce from (6.1)-(6.2) that if $T > 0$ is sufficiently large, then

$$\alpha\|\phi\|_{L^2}^\alpha \int_0^T \mu(s)\rho(s)^{-\frac{\alpha}{2}}\,\mathrm{d}s > 1.$$

Writing explicitly μ and ρ, this means

$$\alpha\|\phi\|_{L^2}^\alpha A(\theta)\int_0^T \exp\Big(-\alpha\int_0^s \frac{\|\nabla\, \mathrm{e}^{s\Delta}\,\phi\|_{L^2}^2}{\|\,\mathrm{e}^{s\Delta}\,\phi\|_{L^2}^2}\Big)\mathrm{d}s > 1. \tag{6.3}$$

It follows that

$$\alpha\|\phi\|_{L^2}^\alpha \int_0^T \theta(\omega s)\exp\Big(-\alpha\int_0^s \frac{\|\nabla\, \mathrm{e}^{s\Delta}\,\phi\|_{L^2}^2}{\|\,\mathrm{e}^{s\Delta}\,\phi\|_{L^2}^2}\Big)\mathrm{d}s > 1, \tag{6.4}$$

provided $|\omega|$ is sufficiently large. We now apply Theorem A.1, this time with $F(t,u) = \theta(\omega t)\|u\|_{L^2}^\alpha$. With this choice of $F(t,u)$, it follows that

$$\Lambda = \sup_{T\ge 0}\int_0^T \theta(\omega s)\exp\Big(-\alpha\int_0^s \frac{\|\nabla\, \mathrm{e}^{s\Delta}\,\phi\|_{L^2}^2}{\|\,\mathrm{e}^{s\Delta}\,\phi\|_{L^2}^2}\Big)\mathrm{d}s. \tag{6.5}$$

We deduce from (6.4) and (6.5) that $\alpha\|\phi\|_{L^2}^\alpha\Lambda > 1$ if $|\omega|$ is sufficiently large. Applying Theorem A.1 (property (iii) or property (iv)), we conclude that u_ω blows up in finite time.

Remark 6.1. Note that the existence of solutions of (1.10) that blow up in finite time follows immediately from Theorem A.1. (Fix $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, $\varphi \not\equiv 0$ and let $\phi = \kappa\varphi$ with $\kappa > 0$ sufficiently large.) It also follows from classical results, see [**13**].

Remark 6.2. Note that we could obtain (with the same proof) conclusions similar to those of Theorem 1.3 for equations slightly more general than (1.9). For example, one could replace the nonlinearity $f(u) = \|u\|_{L^2}^{\alpha} u$ in (1.9) by the more general one

$$f(u) = \Big(\int_\Omega k(x)|u(x)|^q dx\Big)^{\frac{\alpha}{q}} u, \tag{6.6}$$

where $\alpha > 0$, $1 \le q \le 2$ and $k \in L^\infty(\Omega)$, $k \ge 0$, $k \not\equiv 0$. (More generally, one can also consider $2 < q < \infty$ by replacing the space $H = L^2(\Omega)$ by $H = D(L^\ell)$ where ℓ is sufficiently large so that $D(L^\ell) \hookrightarrow L^q(\Omega)$.)

Appendix A. Blowup for an abstract evolution equation

Let H be a Hilbert space with norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$ and let L be a linear, unbounded operator on H, with domain $D(L)$. Assume that L is the generator of a C_0 semigroup $(\mathrm{e}^{tL})_{t\ge 0}$ on H. Let $F \in C([0,\infty) \times H, \mathbb{R})$ and assume that there exists $\alpha > 0$ such that

$$F(t, \lambda x) = \lambda^\alpha F(t,x), \tag{A.1}$$

for all $t \ge 0$, $\lambda \ge 0$ and $x \in H$. Suppose further that the map $u \mapsto F(t,u)u$ is Lipschitz continuous from bounded sets of H onto H, uniformly for t in a bounded interval. Given $\phi \in H$, we consider the equation

$$\begin{cases} u' = Lu + F(t,u)u, \\ u(0) = \phi, \end{cases} \tag{A.2}$$

in the equivalent form

$$u(t) = \mathrm{e}^{tL}\,\phi + \int_0^t \mathrm{e}^{(t-s)L}\, F(s,u(s))u(s)\,\mathrm{d}s. \tag{A.3}$$

Under the above assumptions, it is well known that, for any $\phi \in H$, there exists a unique solution u of (A.3), which is defined on a maximal interval $[0, T_{\max})$, i.e. $u \in C([0,T_{\max}), H)$. Moreover, if $T_{\max} < \infty$, then $\|u(t)\| \to \infty$ as $t \uparrow T_{\max}$. (Blowup alternative.) In addition, if $\phi \in D(L)$, then $u \in C([, T_{\max}), D(L)) \cap C^1([0,T_{\max}), H)$ and u is the solution of (A.2).

Theorem A.1. Let $\varphi \in D(L)$, $\varphi \neq 0$, and suppose (for simplicity) that $\mathrm{e}^{tL}\,\varphi \neq 0$ for all $t \ge 0$. Set

$$\eta(t) = \|\,\mathrm{e}^{tL}\,\varphi\|^{-2}(L\,\mathrm{e}^{tL}\,\varphi, \mathrm{e}^{tL}\,\varphi), \tag{A.4}$$

$$\mu(t) = \|\,\mathrm{e}^{tL}\,\varphi\|^{-\alpha} F(t, \mathrm{e}^{tL}\,\varphi), \tag{A.5}$$

$$\rho(t) = \exp\Big(-2\int_0^t \eta(s)\,\mathrm{d}s\Big) > 0, \tag{A.6}$$

for all $t \ge 0$ and

$$\Lambda = \sup_{T\ge 0} \int_0^T \mu(s)\rho(s)^{-\frac{\alpha}{2}}\mathrm{d}s, \tag{A.7}$$

so that $0 \le \Lambda \le \infty$. Let $\kappa \ge 0$, set $\phi^\kappa = \kappa\varphi$ and let u^κ be the solution of (A.2) with initial value ϕ^κ, defined on the maximal interval $[0, T^\kappa_{\max})$.

(i) If $\Lambda = 0$, then u^κ is global, i.e. $T^\kappa_{\max} = \infty$ for all $\kappa \ge 0$.

(ii) If $\Lambda = \infty$, then u^κ blows up in finite time, i.e. $T^\kappa_{\max} < \infty$ for all $\kappa > 0$.

(iii) If $0 < \Lambda < \infty$ and the supremum in (A.7) is achieved, then u^κ is global if $\kappa < (\alpha\Lambda\|\varphi\|^\alpha)^{-\frac{1}{\alpha}}$ and u^κ blows up in finite time if $\kappa \ge (\alpha\Lambda\|\varphi\|^\alpha)^{-\frac{1}{\alpha}}$. Moreover, in the last case, $T^\kappa_{\max}$ is the smallest positive number T such that

$$\int_0^T \mu(s)\rho(s)^{-\frac{\alpha}{2}} \mathrm{d}s = \frac{1}{\alpha\kappa^\alpha\|\varphi\|^\alpha}. \tag{A.8}$$

(iv) If $0 < \Lambda < \infty$ and the supremum in (A.7) is not achieved, then u^κ is global if $\kappa \le (\alpha\Lambda\|\varphi\|^\alpha)^{-\frac{1}{\alpha}}$ and u^κ blows up in finite time if $\kappa > (\alpha\Lambda\|\varphi\|^\alpha)^{-\frac{1}{\alpha}}$. Moreover, in the last case, $T^\kappa_{\max}$ is the smallest positive number T such that (A.8) holds

Remark A.2. Here are some comments on Theorem A.1.

(i) Note that Theorem A.1 yields some blowup results that are not immediate by the standard techniques. In particular, the operator L is only supposed to be the generator of a C_0 semigroup on H. (L is not assumed to be symmetric).

(ii) There is no need in principle to introduce the parameter κ in Theorem A.1. (One could let $\kappa = 1$ in the statement.) The parameter κ is there to emphasize the fact that the elements η, μ, ρ, Λ are left unchanged if one replaces the initial value φ by $\kappa\varphi$ for $\kappa > 0$.

The proof of Theorem A.1 is based on the following elementary property. (See the proof of Theorem 44.2 (ii) in [**18**] for similar calculations. See also the proof of Theorem 2.1 in [**4**].)

Proposition A.3. Let $\phi \in D(L)$, $f \in C([0,\infty), \mathbb{R})$ and let $u \in C([0,\infty), D(L)) \cap C^1([0,\infty), H)$ be the solution of

$$\begin{cases} u' = Lu + f(t)u, \\ u(0) = \phi. \end{cases} \tag{A.9}$$

It follows that

$$\| \mathrm{e}^{tL} \phi\| u(t) = \|u(t)\| \mathrm{e}^{tL} \phi, \tag{A.10}$$

for all $t \ge 0$.

Proof. Set $w(t) = \Phi(t) \mathrm{e}^{tL} \phi$ where $\Phi(t) = \exp(\int_0^t f(s)\,\mathrm{d}s)$. It follows that $w \in C([0,\infty), D(L)) \cap C^1([0,\infty), H)$, $w(0) = \phi$, and $w_t = Lw + f(t)w$. Therefore $w(t) \equiv u(t)$, so that $u(t) = \Phi(t) \mathrm{e}^{tL} \phi$, and (A.10) easily follows. □

Proof of Theorem A.1. Set $M_\kappa(t) = \|u^\kappa(t)\|^2$ for all $0 \le t < T^\kappa_{\max}$. Taking the scalar product of (A.2) with u^κ, we obtain

$$\frac{1}{2} M_\kappa'(t) = (Lu^\kappa, u^\kappa) + F(t, u^\kappa) M_\kappa(t). \tag{A.11}$$

On the other hand, it follows from Proposition A.3 that

$$u^\kappa(t) = \frac{\|u^\kappa(t)\|}{\|\,\mathrm{e}^{tL}(\kappa\varphi)\|}\,\mathrm{e}^{tL}(\kappa\varphi) = \frac{\|u^\kappa(t)\|}{\|\,\mathrm{e}^{tL}\,\varphi\|}\,\mathrm{e}^{tL}\,\varphi. \tag{A.12}$$

Using the homogeneity property (A.1), we deduce that

$$\begin{aligned}\frac{1}{2}M_\kappa'(t) &= \frac{M_\kappa(t)}{\|\,\mathrm{e}^{tL}\,\varphi\|^2}(L\,\mathrm{e}^{tL}\,\varphi, \mathrm{e}^{tL}\,\varphi) + \frac{M_\kappa(t)^{1+\frac{\alpha}{2}}}{\|\,\mathrm{e}^{tL}\,\varphi\|^\alpha}F(t, \mathrm{e}^{tL}\,\varphi)\\ &= \eta(t)M_\kappa(t) + \mu(t)M_\kappa(t)^{1+\frac{\alpha}{2}}.\end{aligned} \tag{A.13}$$

Integrating the above differential equation, we deduce that

$$[\rho(t)M_\kappa(t)]^{-\frac{\alpha}{2}} = [\kappa\|\phi\|]^{-\alpha} - \alpha\int_0^t \mu(s)\rho(s)^{-\frac{\alpha}{2}}\,\mathrm{d}s, \tag{A.14}$$

for all $0 \le t < T_{\max}^\kappa$.

If $\Lambda \le 0$, we deduce from (A.14) that $\|u^\kappa(t)\| \le \kappa\rho(t)^{-\frac{1}{2}}\|\phi\|$ for all $0 \le t < T_{\max}^\kappa$, so that $T_{\max}^\kappa = \infty$ by the blowup alternative. This proves property (i).

Suppose now $\Lambda > 0$. If $\kappa > (\alpha\Lambda\|\varphi\|^\alpha)^{-\frac{1}{\alpha}}$, then there exists $0 < T < \infty$ such that the right-hand side of (A.14) is negative. Thus we deduce from (A.14) that $T_{\max}^\kappa < T < \infty$. This proves property (ii) and part of properties (iii) and (iv).

Finally, suppose $0 < \Lambda < \infty$. If $\kappa < (\alpha\Lambda\|\varphi\|^\alpha)^{-\frac{1}{\alpha}}$, then we deduce from (A.14) that

$$[\rho(t)M_\kappa(t)]^{-\frac{\alpha}{2}} \ge [\kappa\|\phi\|]^{-\alpha} - \alpha\Lambda \stackrel{\text{def}}{=} \delta > 0. \tag{A.15}$$

Thus we see that $\|u^\kappa(t)\| \le \delta^{-\frac{1}{\alpha}}\rho(t)^{-\frac{1}{2}}$ for all $0 \le t < T_{\max}^\kappa$, so that $T_{\max}^\kappa = \infty$ by the blowup alternative. This proves part of properties (iii) and (iv).

It remains to consider the case $0 < \Lambda < \infty$ and $\kappa = (\alpha\Lambda\|\varphi\|^\alpha)^{-\frac{1}{\alpha}}$. If the supremum in (A.7) is achieved at some $T < \infty$, then the right-hand side of (A.14) vanishes at T, so that $T_{\max}^\kappa \le T$. This completes the proof of property (iii). If the supremum in (A.7) is not achieved, then the right-hand side of (A.14) is positive for all $t \ge 0$, so that $T_{\max}^\kappa = \infty$ by the blowup alternative. This completes the proof of property (iv). □

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