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APPROXIMATE FIXED POINT OF REICH OPERATOR

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Abstract. In the present paper, we study the existence of approximate fixed point for Reich operator together with the property that the ε-fixed points are concentrated in a set with the diameter tends to zero if $\varepsilon \to 0$.

1. Introduction

Fixed point theory has been an important tool for solving various problems in nonlinear functional analysis and as well as useful for proving the existence theorems for nonlinear differential and integral equations. However, in many practical situations, the conditions in the fixed point theorems are too strong, and so the existence of a fixed point is not guaranteed. In that situation, one can content with nearly fixed points what we call approximate fixed points. By an approximate fixed point x of a function f, we mean that $f(x)$ is 'near to' x. The study of approximate fixed point theorems is as interesting as the study of fixed point theorems. Motivated by the article of S. Tijs, A. Torre and R. Brănzei [**9**], M. Berinde [**1**] established some fundamental approximate fixed point theorems in metric space. We investigate the approximate fixed point theorems for Reich operator [**7**] which in turn generalize the results of Berinde [**1**] showing that the existence of approximate fixed point for Reich operator is still guaranteed in spite of the completeness of the underlying space is withdrawn.

2. Some preliminary ideas and definitions

Definition 2.1. Let (X, d) be a metric spce and $f\colon X \to X$, $\varepsilon > 0$, $x_0 \in X$. Then x_0 is an ε-fixed point (approximate fixed point) of f if $d(f(x_0), x_0) < \varepsilon$.

Throughout the paper, we will denote the set of all ε-fixed points of f for a given ε, by

$$F_\varepsilon(f) = \{x \in X |\ d(f(x), x) < \varepsilon\}.$$

Definition 2.2. Let $f\colon X \to X$. Then f has the approximate fixed point property if for all $\varepsilon > 0$

$$F_\varepsilon(f) \neq \phi.$$

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Remark 2.3. The following result (see [**5**]) gives conditions under which the existence of fixed points for a given mapping is equivalent to that of approximate fixed points.

Proposition 2.4. *Let A be a closed subset of a metric space (X,d) and $f: A \to X$ a compact map. Then f has a fixed point if and only if it has the approximate fixed point property.*

Now an important lemma by Berinde [**1**] regarding the existence of ε-fixed point of an operator is being recalled here.

Lemma 2.5 ([**1**])**.** *Let (X,d) be a metric space, $f: X \to X$ such that f is asymptotic regular, i.e.,*

$$d(f^n(x), f^{n+1}(x)) \to 0 \quad as \quad n \to \infty \quad for\ all\ x \in X.$$

Then f has the approximate fixed point property i.e. $\forall \varepsilon > 0$, $F_\varepsilon(f) \neq \phi$.

We also assume that $\delta(A)$ as the diameter of a set $A \neq \phi$, in otherwords,

$$\delta(A) = \sup \{d(x,y) \mid x, y \in A\}.$$

Another important lemma [**1**] will be needed to establish our result. So we recall this.

Lemma 2.6 ([**1**])**.** *Let (X,d) be a metric space, $f: X \to X$ an operator and $\varepsilon > 0$. We assume that:*

(i) $F_\varepsilon(f) \neq \phi$.
(ii) for all $\eta > 0$, there exists $\psi(\eta) > 0$ such that

$$d(x,y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x,y) \leq \psi(\eta), \quad for\ all \quad x, y \in F_\varepsilon(f).$$

Then $\delta(F_\varepsilon(f)) \leq \psi(2\varepsilon)$.

Definition 2.7. A mapping $f: X \to X$ is a Reich operator [**7**] if there exist $a, b, c \geq 0$ with $0 \leq a + b + c < 1$ such that

$$d(f(x), f(y)) \leq a d(x,y) + b d(x, f(x)) + c d(y, f(y)) \quad for\ all\ x, y \in X.$$

3. Main Results

In this section, we present two important results—one is qualitative and the other one is quantitative concerning approximate fixed point theorems for Reich operator.

Theorem 3.1. *Let (X,d) be a metric space and $f: X \to X$ be a Reich operator. Then*

$$F_\varepsilon(f) \neq \phi \qquad for\ all\ \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$ and $x \in X$. Then

$$\begin{aligned} d(f^n(x), f^{n+1}(x)) &= d(f(f^{n-1}(x)), f(f^n(x))) \\ &\le ad(f^{n-1}(x), f^n(x)) + bd(f^{n-1}(x), f(f^{n-1}(x))) \\ &\quad + cd(f^n(x), f(f^n(x))) \\ &= ad(f^{n-1}(x), f^n(x)) + bd(f^{n-1}(x), f^n(x)) \\ &\quad + cd(f^n(x), f^{n+1}(x)) \end{aligned}$$

implies that

$$d(f^n(x), f^{n+1}(x)) \le \left(\frac{a+b}{1-c}\right) d(f^{n-1}(x), f^n(x)).$$

Put $h = \left(\frac{a+b}{1-c}\right)$, so that $0 < h < 1$ as $a + b + c < 1$, and so

$$d(f^n(x), f^{n+1}(x)) \le hd(f^{n-1}(x), f^n(x)) \le \ldots \le h^n d(x, f(x))$$

which implies that $d(f^n(x), f^{n+1}(x)) \to 0$ as $n \to \infty$, $\forall x \in X$.

Then by Lemma 2.5, it follows that $F_\varepsilon(f) \neq \phi$ for all $\varepsilon > 0$. □

Theorem 3.2. *Let (X, d) be a metric space and $f\colon X \to X$ be a Reich operator. Then*

$$\delta(F_\varepsilon(f)) \le \left(\frac{2+b+c}{1-a}\right) \varepsilon.$$

Proof. Let $\varepsilon > 0$. Then the condition (i) of Lemma 2.6 is satisfied immediately from Theorem 3.1. Now we shall show that the condition (ii) of Lemma 2.6 is also satisfied. In order to do that let, $\eta > 0$ and $x, y \in F_\varepsilon(f)$. We also assume that $d(x, y) - d(f(x), f(y)) \le \eta$. We will show that there $\psi(\eta) > 0$ exist. Now we have

$$\begin{aligned} d(x, y) &\le \eta + d(f(x), f(y)) \\ &\le \eta + ad(x, y) + bd(x, f(x)) + cd(y, f(y)) \\ &\le \eta + ad(x, y) + b\varepsilon + c\varepsilon \end{aligned}$$

as $x, y \in X$, $d(x, f(x)) < \varepsilon$ and $d(y, f(y)) < \varepsilon$, which implies that

$$d(x, y) \le \frac{\eta + (b+c)\varepsilon}{1-a}.$$

So for all $\eta > 0$, there exists $\psi(\eta) = \frac{\eta+(b+c)\varepsilon}{1-a} > 0$ such that $d(x, y) - d(f(x), f(y)) \le \eta$. It implies $d(x, y) \le \psi(\eta)$. So by Lemma 2.6, it follows that

$$\delta\left(F_\varepsilon(f)\right) \le \psi(2\varepsilon) \qquad \text{for all } \varepsilon > 0,$$

which means

$$\delta(F_\varepsilon(f)) \le \left(\frac{2+b+c}{1-a}\right) \varepsilon.$$

□

Corollary 3.3. *Suppose that the Reich operator in Theorem 3.1 posseses a fixed point x^*. Then*

(i) x^* *is the unique fixed point of* f*,*

(ii) for each sequence $x_1, x_2, x_3, \ldots$ *with the property that for each* $n \in N$*, the point* x_n *is an* n^{-1} *fixed point we have* $\lim_n x_n = x^*$*.*

Proof. (i) is very clear.
For assertion (ii): It follows from the fact that $x^* \in F_\varepsilon(f)$ for each $\varepsilon > 0$, then by Theorem 3.2,

$$d\left(x_n, x^*\right) \leq \delta\left(F_n^{-1}(f)\right) \leq \left(\frac{2+b+c}{1-a}\right) n^{-1}.$$

Hence $\lim_n d\left(x_n, x^*\right) = 0$. □

Remark 3.4. In Reich operator, taking $b = c = 0$, we get Banach contraction and the Theorem 3.1 reduces to [**1**, Theorem 2.1] of Berinde as a special case.

Remark 3.5. $b = c$ and $a = 0$ imply Kannan operator [**6**] and then the Theorem 3.1 reduces to [**1**, Theorem 2.2] of Berinde.

Remark 3.6. Theorem 3.1 is also a generalization of approximate fixed point theorem for Bianchini's [**3**] and Sehgal's [**8**] contraction mappings.

4. Example

Let $f : [1, \infty) \to [1, \infty)$ be defined by

$$f(x) = x + \frac{1}{x+1} \qquad \text{for all } \ x \in [1, \infty).$$

Then it is easy to check that f is a Reich operator. Take $0 < \varepsilon < \frac{1}{2}$ and select $x_0 \in [1, \infty)$ such that $x_0 > \frac{1-\varepsilon}{\varepsilon}$. Then

$$d(f(x_0), x_0) = |f(x_0) - x_0| = \left|\frac{1}{x_0+1}\right| < \varepsilon.$$

So f has an ε-fixed point which implies that $F_\varepsilon(f) \neq \phi$. On the contrary, there is no fixed point of f in $[1, \infty)$.

5. Conclusions

Weakening the condition by removing the completeness of underlying space still guarantees the existence of ε-fixed points for such operators and the fact is that the diameter of the set containing ε-fixed points goes to zero when ε tends to zero. In the light of this note we hope that there is a potential area in which approximate fixed point theorems for various operators can be studied further. Also the study of approximate fixed point for multivalued mappings and well-posed fixed point problems could be interesting.

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