Acta Math. Univ. Comenianae
Vol. LXXXII, 1 (2013), pp. 11–20

ON BIEMBEDDING AN IDEMPOTENT LATIN SQUARE WITH ITS TRANSPOSE

D. M. DONOVAN, T. S. GRIGGS AND C. PSOMAS

Abstract. Let L be an idempotent Latin square of side n, thought of as a set of ordered triples (i, j, k) where $L(i, j) = k$. Let I be the set of triples (i, i, i). We consider the problem of biembedding the triples of $L \setminus I$ with the triples of $L' \setminus I$, where L' is the transpose of L, in an orientable surface. We construct such embeddings for all doubly even values of n.

1. Introduction

A triangular embedding of a complete regular tripartite graph $K_{n,n,n}$ in a surface is face two-colourable if and only if the surface is orientable [**4**]. In this case, the faces of each colour class can be regarded as the triples of a *transversal design* $\mathrm{TD}(3, n)$, of order n and block size 3. Such a design comprises a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a $3n$-element set (the points), $\mathcal{G}$ is a partition of V into three parts (the groups) each of cardinality n, and $\mathcal{B}$ is a collection of 3-element subsets (the blocks) of V such that each 2-element subset of V is either contained in exactly one block of $\mathcal{B}$, or in exactly one group of $\mathcal{G}$, but not both. Two $\mathrm{TD}(3, n)$s, $(V, \{G_1, G_2, G_3\}, \mathcal{B})$ and $(V', \{G'_1, G'_2, G'_3\}, \mathcal{B}')$ are said to be *isomorphic* if, for some permutation π of $\{1, 2, 3\}$, there exist bijections $\alpha_i : G_i \to G'_{\pi(i)}$, $i = 1, 2, 3$, that map blocks of $\mathcal{B}$ to blocks of $\mathcal{B}'$. A Latin square of side n determines a $\mathrm{TD}(3, n)$ by assigning the row labels, the column labels, and the entries as the three groups of the design. Two Latin squares are said to be in the same *main class* if the corresponding transversal designs are isomorphic. A question that naturally arises is: which pairs of (main classes of) Latin squares may be biembedded?

This question seems to be difficult. On the existence side, recursive constructions are given in [**1, 5, 7**]. Of particular interest are biembeddings of Latin squares which are the Cayley tables of groups and other algebraic structures. An infinite class of biembeddings of Latin squares representing the Cayley tables of cyclic groups of order n is known for all $n \geq 2$. This is the family of regular biembeddings constructed using a voltage graph based on a dipole with n parallel edges embedded in a sphere [**10**], or alternatively directly from the Latin squares defined by $C_n(i, j) = i+j \pmod n$, and $C'_n(i, j) = i+j-1 \pmod n$ [**4**]. A regular

Received July 2, 2011; revised September 28, 2012.
2010 *Mathematics Subject Classification.* Primary 05B15, 05C10.
Key words and phrases. Latin square; idempotent; SOLS; orientable surface; biembedding.

biembedding of a Latin square of side n has the greatest possible symmetry, with full automorphism group of order $12n^2$, the maximum possible value. Recently, two other families of biembeddings of the Latin squares representing the Cayley tables of cyclic groups, also with a high degree of symmetry, have been constructed [**1, 2**]. Enumeration results for biembeddings of Latin squares of side 3 to 7 are given in [**4**] and for groups of order 8 in [**6**]. In [**8**], it was shown that with the single exception of the group C_2^2, the Cayley table of each Abelian group appears in some biembedding.

In this paper, we consider a slightly different but related aspect of biembeddings of Latin squares. Let L be a Latin square of side n, which we will think of as a set of ordered triples (i, j, k) where entry k occurs in row i, column j of L, $k = L(i, j)$. Let L' be the *transpose* of L, i.e. $(i, j, k) \in L'$ if and only if $(j, i, k) \in L$. Clearly no biembedding of L with L' exists because triples (i, i, k) occur in both squares. However, suppose that L is *idempotent*, i.e. $(i, i, i) \in L$ for all i. Denote the set of idempotent triples by I. Then it may be possible to biembed the triples $L \setminus I$ with the triples $L' \setminus I$ and it is this question which is the focus of what follows.

So, given an idempotent Latin square L of side n, we denote the set of row labels by $R = \{0_r, 1_r, \ldots, (n-1)_r\}$, the set of column labels by $C = \{0_c, 1_c, \ldots, (n-1)_c\}$, the set of entries by $E = \{0_e, 1_e, \ldots, (n-1)_e\}$, and the set of idempotent triples by $I = \{\{i_r, i_c, i_e\} : i = 0, 1, \ldots, n-1\}$. Now consider the sets of triples $L \setminus I$ (the black triples) and $L' \setminus I$ (the white triples) and glue them together along common sides, $\{i_r, j_c\}, i \neq j, \{j_c, k_e\}, j \neq k, \{k_e, i_r\}, k \neq i$. The resulting topological space is not necessarily a surface but is certainly a pseudosurface which we will call the *transpose pseudosurface* of $L \setminus I$ and denote by $S(L \setminus I)$. Within this framework, the main interest is when $S(L \setminus I)$ is a surface, in which case we say that the idempotent Latin square L biembeds with its transpose and write $(L \setminus I) \bowtie (L' \setminus I)$.

From a graph theoretic viewpoint, a biembedding of an idempotent Latin square with its transpose, as described above, gives a face two-colourable triangular embedding of a complete regular tripartite graph $K_{n,n,n}$ with the removal of a triangle factor. For the same reason as applies without the removal of a triangular factor, the surface is orientable. In such a biembedding, the number of vertices, $V = 3n$, the number of edges, $E = 3(n^2 - n)$, and the number of faces, $F = 2(n^2 - n)$. Therefore, using Euler's formula, $V + F - E = 4n - n^2$ which is even if and only if n is even. In the next section, we construct biembeddings of idempotent Latin squares with their transpose for all doubly even values of n. We leave the problem for singly even values for future investigation. In section 3, we consider the situation when the transpose L' is mutually orthogonal to L, i.e. the Latin square L is a self-orthogonal Latin square (SOLS). Biembeddings of a self-orthogonal Latin square L with its transpose are constructed for all $n = 2^m, m \geq 2$.

We represent the biembeddings by means of *rotation schemes*. The *rotation* about a point i_r is defined to be the set of cycles
$(j_c^1 k_e^1\, j_c^2 k_e^2 \ldots j_c^{a_1-1} k_e^{a_1-1})(j_c^{a_1} k_e^{a_1} \ldots j_c^{a_2-1} k_e^{a_2-1}) \ldots (j_c^{a_{m-1}} k_e^{a_{m-1}} \ldots j_c^{a_m-1} k_e^{a_m-1})$
where $k^s = L(i, j^s) = L'(i, j^{s+1})$, $s \in \{1, 2, \ldots, n-1\} \setminus \{a_1 - 1, a_2 - 1, \ldots, a_m - 1\}$ and $k^{a_t-1} = L(i, j^{a_t-1}) = L'(i, j^{a_{t-1}})$, $1 \leq t \leq m$, $1 \leq m \leq n-1$ with $a_0 = 1$

and $a_m = n$. The cycles are the order of vertices adjacent to i_r as determined by the biembedding. The rotation about a point j_c or k_e is defined analogously. The two Latin squares L and L' are biembedded in a surface if and only if the rotation about each point is a single cycle. The biembedding is in an orientable surface because the rotations about points j_c and k_e can be ordered so that whenever the cycle about a vertex i_r contains the sequence $\dots j_c k_e \dots$ then the cycle about j_c contains the sequence $\dots k_e i_r \dots$ and the cycle about k_e contains the sequence $\dots i_r j_c \dots$.

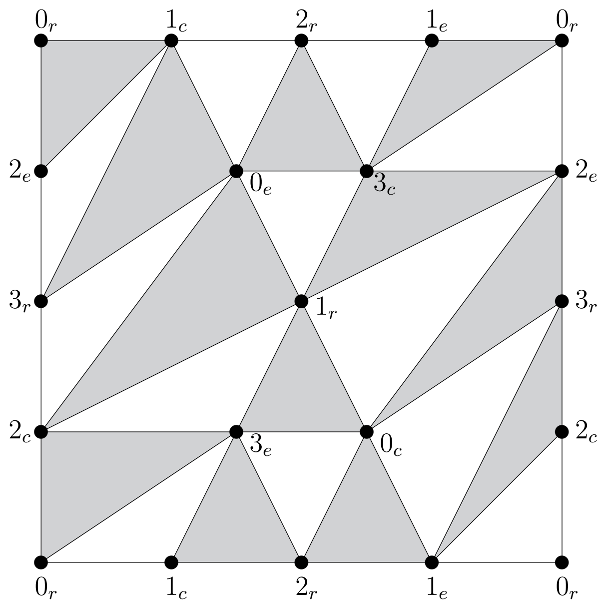
To conclude the introduction, below is an example which illustrates some of the ideas presented in this section.

Example 1.1. There are two idempotent Latin squares of side 4 each of which is the transpose of the other.

	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

	0	1	2	3
0	0	3	1	2
1	2	1	3	0
2	3	0	2	1
3	1	2	0	3

These biembed in the torus as shown.



The rotation scheme is

$$\begin{array}{lll}
0_r : (1_c 2_e 3_c 1_e 2_c 3_e) & 0_c : (1_e 2_r 3_e 1_r 2_e 3_r) & 0_e : (1_r 2_c 3_r 1_c 2_r 3_c) \\
1_r : (2_c 0_e 3_c 2_e 0_c 3_e) & 1_c : (2_e 0_r 3_e 2_r 0_e 3_r) & 1_e : (2_r 0_c 3_r 2_c 0_r 3_c) \\
2_r : (3_c 0_e 1_c 3_e 0_c 1_e) & 2_c : (3_e 0_r 1_e 3_r 0_e 1_r) & 2_e : (3_r 0_c 1_r 3_c 0_r 1_c) \\
3_r : (0_c 2_e 1_c 0_e 2_c 1_e) & 3_c : (0_e 2_r 1_e 0_r 2_e 1_r) & 3_e : (0_r 2_c 1_r 0_c 2_r 1_c)
\end{array}$$

2. Idempotent Latin squares of doubly even order

In order to construct a Latin square of doubly even order which biembeds with its transpose, we use the concept of a near-Hamiltonian factorization of a complete directed graph together with known triangulations of complete (undirected) graphs in orientable surfaces. Although the main results are when the side of the Latin square $n = 4m$, $m \geq 1$, some of the theory is more general and so to begin, we do not place this restriction on n. Let K_n (resp. K_n^*) be the complete undirected (resp. directed) graph on a set of n vertices, $\{0, 1, \dots, n-1\}$. A *near-Hamiltonian circuit* of K_n^* is an ordered $(n-1)$-cycle $(x_1, x_2, \dots, x_{n-1})$ where $x_i \neq x_j$ if $i \neq j$. A *near-Hamiltonian factorization* F of K_n^* is a partition of the directed edges of K_n^* into near-Hamiltonian circuits. A straightforward counting argument shows that F contains n near-Hamiltonian circuits and that each vertex i, $0 \leq i \leq n-1$, is absent from precisely one circuit.

Given a near-Hamiltonian factorization F of K_n^*, an idempotent Latin square L_F of side n can be constructed as follows,

1. $L_F(i,i) = i$, $0 \leq i \leq n-1$,
2. $L_F(i,j) = k$, $0 \leq i \leq n-1$, $0 \leq j \leq n-1$, $i \neq j$, where the directed edge (i,j) occurs in the $(n-1)$-cycle which does not contain k.

We then have the following result.

Lemma 2.1. *Let F be a near-Hamiltonian factorization of the complete directed graph K_n^*, and let L_F be the Latin square constructed from F as above. Let $S(L_F)$ be the transpose pseudosurface of L_F. Then the rotation about every entry point k_e, $0 \leq k \leq n-1$, is a single cycle of length $2n-2$ if n is even and two cycles each of length $n-1$ if n is odd.*

Proof. Consider the near-Hamiltonian circuit not containing k. Suppose that it is $(x_1, x_2, \dots, x_{n-1})$. Then entry k occurs in the following (row, column) pairs of L_F: $(x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_1), (k, k)$ and in the following (row, column) pairs of L'_F: $(x_2, x_1), (x_3, x_2), \dots, (x_1, x_{n-1}), (k, k)$. The rotation scheme about k_e is then

$((x_1)_r(x_2)_c(x_3)_r(x_4)_c \dots (x_{n-2})_c(x_{n-1})_r(x_1)_c(x_2)_r \dots (x_{n-2})_r(x_{n-1})_c)$

if n is even, and

$((x_1)_r(x_2)_c(x_3)_r(x_4)_c \dots (x_{n-2})_r(x_{n-1})_c)((x_1)_c(x_2)_r(x_3)_c(x_4)_r \dots (x_{n-2})_c(x_{n-1})_r)$

if n is odd. □

A source of near-Hamiltonian factorizations of complete directed graphs K_n^* is provided by triangulations of the complete graph K_n in an orientable surface. It is well-known that these exist precisely when $n \equiv 0, 3, 4, 7 \pmod{12}$ [**9**]. Given such a triangulation on vertex set $\{0, 1, \dots, n-1\}$, first choose an arbitrary but fixed orientation. A near-Hamiltonian circuit avoiding a point is obtained by the rotation about that point in the selected orientation, and the set of all such near-Hamiltonian circuits forms a near-Hamiltonian factorization. Using this construction, we then have the following result.

Lemma 2.2. *Let $n \equiv 0, 3, 4, 7 \pmod{12}$, and T be a triangulation of the complete graph K_n in an orientable surface. Let $F(T)$ be the near-Hamiltonian factorization of K_n^* constructed as above. Let $L_{F(T)}$ be the Latin square constructed from $F(T)$ and $S(L_{F(T)})$ the transpose pseudosurface of $L_{F(T)}$. Then the rotation about every row point i_r, $0 \le i \le n-1$, and every column point j_c, $0 \le j \le n-1$, is a single cycle of length $2n-2$ if n is even, and two cycles each of length $n-1$ if n is odd.*

Proof. The Latin square L constructed from the triangulation T has the property that if $L(i,j) = k$ then $L(j,k) = i$ and $L(k,i) = j$. It follows that the rotation about a row point i_r (resp. column point j_c) can be obtained from the rotation about i_e (resp. j_e) by applying the permutations $(e\ r\ c)$ (resp. $(e\ c\ r)$) to the suffixes. □

The following theorem is now an immediate consequence of Lemmas 2.1 and 2.2.

Theorem 2.1. Let $n \equiv 0, 4 \pmod{12}$. Then there exists an idempotent Latin square L of side n which biembeds with its transpose, i.e. $(L \setminus I) \bowtie (L' \setminus I)$.

In the cases where $n \equiv 3, 7 \pmod{12}$, the transpose pseudosurface $S(L_{F(T)})$ constructed as in Lemma 2.2, although not a surface, does exhibit some regularity in that every point is a pinch point and the rotation about each point consists of two cycles each of length $n-1$. Moreover, if we separate each pinch point, the pseudosurface fractures into two orientable surfaces having isomorphic triangulations. Let $C_{i,\alpha,\beta}$, $0 \le i \le n-1$, $\alpha, \beta \in \{r, c, e\}, \alpha \ne \beta$, represent the cycle $((x_{i,1})_\alpha (x_{i,2})_\beta \ldots (x_{i,n-2})_\alpha (x_{i,n-1})_\beta)$. Then the rotation about a point $(x_i)_e$ is $C_{i,r,c}C_{i,c,r}$, about a point $(x_i)_r$ is $C_{i,c,e}C_{i,e,c}$, and about a point $(x_i)_c$ is $C_{i,e,r}C_{i,r,e}$. Now separate each entry point $(x_i)_e$ into two points $(x_i)_e^0$ and $(x_i)_e^1$ so that the rotation about $(x_i)_e^0$ is $C_{i,r,c}$ and about $(x_i)_e^1$ is $C_{i,c,r}$. The row and column points may then also be separated and labelled appropriately so that the rotation about $(x_i)_e^0$ is $C^0_{i,r,c}$, $(x_i)_r^0$ is $C^0_{i,c,e}$, and $(x_i)_c^0$ is $C^0_{i,e,r}$ and about $(x_i)_e^1$ is $C^1_{i,c,r}$, $(x_i)_r^1$ is $C^1_{i,e,c}$, and $(x_i)_c^1$ is $C^1_{i,r,e}$.

It remains to deal with the case $n \equiv 8 \pmod{12}$. We use a triangulation of the complete graph K_{n-1} in an orientable surface to first construct a near-Hamiltonian factorization F of K_{n-1}^* and then augment this to obtain a near-Hamiltonian factorization $\bar{F}$ of K_n^*.

Construction of $\bar{F}$

Let $n \equiv 8 \pmod{12}$. Then there exists a triangulation T of the complete graph K_{n-1} in an orientable surface, having a cyclic automorphism [**9, 11, 12**]. Without loss of generality assume that the vertex set of K_{n-1} is $N = \{0, 1, 2, \ldots, n-2\}$ and the cyclic automorphism is generated by the mapping $i \mapsto i+1 \pmod{n-1}$. Let $F(T) = \{C_0, C_1, \ldots, C_{n-2}\}$ be the near-Hamiltonian factorization of K_{n-1}^* constructed from T as above, where $C_i = ((x_1+i)\ (x_2+i)\ \ldots\ (x_{n-2}+i))$, $0 \le i \le n-2$, is the near-Hamiltonian circuit which avoids the vertex i. Now choose l, $1 \le l \le n-2$, relatively prime to $n-1$, (in fact we can choose $l = 1$). Then, because T has a cyclic automorphism, there exists j, $1 \le j \le n-2$, such that $x_{j+1}-x_j = l$, where if $j = n-2$, $x_{j+1} = x_1$. Introduce a new vertex ∞ and let $\bar{C}_i = ((x_1+i)\ (x_2+i)\ \ldots\ (x_j+i)\ \infty\ (x_{j+1}+i)\ \ldots\ (x_{n-2}+i))$. Further let $\bar{C}_\infty = (0\ l\ 2l\ \ldots\ (n-2)l)$, arithmetic modulo $n-1$. Let $\bar{F}(T) = \{\bar{C}_0, \bar{C}_1, \ldots, \bar{C}_{n-2}, \bar{C}_\infty\}$. Then $\bar{F}(T)$ is a near-Hamiltonian factorization of K_n^* on vertex set $N \cup \{\infty\}$. We can now prove the following theorem.

Theorem 2.2. Let $n \equiv 8 \pmod{12}$. Then there exists an idempotent Latin square L of side n which biembeds with its transpose, i.e. $(L \setminus I) \bowtie (L' \setminus I)$.

Proof. Let $\bar{F}(T)$ be a near-Hamiltonian factorization of the complete directed graph K_n^* obtained by the triangulation T of the complete graph K_{n-1} having a cyclic automorphism, as above. Let $L_{\bar{F}(T)}$ be the Latin square constructed from $\bar{F}(T)$ and $S(L_{\bar{F}(T)})$ the transpose pseudosurface. We need to prove that the rotations about row points, column points and entry points are all single cycles.
Entry points. This follows immediately from Lemma 2.1.
Row points. Let $x_p = l$ and $x_q = n-1-l$. The rotation about the point 0_r is

$$(\infty_c (x_{p+1})_e \ldots (x_{n-2})_e (x_1)_c \ldots (x_p)_c \infty_e (x_q)_c \ldots (x_{n-2})_c (x_1)_e \ldots (x_{q-1})_e).$$

The rotation about the point i_r, $i \neq 0$, is obtained by adding i, modulo $n-1$. The rotation about the point ∞_r is

$$(0_c (x_{q-1})_e (n-1-l)_c (x_{q-1}-l)_e (n-1-2l)_c (x_{q-1}-2l)_e \ldots l_c (x_{q-1}+l)_e).$$

Column points. With the same definition of p and q as for the row points, the rotation about the point 0_c is

$$(\infty_e (x_q)_r \ldots (x_{n-2})_r (x_1)_e \ldots (x_{q-1})_e \infty_r (x_{p+1})_e \ldots (x_{n-2})_e (x_1)_r \ldots (x_p)_r).$$

Again the rotation about the point i_c, $i \neq 0$, is obtained by adding i, modulo $n-1$. The rotation about the point ∞_c is

$$((x_{p+1})_e 0_r (x_{p+1}-l)_e (n-1-l)_r (x_{p+1}-2l)_e (n-1-2l)_r \ldots (x_{p+1}+l)_e l_r).$$ □

Example 2.1. Consider the triangulation of the complete graph K_7 on vertex set $\{0,1,2,3,4,5,6\}$ in a torus, where the triangles are given by the sets $\{i, 1+i, 3+i\}$ and $\{i, 3+i, 2+i\}$, $0 \leq i \leq 6$. The rotation C_i about a point i is $(1+i\ 3+i\ 2+i\ 6+i\ 4+i\ 5+i)$. Choose $l = 2$. Then the rotation $\bar{C}_i$ is $(1+i\ \infty\ 3+i\ 2+i\ 6+i\ 4+i\ 5+i)$ and $\bar{C}_\infty = (0\ 2\ 4\ 6\ 1\ 3\ 5)$. The Latin square of order 8 formed from this near-Hamiltonian factorization is

	0	1	2	3	4	5	6	∞
0	0	3	∞	2	5	1	4	6
1	5	1	4	∞	3	6	2	0
2	3	6	2	5	∞	4	0	1
3	1	4	0	3	6	∞	5	2
4	6	2	5	1	4	0	∞	3
5	∞	0	3	6	2	5	1	4
6	2	∞	1	4	0	3	6	5
∞	4	5	6	0	1	2	3	∞

The rotations about the various points are as follows.

$i_e : ((1+i)_r \infty_c (3+i)_r (2+i)_c (6+i)_r (4+i)_c (5+i)_r (1+i)_c \infty_r (3+i)_c (2+i)_r (6+i)_c (4+i)_r (5+i)_c)$

$\infty_e : (0_r 2_c 4_r 6_c 1_r 3_c 5_r 0_c 2_r 4_c 6_r 1_c 3_r 5_c)$

$i_r : ((1+i)_c(3+i)_e(2+i)_c\infty_e(5+i)_c(1+i)_e(3+i)_c(2+i)_e(6+i)_c(4+i)_e\infty_c(6+i)_e(4+i)_c(5+i)_e)$

$\infty_r : (0_c4_e5_c2_e3_c0_e1_c5_e6_c3_e4_c1_e2_c6_e)$

$i_c : ((1+i)_e(3+i)_r(2+i)_e(6+i)_r(4+i)_e\infty_r(6+i)_e(4+i)_r(5+i)_e(1+i)_r(3+i)_e(2+i)_r\infty_e(5+i)_r)$

$\infty_c : (0_e1_r5_e6_r3_e4_r1_e2_r6_e0_r4_e5_r2_e3_r)$

3. Self-orthogonal Latin squares

In this section, we present a finite field construction to biembed a self-orthogonal Latin square (SOLS) with its transpose in an orientable surface. First recall the definitions. Let L, M be Latin squares of side n. Then L and M are said to be orthogonal if $L(i_1, j_1) = L(i_2, j_2)$ and $M(i_1, j_1) = M(i_2, j_2)$ implies $i_1 = i_2$ and $j_1 = j_2$. If $M = L'$, the transpose of L, then L is said to be a self-orthogonal Latin square. In a SOLS, the main diagonal is a transversal and without loss of generality, by renaming the entries, it can be assumed that L is idempotent.

The construction is not new, see for example Construction 5.44 in [**3**], and applies for any finite field except those of order 2 or 3. We present it in this more general form but by the calculation using Euler's formula given in the introduction, a biembedding can exist only for even values.

Let $\omega \notin \{0, -1, 1\}$ be a generator of the cyclic multiplicative group of $\mathrm{GF}(p^m)$. Define $L(i,j) = (i + \omega j)/(1 + \omega)$. Then it is easily verified that L is a self-orthogonal Latin square with the rows, columns, and entries indexed by the elements of the Galois field, which in what follows it will be convenient to represent by $0, 1, \omega, \ldots, \omega^{n-2}$. Further let $\zeta^{(k)} = \omega^k/(1+\omega)$, $0 \le k \le n-2$.

We now restrict our attention to Galois fields $\mathrm{GF}(2^m)$, $m \ge 2$. By considering the rotations about each of the row, column and entry points we show that $(L \backslash I) \bowtie (L' \setminus I)$.

(1) Row 0 of L and column 0 of L' are as follows.

	0	1	ω	ω^2	$\ldots$	ω^{n-3}	ω^{n-2}
0	0	$\zeta^{(1)}$	$\zeta^{(2)}$	$\zeta^{(3)}$	$\ldots$	$\zeta^{(n-2)}$	$\zeta^{(0)}$

Row 0 of L' and column 0 of L are as follows.

	0	1	ω	ω^2	$\ldots$	ω^{n-3}	ω^{n-2}
0	0	$\zeta^{(0)}$	$\zeta^{(1)}$	$\zeta^{(2)}$	$\ldots$	$\zeta^{(n-3)}$	$\zeta^{(n-2)}$

Clearly the rotation about the points 0_r and 0_c are single cycles.

(2) Row ω^k of L and column ω^k of L', $0 \le k \le n-2$, are as follows.

	0	$\omega^0 = 1$	ω	ω^2
ω^k	$\zeta^{(k)}$	$\zeta^{(k)} + \zeta^{(1)}$	$\zeta^{(k)} + \zeta^{(2)}$	$\zeta^{(k)} + \zeta^{(3)}$

	$\ldots$	ω^{k-1}	ω^k	ω^{k+1}
ω^k	$\ldots$	$\zeta^{(k)} + \zeta^{(k)} = 0$	$\zeta^{(k)} + \zeta^{(k+1)} = \omega^k$	$\zeta^{(k)} + \zeta^{(k+2)}$

	$\ldots$	ω^{n-3}	ω^{n-2}
ω^k	$\ldots$	$\zeta^{(k)} + \zeta^{(n-2)}$	$\zeta^{(k)} + \zeta^{(0)}$

Row ω^k of L' and column ω^k of L, $0 \le k \le n-2$, are as follows.

$$\begin{array}{c|cccc} & 0 & \omega^0 = 1 & \omega & \omega^2 \\ \hline \omega^k & \zeta^{(k+1)} & \zeta^{(k+1)} + \zeta^{(0)} & \zeta^{(k+1)} + \zeta^{(1)} & \zeta^{(k+1)} + \zeta^{(2)} \end{array}$$

$$\begin{array}{c|cccc} & \dots & \omega^{k-1} & \omega^k & \omega^{k+1} \\ \hline \omega^k & \dots & \zeta^{(k+1)} + \zeta^{(k-1)} & \zeta^{(k+1)} + \zeta^{(k)} = \omega^k & \zeta^{(k+1)} + \zeta^{(k+1)} = 0 \end{array}$$

$$\begin{array}{c|ccc} & \dots & \omega^{n-3} & \omega^{n-2} \\ \hline \omega^k & \dots & \zeta^{(k+1)} + \zeta^{(n-3)} & \zeta^{(k+1)} + \zeta^{(n-2)} \end{array}$$

For each k, $0 \le k \le n-2$, define $q_0 = q_0(k)$ by the equation $\zeta^{(k)} = \zeta^{(k+1)} + \zeta^{(q_0)}$, i.e. $\omega^{q_0} = \omega^k(1-\omega)$. Further, for $1 \le i \le n-2$, define $q_i = q_i(k)$ by the equations $\zeta^{(q_i)} = \zeta^{(q_0)}(1 + \omega + \dots + \omega^i)$, i.e. $\omega^{q_i} = \omega^k(1-\omega^{i+1})$. Note that for $0 \le i \le n-2$, the values ω^{q_i} are distinct, as are the values $\zeta^{(q_i)}$. Moreover $\omega^{q_{n-2}} = 0$. The rotation about a row point ω^k_r is a single cycle as follows.
$(0_c\,(\zeta^{(k+1)} + \zeta^{(q_0)})_e\,\omega_c^{q_0}\,(\zeta^{(k+1)} + \zeta^{(q_1)})_e\,\omega_c^{q_1}\,(\zeta^{(k+1)} + \zeta^{(q_2)})_e\,\dots\,\omega_c^{q_{n-3}}\,(\zeta^{(k+1)} + \zeta^{(q_{n-2})})_e)$
The rotation about a column point ω^k_c is similar and is also a single cycle.

(3) In L, entry 0 occurs in cells $(i, -\frac{i}{\omega})$, and in L' in cells $(-\frac{i}{\omega}, i)$, $1 = \omega^0 \le i \le \omega^{n-2}$. The rotation about the point 0_e is therefore,
$(1_r(-\frac{1}{\omega})_c(\frac{1}{\omega^2})_r(-\frac{1}{\omega^3})_c \dots (\frac{1}{\omega^{n-2}})_r(-\frac{1}{\omega^{n-1}})_c(\frac{1}{\omega})_r(-\frac{1}{\omega^2})_c \dots (\frac{1}{\omega^{n-3}})_r(-\frac{1}{\omega^{n-2}})_c)$
i.e. $(1_r(\frac{1}{\omega})_c(\frac{1}{\omega^2})_r(\frac{1}{\omega^3})_c \dots (\frac{1}{\omega^{n-2}})_r 1_c(\frac{1}{\omega})_r(\frac{1}{\omega^2})_c \dots (\frac{1}{\omega^{n-3}})_r(\frac{1}{\omega^{n-2}})_c)$
which is a single cycle.

(4) In L, entry ω^k, $0 \le k \le n-2$, occurs in cells $(0, \omega^k + \omega^{k-1})$ and $(\omega^i, \omega^k + \omega^{k-1} - \omega^{i-1})$, $0 \le i \le n-2$. Similarly in L', entry ω^k occurs in cells $(0, \omega^k + \omega^{k+1})$ and $(\omega^k + \omega^{k-1} - \omega^{i-1}, \omega^i)$, $0 \le i \le n-2$.

The rotation about the point ω^k_e, where k is even is
$(0_r\ (\omega^k + \omega^{k-1})_c\ (\omega^k - \omega^{k-2})_r\ (\omega^k + \omega^{k-3})_c\ (\omega^k - \omega^{k-4})_r\ (\omega^k + \omega^{k-5})_c\ \dots$
$(\omega^k - \omega^2)_r\ (\omega^k + \omega)_c\ (\omega^k - 1)_r\ (\omega^k + \omega^{n-2})_c\ (\omega^k - \omega^{n-3})_r\ \dots$
$(\omega^k - \omega^{k+1} = \omega^k + \omega^{k+1})_r\ 0_c\ (\omega^k + \omega^{k-1} = \omega^k - \omega^{k-1})_r(\omega^k + \omega^{k-2})_c\ \dots$
$(\omega^k - \omega)_r\ (\omega^k + 1)_c\ (\omega^k - \omega^{n-2})_r\ (\omega^k + \omega^{n-3})_c\ \dots\ (\omega^k + \omega^{k+1})_c)$
and where k is odd is
$(0_r\ (\omega^k + \omega^{k-1})_c\ (\omega^k - \omega^{k-2})_r\ (\omega^k + \omega^{k-3})_c\ (\omega^k - \omega^{k-4})_r\ (\omega^k + \omega^{k-5})_c\ \dots$
$(\omega^k - \omega)_r\ (\omega^k + 1)_c\ (\omega^k - \omega^{n-2})_r\ (\omega^k + \omega^{n-3})_c\ (\omega^k - \omega^{n-4})_r\ \dots$
$(\omega^k - \omega^{k+1} = \omega^k + \omega^{k+1})_r\ 0_c\ (\omega^k + \omega^{k-1} = \omega^k - \omega^{k-1})_r(\omega^k + \omega^{k+1})_c\ \dots$
$(\omega^k - \omega^2)_r\ (\omega^k + \omega)_c\ (\omega^k - 1)_r\ (\omega^k + \omega^{n-2})_c\ \dots\ (\omega^k + \omega^{k-1})_c)$
in either case, a single cycle.

It is worth remarking that for a Galois field $\mathrm{GF}(p^m)$ where p is an odd prime and $m \geq 1$, except for $(p, m) = (3, 1)$, the rotation about all row points and all column points is also a single cycle. The proof is precisely as given above for $\mathrm{GF}(2^m)$, $m \geq 2$, except for the observations that $\zeta^{(k)} + \zeta^{(k)} = \zeta^{(k+1)} + \zeta^{(k+1)} = 0$ which in fact play no part in the proof. However the proof that the rotation about all entry points is a single cycle, does rely on the field having characteristic 2. Otherwise, we find that the rotation about all entry points is two cycles each of length $p^m - 1$. Thus in these cases, although the transpose pseudosurface $S(L \setminus I)$ is not a surface, it does exhibit some regularity.

Example pr1.1 provides an example of a self-orthogonal Latin square which can be biembedded with its transpose. A further example is given below.

Example 3.1. Let $F = \mathrm{GF}(2^3)$ with irreducible polynomial $x^3 = x+1$. Choose $\omega = x$. Then $(x, x^2, x^3, x^4, x^5, x^6, x^7) = (x, x^2, x+1, x^2+x, x^2+x+1, x^2+1, 1)$. The Latin square L, obtained from the construction described in this section is

	0	1	x	x^2	$x+1$	x^2+x	x^2+x+1	x^2+1
0	0	x^2+x+1	x^2+1	1	x	x^2	$x+1$	x^2+x
1	x^2+x	1	$x+1$	x^2+x+1	x^2	x	x^2+1	0
x	x^2+x+1	0	x	x^2+x	x^2+1	$x+1$	x^2	1
x^2	x^2+1	x	0	x^2	x^2+x+1	1	x^2+x	$x+1$
$x+1$	1	x^2+x	x^2	0	$x+1$	x^2+1	x	x^2+x+1
x^2+x	x	x^2+1	x^2+x+1	$x+1$	0	x^2+x	1	x^2
x^2+x+1	x^2	$x+1$	1	x^2+1	x^2+x	0	x^2+x+1	x
x^2+1	$x+1$	x^2	x^2+x	x	1	x^2+x+1	0	x^2+1

The rotation scheme is

$$0_r : (1_c(x^2+x)_e(x^2+1)_c(x+1)_e(x^2+x+1)_c x_e^2(x^2+x)_c x_e(x+1)_c 1_e x_c^2(x^2+1)_e x_c(x^2+x+1)_e)$$
$$1_r : (0_c(x^2+x+1)_e x_c^2 x_e(x^2+x)_c(x^2+1)_e(x^2+x+1)_c(x+1)_e x_c 0_e(x^2+1)_c x_e^2(x+1)_c(x^2+x)_e)$$
$$x_r : ((x+1)_c x_e^2(x^2+x+1)_c 1_e(x^2+1)_c(x^2+x)_e x_c^2 0_e 1_c(x+1)_e(x^2+x)_c(x^2+x+1)_e 0_c(x^2+1)_e)$$
$$x_r^2 : ((x^2+1)_c x_e 1_c(x^2+x+1)_e(x+1)_c 0_e x_c(x^2+x)_e(x^2+x+1)_c(x^2+1)_e 0_c 1_e(x^2+x)_c(x+1)_e)$$
$$(x+1)_r : (x_c(x^2+1)_e(x^2+x)_c 0_e x_c^2(x^2+x+1)_e(x^2+1)_c 1_e 0_c x_e(x^2+x+1)_c(x^2+x)_e 1_c x_e^2)$$
$$(x^2+x)_r : ((x^2+x+1)_c 0_e(x+1)_c(x^2+1)_e 1_c x_e 0_c x_e^2(x^2+1)_c(x^2+x+1)_e x_c(x+1)_e x_c^2 1_e)$$
$$(x^2+x+1)_r : ((x^2+x)_c 1_e x_c x_e^2 0_c(x+1)_e 1_c(x^2+1)_e x_c^2(x^2+x)_e(x+1)_c x_e(x^2+1)_c 0_e)$$
$$(x^2+1)_r : (x_c^2(x+1)_e 0_c(x^2+x)_e x_c 1_e(x+1)_c(x^2+x+1)_e(x^2+x)_c x_e^2 1_c 0_e(x^2+x+1)_c x_e)$$

$$0_c : (1_r(x^2+x+1)_e x_r(x^2+1)_e x_r^2 1_e(x+1)_r x_e(x^2+x)_r x_e^2(x^2+x+1)_r(x+1)_e(x^2+1)_r(x^2+x)_e)$$
$$1_c : (0_r(x^2+x)_e(x+1)_r x_e^2(x^2+1)_r 0_e x_r(x+1)_e(x^2+x+1)_r(x^2+1)_e(x^2+x)_r x_e x_r^2(x^2+x+1)_e)$$
$$x_c : ((x+1)_r(x^2+1)_e 0_r(x^2+x+1)_e(x^2+x)_r(x+1)_e 1_r 0_e x_r^2(x^2+x)_e(x^2+1)_r 1_e(x^2+x+1)_r x_e^2)$$
$$x_c^2 : ((x^2+1)_r(x+1)_e(x^2+x)_r 1_e 0_r(x^2+1)_e(x^2+x+1)_r(x^2+x)_e x_r 0_e(x+1)_r(x^2+x+1)_e 1_r x_e)$$
$$(x+1)_c : (x_r x_e^2 1_r(x^2+x)_e(x^2+x+1)_r x_e 0_r 1_e(x^2+1)_r(x^2+x+1)_e x_r^2 0_e(x^2+x)_r(x^2+1)_e)$$
$$(x^2+x)_c : ((x^2+x+1)_r 1_e x_r^2(x+1)_e x_r(x^2+x+1)_e(x^2+1)_r x_e^2 0_r x_e 1_r(x^2+1)_e(x+1)_r 0_e)$$
$$(x^2+x+1)_c : ((x^2+x)_r 0_e(x^2+1)_r x_e(x+1)_r(x^2+x)_e x_r^2(x^2+1)_e 1_r(x+1)_e 0_r x_e^2 x_r 1_e)$$
$$(x^2+1)_c : ((x_r^2 x_e(x^2+x+1)_r 0_e 1_r x_e^2(x^2+x)_r(x^2+x+1)_e(x+1)_r 1_e x_r(x^2+x)_e 0_r(x+1)_e)$$

$0_e : (1_r x_c x_r^2 (x+1)_c (x^2+x)_r (x^2+x+1)_c (x^2+1)_r 1_c x_r x_c^2 (x+1)_r (x^2+x)_c (x^2+x+1)_r (x^2+1)_c)$
$1_e : (0_r (x+1)_c (x^2+1)_r x_c (x^2+x+1)_r (x^2+x)_c x_r^2 0_c (x+1)_r (x^2+1)_c x_r (x^2+x+1)_c (x^2+x)_r x_c^2)$
$x_e : ((x+1)_r 0_c (x^2+x)_r 1_c x_r^2 (x^2+1)_c (x^2+x+1)_r (x+1)_c 0_r (x^2+x)_c 1_r x_c^2 (x^2+1)_r (x^2+x+1)_c)$
$x_e^2 : ((x^2+1)_r (x^2+x)_c 0_r (x^2+x+1)_c x_r (x+1)_c 1_r (x^2+1)_c (x^2+x)_r 0_c (x^2+x+1)_r x_c (x+1)_r 1_c)$
$(x+1)_e : (x_r 1_c (x^2+x+1)_r 0_c (x^2+1)_r x_c^2 (x^2+x)_r x_c 1_r (x^2+x+1)_c 0_r (x^2+1)_c x_r^2 (x^2+x)_c)$
$(x^2+x)_e : ((x^2+x+1)_r x_c^2 x_r (x^2+1)_c 0_r 1_c (x+1)_r (x^2+x+1)_c x_r^2 x_c (x^2+1)_r 0_c 1_r (x+1)_c)$
$(x^2+x+1)_e : ((x^2+x)_r (x^2+1)_c (x+1)_r x_c^2 1_r 0_c x_r (x^2+x)_c (x^2+1)_r (x+1)_c x_r^2 1_c 0_r x_c)$
$(x^2+1)_e : (x_r^2 (x^2+x+1)_c 1_r (x^2+x)_c (x+1)_r x_c 0_r x_c^2 (x^2+x+1)_r 1_c (x^2+x)_r (x+1)_c x_r 0_c)$

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D. M. Donovan, Centre for Discrete Mathematics and Computing University of Queensland, St Lucia 4072, Australia

T. S. Griggs, Department of Mathematics and Statistics, The Open University, Walton Hall Milton Keynes MK7 6AA, United Kingdom

C. Psomas, Department of Mathematics and Statistics, The Open University, Walton Hall Milton Keynes MK7 6AA, United Kingdom