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FREE ENERGIES AS INVARIANTS OF TEICHMÜLLER LIKE STRUCTURES

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Abstract. A Teichmüller like structure on the space of d-degree holomorphic maps on the circle S^1, marked by conjugations to the map $z \mapsto z^d$, can be defined. Here we introduce a definition of free energy associated with this kind of dynamics as an invariant of equivalence classes in the Teichmüller space. This quantity encodes a length spectrum of rotation cycles in S^1.

1. Introduction

The free energy is a map defined as average limit of a partition function for configurations of a system. In lattice Statistical Mechanics, the partition function is usually defined from admissible sequences of spins, whereas in the area of Dynamical Systems we may have an analogous function taken as configuration orbits of the dynamics. So, free energy plays a relevant role whatever of these or even other, areas considered.

Free energy rigidity properties for finite range potentials were established in [**8**] whereas in [**4**] we analyzed the rigidity problem but including long range potential. In both works a statistical mechanics point of view is taken.

A geometric free energy was introduced by Pollicott and Weiss [**9**]. The partition function there is defined from the sum over closed geodesics in hyperbolic manifolds. Here we shall consider a free energy which may be seen as a sort of a mix between dynamical and geometric free energies.

Let $\mathcal{B}_d$ be the space of proper holomorphic maps $f\colon H^2 \to H^2$ which can be expressed as Blaschke products

$$f(z) = z \prod_{j=2}^{d} \left(\frac{z - c_i}{1 - \overline{c_i} z} \right), \qquad c_i \in H^2 \tag{1.1}$$

and where H^2 is a hyperbolic disc model. These applications have the property that $f\,|_{S^1}$ preserves the Lebesgue measure and $|f'| > 1$ on the circle S^1. We also call these restrictions Blaschke products.

A *covering* of a circle map f is a map $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that $\pi \circ \widetilde{f} = f \circ \pi$ where $\pi\colon \mathbb{R} \to S^1$ is the map $\pi(t) = \exp(2\pi i t)$. Let $\mathrm{Cov}_d(\mathbb{R})$ be the space of

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d-degree coverings, i.e., $\widetilde{f}(t+1) = \widetilde{f}(t) + d$ for any real t. Let $C_d(\mathbb{R})$ be the subspace of $\mathrm{Cov}_d(\mathbb{R})$ which consists of analytic homeomorphisms $\widetilde{f}$, covering of expanding Blaschke products. We shall call $C_d(S^1)$ to the space of expanding Blaschke products whose coverings are in $C_d(\mathbb{R})$.

Any $f \in C_d(S^1)$ is conjugated to the map $p_d(z) = z^d$ by the *marking map* $\phi_f \colon S^1 \to S^1$ [**6**]. The marking map satisfies

$$\widetilde{\phi_f}(t) = \lim_{n\to\infty} \frac{1}{d^n} \widetilde{f^n}(t),$$

which limit does exist [**6**]. The *Teichmüller space* $\tau(\mathcal{C}_d(S^1))$ is formed by the equivalence classes $[(f, \phi_f)]$ for the relation $(f, \phi_f) \sim (g, \phi_g)$ if $\phi\colon = \phi_f \circ \phi_g^{-1}$ is a diffeomorphism which conjugates f and g.

For these spaces, the mapping class group $\mathcal{M}_d$ for this Teichmüller structure is isomorphic to $\mathbf{Z}_{d-1}$ which in turn is isomorphic to the automorphism group of p_d.

An n-periodic point C of $p_d(z) = z^d$ will be called *n-cycle* and its period will be denoted by $|C|$. Following [**7**] the length of an n-cycle C with respect to $f \in \mathcal{C}_d(S^1)$ is defined as

$$L(C, f) := \log |(f^n)'(z)|\,, \text{with } z \in C \tag{1.2}$$

This gives the following length spectra

$$\mathcal{S}_f^N = \{L(C, f) : |C| = n\} \quad \textit{non-marked spectrum}$$

and

$$\mathcal{S}_f^M = \{(L(C, f), |C|) : |C| = n\} \quad \textit{marked spectrum.}$$

The cycles, as studied by McMullen in [**7**], have geometric and topological behaviors comparable to closed geodesics in hyperbolic surfaces. The *degree* of a cycle C is the least $s > 0$ such that $p_d\mid_C$ can be extended to an s-degree topological covering map on the circle. The cycles whose degree is 1 are called *simple cycles*, equivalently simple cycles are those that $p_d\mid_C$ preserves its cyclic ordering. Precisely, these kind of points present similar facts to closed simple geodesics. For instance, if a cycle C verifies $L(C, f) < \log 2$, then it is a simple cycle. The counterpart for geodesics in hyperbolic surfaces of this result is that any closed geodesic in a genus g-surface with length less than $\log(3 + 2\sqrt{2})$ is simple [**7**].

In this article we shall consider a free energy encoding marked length spectra of cycles. In [**9**] a free energy encoding marked length spectra of closed geodesics was introduced, thus our objective is to analyze facts of the free energy of herein comparing with the partition function for length of geodesics [**9**]. We will specially pointed out the invariance for the Teichmüller structures above mentioned.

A "dual free energy" will be defined with partition function summing over periodic sequences in a "dual symbolic space". An orientation-preserving map $f\colon S^1 \to S^1$ of degree $d \geq 2$ with $f(1) = 1$ admits a Markov partition and consequently a symbolic space Σ. Its *dual space* is defined as

$$\Sigma^* = \{\omega^* = \ldots k_n \ldots k_1 k_0 : \omega = k_0 k_1 \ldots k_n \ldots \in \Sigma\}\,.$$

Let $H_n = k_0 k_1 \dots k_{n-1}$ be the truncation of a sequence in Σ to its first n symbols and $H_n^* = k_{n-1} \dots k_1 k_0$ be the dual sequence, i.e., $\omega^* = \dots k_{n-1} \dots k_1 k_0 \in \Sigma^*$.

In [**1**] Jiang introduced a *dual derivative*

$$D^*(f)\colon \Sigma^* \to \mathbb{R}$$

on the dual space (the definition will be displayed in the next section). The dynamics on Σ^* are given by dual Bernoulli shift defined by

$$\sigma^*(\omega^* = \dots k_n \dots k_1 k_0) = \dots k_n \dots k_1.$$

The cycles in this context will be periodic sequences. This symbolic space and this derivative can be obtained for a larger class of holomorphic d-degree maps, namely on the class *uniformly symmetric maps* [**1**]. A circle homeomorphism f with lifting $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ is called uniformly symmetric if

$$\frac{1}{1+\varepsilon(t)} \leq \left| \frac{\widetilde{f}^{-n}(x+t) - \widetilde{f}^{-n}(x)}{\widetilde{f}^{-n}(x) - \widetilde{f}^{-n}(x-t)} \right| \leq 1 + \varepsilon(t) \tag{1.3}$$

for some bounded function $\varepsilon(t)$ and for any $t > 0$, $x \in \mathbb{R}$. Uniformly symmetric maps may not be differentiable. If $\mathcal{US}(S^1)$ denotes the set of uniformly symmetric homeomorphisms on the circle, then the Teichmüller space $\tau(\mathcal{US}(S^1))$ with base point p_d is given by equivalence classes $[(f, \phi_f)]$, but now with the relation $(f, \phi_f) \sim (g, \phi_g)$ if $\phi := \phi_f \circ \phi_g^{-1}$ is symmetric.

For $f \in \mathcal{C}_d(S^1)$ and so $\widetilde{f} \in \mathcal{C}_d(\mathbb{R})$, the potential $\Psi(\omega^*) = -\log D^*(f)(\omega^*)$ has a unique Gibbs state. For the broader class $\mathcal{US}$, there is not an exponential convergence of $D^*(f)(H_n^*)$ to $D^*(f)(\omega^*)$ like in the case $f \in \mathcal{C}_d(S^1)$ and the thermodynamic formalism is not enough to ensure Gibbs inequalities.

The dual length of f with respect to an $n-$cycle ω^* is defined as

$$L^*(f, \omega^*) = S_n(D^*(f))(\omega^*) := \sum_{i=0}^{n-1} D^*(f)((\sigma^*)^i(\omega^*)) \quad \text{(statistical sum).} \tag{1.4}$$

A dual free energy will be defined encoding the spectrum formed by the length L^*.

This article is in the line of a previous one [**5**] in which we obtained relationships between Teichmüller structures and thermodynamics objects for conformal iterated schemes of d-proper holomorphic maps.

2. Free energies

Let $f \in \mathcal{C}_d(S^1)$ the set of n-cycles with respect to f be denoted by $\mathcal{C}_{n,f}$. We consider the partition function

$$Z_{n,f}(q) := \sum_{|C|=n} \exp(-qL(C, f)), \tag{2.1}$$

where q is interpreted as the inverse of the temperature and the free energy

$$T_f(q) = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,f}(q). \tag{2.2}$$

Definition. Two maps $\varphi, \psi\colon X \to \mathbb{R}$ are cohomologous with respect to a dynamical map $f\colon X \to X$ if there exits a function $h\colon X \to X$ such that $\varphi = \psi + h - h \circ f$.

Thus, two cohomologous functions have the same statistical sum when evaluated in periodic points by a direct calculation.

Lemma 1. *If (f, ϕ_f), (g, ϕ_g) belong to the same class in $\tau(\mathcal{C}_d)$ then $T_f = T_g$.*

Proof. Since $(f, \phi_f) \sim (g, \phi_g)$, we have $f(\phi(z)) = \phi(g(z))$ where $\phi = \phi_f \circ \phi_g^{-1}$. Then $f'(\phi(z))\phi'(z) = \phi'(g(z))g'(z)$. Let $\varphi_f = \log|f'|$ and $\varphi_g = \log|g'|$, so $\varphi_f = \varphi_g + \log|\phi'| - \log|\phi' \circ f|$, so that φ_f and φ_g are cohomologous with $h = \log|\phi'|$. Hence for any cycle $L(C, f) = L(C, g)$, C and $T_f = T_g$. $\square$

By the Livsic theorem, the iplication $L(C, f) = L(C, g) \Rightarrow \mathcal{S}_f^M = \mathcal{S}_g^M$ is valid.

Next we shall consider a Poincaré series in the sense of [**9**], but with the length spectrum of cycles instead of geodesics.

Let

$$P(q, r) = \sum_{C \in \mathcal{C}_{n,f}} \exp\left[-qL(C, f) - r\,|C|\right], \tag{2.3}$$

or

$$P(q, r) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{C \in \mathcal{C}_{n,f}} \exp\left[-qL(C, f) - r\,|C|\right] \right). \tag{2.4}$$

This series converges if

$$L := \lim_{n\to\infty} \left[\sum_{C \in \mathcal{C}_{n,f}} \exp(-qL(C, f)) \right]^{\frac{1}{n}} (\exp(-rn))^{\frac{1}{n}} < 1,$$

thus

$$\lim_{n\to\infty} \frac{1}{n} \log Z_{n,f}(q) - r = \log L < 0.$$

The region of convergence of the series is $\Omega = \{(q, r) : T_f(q) < r\}$.

The potential $\varphi_f = \log|f'|$ has a unique Gibbs state μ_f [**10**], now a Gibbs state can be associated with to any pair (f, ϕ_f) and since cohomologous maps have the same Gibbs state [**2**], pairs in the same equivalence class of $\tau(\mathcal{C}_d(S^1))$ share its Gibbs state.

Next we display a result similar to one in [**9**] for geodesics.

Theorem 1. *Let μ_q be the Gibbs state for the potential $-q\log|f'|$, $f \in \mathcal{C}_d(S^1)$, with a covering $\widetilde{f} \in \mathcal{C}_d(\mathbb{R})$, the behaviors of the Gibbs states for "zero temperature" are*

$$\lim_{q\to+\infty} \int \log|f'| \mathrm{d}\mu_q = \inf\left\{ \frac{L(C, f)}{|C|} : C \textit{ is a cycle} \right\}$$

and

$$\lim_{q\to-\infty}\int \log|f'|\mathrm{d}\mu_q = \sup\left\{\frac{L(C,f)}{|C|} : C \textit{ is a cycle}\right\}.$$

Proof. By the variational principle, we have $h_{\mu_q}(f) - q\int \log|f'|\mathrm{d}\mu_q \geq h_\mu(f) - q\int \log|f'|\mathrm{d}\mu$ for any measure μ where $h_\mu(f)$ is the measure-theoretic entropy. Thus for $q>0$,

$$\frac{h_{\mu_q}(f)}{q} - \int \log|f'|\mathrm{d}\mu_q \geq \frac{h_\mu(f)}{q} + \int \log|f'|\mathrm{d}\mu$$

and so

$$\begin{aligned}\int \log|f'|\mathrm{d}\mu_q &\leq \frac{h_{\mu_q}(f) - h_\mu(f)}{q} + \int \log|f'|\mathrm{d}\mu \\ &\leq \frac{2h}{q} + \int \log|f'|\mathrm{d}\mu \quad (h = \text{topological entropy}).\end{aligned}$$

Then $\int \log|f'|\mathrm{d}\mu_q \leq \dfrac{2h}{q} + \inf\limits_\mu \left\{\int \log|f'|\mathrm{d}\mu\right\}$. By the ergodic theorem, $\dfrac{1}{|C|}L(C,f)$ tends to $\int \log|f'|\mathrm{d}\mu$ as $|C|\to\infty$, $\mu-a.e.$ for any ergodic measure μ. Now we have

$$\lim_{q\to+\infty}\int \log|f'|\mathrm{d}\mu_q = \inf_\mu\left\{\int \log|f'|\mathrm{d}\mu\right\} = \inf\left\{\frac{L(C,f)}{|C|} : C \text{ is a cycle}\right\}.$$

The demonstration for the other limit is totally similar. □

Then for any cycle C, we have

$$\begin{aligned}A_1 := \inf\left\{\frac{L(C,f)}{|C|} : C \text{ is a cycle}\right\} &\leq \frac{L(C,f)}{|C|} \\ &\leq A_2 := \sup\left\{\frac{L(C,f)}{|C|} : C \text{ is a cycle}\right\},\end{aligned}$$

similar to a known result by Milnor for closed geodesics.

The value $\int \log|f'|\mathrm{d}\mu_q$ is precisely $T_f'(q)$ [**2**], so that for high and low temperatures, the free energy behaves as $\lim\limits_{q\to+\infty} T_f'(q) = \inf\left\{\dfrac{L(C,f)}{|C|} : C \text{ is a cycle}\right\}$ and $\lim\limits_{q\to-\infty} T_f(q) = \sup\left\{\dfrac{L(C,f)}{|C|} : C \text{ is a cycle}\right\}$.

By the Anosov closing lemma, for any $t\in[A_1,A_2]$ and for any $\varepsilon>0$, there is a cycle C such that

$$\left|\frac{L(C,f)}{|C|} - t\right| < \varepsilon.$$

Next we shall introduce a *dual free energy.* We begin by presenting the definition of the *dual derivative* according to [**1**]. Let $f\colon S^1\to S^1$ be an orientation-preserving of degree $d\geq 2$ with a fixed point in $z=1$. There is a Markov partition $\mathcal{J}=\{J_0,J_1,\dots,J_{d-1}\}$ for (S^1,f) where intervals are obtained by the intersection of $f^{-1}(1)$ with the circle. Let $\mathcal{I}=\{I_0,I_1,\dots,I_{d-1}\}$ be the partition of $I=[0,1]$

obtained by lifting any J_i to I_i by the cover map $\pi(x) = \exp(2\pi i x)$. The name of length n of a point $z \in S^1$ is the string $H_n = k_0 k_1 \ldots k_{n-1}$ such that $f^\ell(z) \in J_{k_\ell}$. let $\mathcal{J}^n$ be the partition by sets J_{H_n} formed by points with the same name with respect to $\mathcal{J}$ and f. The number of intervals of $\mathcal{J}^n$ is $(d-1)^n$ and $\mathcal{J}^n$ is also a Markov partition and by $\mathcal{I}^n$, the lift of $\mathcal{J}^n$ is denoted to I. The strings which give the names of infinite length corresponding to points in S^1 or in $[0,1]$ originate a symbolic space with alphabet $\Omega = \{0, 1 \ldots, d-1\}$,

$$\Sigma = \{\omega = k_0 k_1 \ldots k_{n-1} \ldots : k_i \in \Omega\}.$$

Its dual space is defined as

$$\Sigma^* = \{\omega^* = \ldots k_{n-1} \ldots k_1 k_0 : k_i \in \Omega\}$$

and the dual shift on Σ^* is

$$\sigma^*(\omega^* = \ldots k_{n-1} \ldots k_1 k_0) = \ldots k_{n-1} \ldots k_1.$$

Let $H_n = k_0 k_1 \ldots k_{n-1}$ be the truncation of a sequence in Σ to its first n symbols and $H_n^* = k_{n-1} \ldots k_1 k_0$ be the dual sequence, i.e., $\omega^* = \ldots k_{n-1} \ldots k_1 k_0 \in \Sigma^*$. Let us call K_{n-1}^* to the last $n-1$ symbols of $\sigma^*(\omega^*)$, i.e., $K_n^* = k_{n-1} \ldots k_1$, which is the dual of the string $K_{n-1} = k_1 \ldots k_{n-1}$. The cylinder containing the string $H_n^* = k_{n-1} \ldots k_1 k_0$ is

$$C(H_n^*) = \left\{\omega^* = \ldots \overline{k_n} k_{n-1} \ldots k_1 k_0 : k_i \in \Omega\right\}.$$

Now we define

$$D^*(f)(H_n^*) := \frac{\ell(I_{K_{n-1}})}{\ell(H_n)}, \tag{2.5}$$

where ℓ denotes the length of the interval. Finally the dual derivative of f is

$$\begin{gathered} D^*(f) : \Sigma^* \to \mathbb{R} \\ D^*(f)(\omega^*) = \lim_{n\to\infty} D^*(f)(H_n^*) \end{gathered} \tag{2.6}$$

where this convergence is exponential.

There is an unique Gibbs state μ^* associated with the potential

$$\omega^* \longmapsto -\log D^*(f)(\omega^*).$$

This measure is defined on the cylinders. Since the exponential convergence for any natural n, there are constants $C > 0$ and $0 < r < 1$ such that

$$\left| \frac{\mu^*(C(H_n^*))}{\mu^*(C(K_{n-1}^*))} - D^*(f)(\omega^*) \right| \leq C r^n.$$

From this, Gibbs inequalities are obtained.

The Teichmüller structures on the spaces of circle maps defined earlier are described by the dual derivative, this means [**1**]

$$\tau(\mathcal{C}_d(S^1)) = \left\{D^*(f) : f \in \mathcal{C}_d(S^1)\right\}$$
$$\tau(\mathcal{US}(S^1)) = \left\{D^*(f) : f \in \mathcal{US}(S^1)\right\}.$$

Next we introduce the *dual free energy*. Let $\mathcal{C}_n^*$ be the set of sequences in Σ^* with period n. In this symbolic context the sequences will be called cycles, the period of a sequence ω^* will be denoted by $|\omega^*|$. The length of $\omega^* \in \mathcal{C}_n^*$ with respect to f is defined as $L^*(\omega^*, f) := S_n(\psi)(\omega^*)$. The partition function is

$$Z_{n,f}^*(q) := \sum_{\omega^* \in \mathcal{C}_n^*} \exp(-qL^*(\omega^*, f)) \tag{2.7}$$

and the dual free energy

$$T_f^*(q) = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,f}^*(q). \tag{2.8}$$

The dual length spectra are

$$\mathcal{S}_f^{*N} = \{L^*(\omega^*, f) : \omega^* \in \mathcal{C}_n^*\} \quad \text{and} \quad \mathcal{S}_f^{*M} = \{(L^*(\omega^*, f), \omega^*) : \omega^* \in \mathcal{C}_n^*\}.$$

Now we can present a similar dual result.

Theorem 2. *Let μ_q^* be the Gibbs state for the potential*

$$-q\psi = -q \log D^*(f)(\omega^*), \qquad f \in \mathcal{C}_d(S^1),$$

with a covering $\widetilde{f} \in \mathcal{C}_d(\mathbb{R})$. Then we have the following behaviors

$$\lim_{q\to+\infty} \int \psi \mathrm{d}\mu_q^* = \inf \left\{ \frac{L^*(\omega^*, f)}{|\omega^*|} : \omega^* \textit{ periodic sequence} \right\}$$

and

$$\lim_{q\to-\infty} \int \psi \mathrm{d}\mu_q^* = \sup \left\{ \frac{L^*(\omega^*, f)}{|\omega^*|} : \omega^* \textit{ periodic sequence} \right\}.$$

Proof. By the variational principle, we have for any $f-$ invariant measure μ,

$$h_{\mu_q}(\sigma^*) - q \int \log D^*(f) \mathrm{d}\mu_q^* \geq h_\mu(\sigma^*) - q \int \log D^*(f) \mathrm{d}\mu.$$

If $q > 0$, then

$$\frac{h_{\mu_q}(\sigma^*)}{q} - \int \log D^*(f) \mathrm{d}\mu_q^* \geq \frac{h_\mu(\sigma^*)}{q} + \int \log D^*(f) \mathrm{d}\mu,$$

which leads to

$$\int \log D^*(f) \mathrm{d}\mu_q^* \leq \frac{h_{\mu_q}(\sigma^*) - h_\mu(\sigma^*)}{q} + \int \log D^*(f) \mathrm{d}\mu \leq \frac{2h}{q} + \int \log D^*(f) \mathrm{d}\mu.$$

So that

$$\int \log D^*(f) \mathrm{d}\mu_q^* \leq \frac{2h}{q} + \inf_\mu \left\{ \int \log D^*(f) \mathrm{d}\mu \right\}.$$

By the ergodic theorem, $\dfrac{1}{|\omega^*|} L^*(\omega^*, f) = \dfrac{1}{|\omega^*|} S_n(\psi)(\omega^*)$ converges to $\int \log D^*(f) \mathrm{d}\mu$, for $|\omega^*| \to \infty$, $\mu - a.e.$ for any ergodic measure μ. Further we obtain

$$\begin{aligned} \lim_{q\to+\infty} \int \log |f'| \mathrm{d}\mu_q^* &= \inf_\mu \left\{ \int \log D^*(f) \mathrm{d}\mu \right\} \\ &= \inf \left\{ \frac{L^*(\omega^*, f)}{|\omega^*|} : \omega^* \text{ periodic sequence} \right\} \end{aligned}$$

In analogous way the equality for the other limit is demonstrated. □

For uniformly symmetric maps Jiang developed a theory for obtaining a "type Gibbs measure" associated with the potential $\omega^* \to -\log D^*(f)(\omega^*)$, $f \in \mathcal{US}$, which involves quasiconformal mappings. The convergence is not in general exponential as in the setting of the above theorem. Since this fact, results used in Theorem 2 do not have be guaranteed. We are deeply grateful to the referee for pointing out about this issue.

We shall consider now a special circle map introduced by Los whose construction follows the ideas of Bowen and Series to define boundary hyperbolic maps associated with an action of a Fuchsian group Γ on the hyperbolic disc. The new in the construction of Los is that he does not consider geometric conditions on the fundamental region for the action of Γ on H^2, but keeps the restrictions on the presentation of the group. Besides, the constructions for defining the map are more combinatorial than geometric. The objective of introducing such a map was to compute the volume entropy, i.e., the growing rate of the ball in the word metric for a presentation of the group. The main result in [**3**] is that the volume entropy for a presentation of Γ equals the topological entropy of the Bowen-Series like map defined in that article. This leads to a method for minimizing the volume entropy among the geometric presentations of the group. Herein we shall introduce a *combinatorial free energy* which will be compared with free energy associated the Los map. Next we give a brief background, for more details the article by Los is available in the web ....

Let Γ be a hyperbolic co-compact surface group with a finite presentation $\mathcal{P}$ given by a symmetric set of generators $S = \left\{s_1^{\pm 1}, \dots, s_m^{\pm 1}\right\}$ and relators $R = \{r_1, \dots, r_k\}$. The length of $\gamma \in \Gamma \equiv \mathcal{P} = \langle S, R\rangle$ denoted by $|\gamma|$ is the minimal number of elements of S needed to express γ. The word metric is defined as $d_S(\gamma_1, \gamma_2) = \left|\gamma_1\gamma_2^{-1}\right|$ and the ball $B_{n,S}$ is $\{\gamma : d_S(\gamma, \mathrm{id}) = n\}$. Recall the Cayley graph $\mathcal{G}(\Gamma, \mathcal{P})$ for a group Γ with presentation $\mathcal{P}$ which is the graph with vertices from the elements of Γ and there is an edge between γ_1 and γ_2. If $\gamma_1\gamma_2^{-1} = \mathrm{id}$, relators represent a closed path in the Cayley graph. The two-complex $\mathcal{G}^{(2)}$ is the two-dimensional complex whose 1-skeleton is $\mathcal{G}$ and where the two-cells are attached to a closed path in $\mathcal{G}$. A presentation $\mathcal{P}$ is called *geometric* if the complex $\mathcal{G}^{(2)}$ is planar.

The presentation $\mathcal{P} = \langle S, R\rangle$ uniquely defines a partition $\{I_{s_i}\}_{s_i \in S}$ of S^1 and the Los map associated with a geometric presentation $\mathcal{P}$ is defined as

$$f_{\mathcal{P}} \colon S^1 \to S^1, \qquad f_{\mathcal{P}}(z) \;= s_i^{-1} z \quad \text{for} \quad z \in I_{s_i} \tag{2.9}$$

By $T_{\mathcal{P}}(q)$, let us denote the free energy for the map $f_{\mathcal{P}}$.

A refinement of the partition $\{I_{s_i}\}$ leads to a Markov partition in such a way that $f_{\mathcal{P}}$ becomes a Markov and strictly expanding map. There are subdivisions L_{s_i}, R_{s_i} of I_{s_i} such that the partition by subdivision points

$$\mathcal{S} := \bigcup_{s_i \in S} (L_{s_i} \cup R_{s_i} \cup \partial I_{s_i})$$

is uniquely determined by $\mathcal{P}$ and $f_{\mathcal{P}}$ is $\mathcal{S}-$invariant. The map $f_{\mathcal{P}}$ satisfies the Markov condition with respect to $\mathcal{S}$, i.e., $f_{\mathcal{P}}$ is an homeomorphism in each interval of the partition and maps extremes to extremes. Thus $\mathcal{S}$ determines a Markov partition for $f_{\mathcal{P}}$. Thus the orbits $\left\{f^n_{\mathcal{P}}(z) : n \in \mathbf{N},\ z \in S^1\right\}$ can be coded by a given symbolic Markov space Σ with an alphabet constituted by generators of Γ, the coding is given as

$$\chi : S^1 - \cup\partial I_{s_i} \to \Sigma, \qquad \chi(z) = s_{i_0}s_{i_1}\dots, \quad \text{with } f^j_{\mathcal{P}}(z) \in I_{s_{i_j}}.$$

Since $\ell(\bigcap_{n=0}^{\infty} f^{-j}_{\mathcal{P}}(I_{s_i})) \to 0$ as $n \to \infty$, the coding map χ is injective [**3**].

Now any point z assigns a sequence $\omega = s_{i_0}s_{i_1}\dots,\ s_{i_j} \in S$. The restriction of the coding sequence to the n-first symbols is called the *n-prefix.* Let $D_{n,S}$ be the number of n-prefix for a geometric presentation $\mathcal{P}$, then for n enough large, $D_{n,S} \approx \mathrm{card}B_{n,S}$ and restriction on sequences in $\chi(S^1 - \cup\partial I_{s_i})$ are equal for all n [**3**].

For $\omega \in \chi(S^1 - \cup\partial I_{s_i})$, let $\psi(\omega) = \log|f'(\chi^{-1}(\omega))|$ and the partition function

$$Z_{n,\mathrm{comb}}(q) := \sum_{\omega\in B_{n,S}\cap\chi(S^1-\cup\partial I_{s_i})} \exp(-qS_n(\psi)(\omega)).$$

The *combinatorial free energy* for a geometric presentation $\mathcal{P}$ is

$$T_{\mathrm{comb},\mathcal{P}}(q) = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,\mathrm{comb}}(q)$$

with $T_{\mathrm{comb}}(0) =$ volume entropy of $\Gamma \equiv \mathcal{P} = \langle S, R\rangle$.

To compare the combinatorial free energy with $T_{\mathcal{P}}(q)$ firstly, we get $\mathrm{card}B_{n,S} = \mathrm{card}D_{n,S} = \|A^n\|$, where A is the transition matrix for the Markov partition [**11**] which in turn is equal to the number of cycles of length n [**10**].

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