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AN INTEGRODIFFERENTIAL EQUATION WITH FRACTIONAL DERIVATIVES IN THE NONLINEARITIES

ZHENYU GUO AND MIN LIU

Abstract. An integrodifferential equation with fractional derivatives in the nonlinearities is studied in this article, and some sufficient conditions for existence and uniqueness of a solution for the equation are established by contraction mapping principle.

1. Introduction

This article is concerned with the existence and uniqueness of a solution of the following integrodifferential equation with fractional derivatives in the nonlinearities:

$$
\begin{aligned}
u''(t) = \ & Au(t) + f\Big(t, u(t), {}^c D^{\alpha_1} u(t), \cdots, {}^c D^{\alpha_m} u(t)\Big) \\
& + \int_0^t g\Big(t, s, u(s), {}^c D^{\beta_1} u(s), \cdots, {}^c D^{\beta_n} u(s)\Big) \mathrm{d}s, \quad t > 0, \\
u(0) = u_0 \in X&, \qquad u'(0) = u_1 \in X,
\end{aligned} \tag{1}
$$

where A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \geq 0$ of bounded linear operators on a Banach space X with norm $\|\cdot\|$, f and g are nonlinear mappings from $\mathbb{R}^+ \times X^m$ to X and $\mathbb{R}^+ \times \mathbb{R}^+ \times X^n$ to X, respectively, $0 < \alpha_i, \beta_j < 1$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$, u_0 and u_1 are given initial data in X.

Recently, fractional order differential equations and systems have been payed much attention, of examples, the monograph of Kilbas et al. [**10**], and the papers by Anguraj et al. [**1**], Benchohra et al. [**2**]–[**4**], Guo and Liu [**5**]–[**7**], Hernandez [**8**], Hernandez et al. [**9**] Kirane et al. [**11**], Tatar [**12**]–[**15**] and the references therein.

Applying the Banach contraction principle, we obtain a result of uniqueness of a solution for problem (1). To simplify our task, we will treat the following simpler

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problem

$$
\begin{aligned}
u''(t) = {} & Au(t) + f\Big(t, u(t), {}^{c}D^{\alpha}u(t)\Big) \\
& + \int_0^t g\Big(t, s, u(s), {}^{c}D^{\beta}u(s)\Big)\mathrm{d}s, \quad t > 0, \\
u(0) = {} & u_0 \in X, \qquad u'(0) = u_1 \in X.
\end{aligned} \tag{2}
$$

The general case can be derived easily.

2. Preliminaries

Let us recall a basic definition in fractional calculus, which can be found in the literature.

Definition 2.1. The Caputo fractional derivative of order $0 < \alpha < 1$ is defined by

$$
{}^{c}D^{\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)dt, \tag{3}
$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Now list the following hypotheses for convenience

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators in the Banach space X.

The associated sine family $S(t), t \in \mathbb{R}$ is defined by

$$
S(t)x := \int_0^t C(s)x\mathrm{d}s, \qquad t \in \mathbb{R}, x \in X. \tag{4}
$$

For $C(t)$ and $S(t)$, it is known (see [**16**]) that there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$
|C(t)| \leq M\,\mathrm{e}^{\omega|t|}, \qquad |S(t) - S(t_0)| \leq M\Big|\int_{t_0}^t \mathrm{e}^{\omega|s|}\,\mathrm{d}s\Big|, \qquad t, t_0 \in \mathbb{R}. \tag{5}
$$

Let $X_A = D(A)$ endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$.

(H2) $f\colon \mathbb{R}^+ \times X_A \times X \to X$ is continuously differentiable,

(H3) $g\colon \mathbb{R}^+ \times \mathbb{R}^+ \times X_A \times X \to X$ is continuous and continuously differentiable with respect to its first variable,

(H4) f, f' (the total derivative of f),g and g_1 (the partial derivative of g with respect to its first variable) are Lipschitz continuous with respect to the last two variables, that is

$$
\begin{aligned}
\|f(t, x_1, y_1) - f(t, x_2, y_2)\| &\leq L_f\big(\|x_1 - x_2\|_A + \|y_1 - y_2\|\big), \\
\|f'(t, x_1, y_1) - f'(t, x_2, y_2)\| &\leq L_{f'}\big(\|x_1 - x_2\|_A + \|y_1 - y_2\|\big), \\
\|g(t, s, x_1, y_1) - g(t, s, x_2, y_2)\| &\leq L_g\big(\|x_1 - x_2\|_A + \|y_1 - y_2\|\big), \\
\|g_1(t, s, x_1, y_1) - g_1(t, s, x_2, y_2)\| &\leq L_{g_1}\big(\|x_1 - x_2\|_A + \|y_1 - y_2\|\big)
\end{aligned} \tag{6}
$$

for some positive constants L_f, $L_{f'}$,L_g and L_{g_1}.

Lemma 2.2 ([**16**])**.** *Assume that* (H1) *is satisfied. Then*

(i) $S(t)X \subset E, \;\; t \in \mathbb{R}$,
(ii) $S(t)E \subset X_A, \;\; t \in \mathbb{R}$,
(iii) $(\mathrm{d}/\mathrm{d}t)C(t)x = AS(t)x, x \in E, \;\; t \in \mathbb{R}$,
(iv) $(\mathrm{d}^2/\mathrm{d}t^2)C(t)x = AC(t)x = C(t)Ax, \;\; x \in X_A, \; t \in \mathbb{R}$, *where*

$$E := \{x \in X : C(t)x \text{ is once continuously differentiable on } \mathbb{R}\}. \tag{7}$$

Lemma 2.3 ([**16**])**.** *Assume that* (H1) *holds,* $v\colon \mathbb{R} \to X$ *is a continuously differentiable function and* $q(t) = \int_0^t S(t-s)v(s)\mathrm{d}s$*. Then,* $q(t) \in X_A$*,* $q'(t) = \int_0^t C(t-s)v(s)\mathrm{d}s$ *and* $q''(t) = \int_0^t C(t-s)v'(s)\mathrm{d}s + C(t)v(0) = Aq(t) + v(t)$*.*

Definition 2.4. A function $u(\cdot) \in C^2(I, X)$ is called a classical solution of problem (2) if $u(t) \in X_A$ satisfies the equation in (2) and the initial conditions are verified.

Definition 2.5. A continuously differentiable solution of the integrodifferential equation

$$\begin{aligned} u(t) = C(t)u_0 + S(t)u_1 &+ \int_0^t S(t-s)f\Big(s, u(s), {}^c D^{\alpha}u(s)\Big)\mathrm{d}s \\ &+ \int_0^t S(t-s)\int_0^s g\Big(s, \tau, u(\tau), {}^c D^{\beta}u(\tau)\Big)\mathrm{d}\tau\mathrm{d}s \end{aligned} \tag{8}$$

is called a mild solution of problem (2).

3. Main results

In this section, the theorem of existence and uniqueness of a solution for equation (2) will be given.

Theorem 3.1. *Assume that* (H1)–(H4) *hold. If* $u_0 \in X_A$*,* $u_1 \in E$ *and* $L_f < 1$*, then there exist* $T > 0$ *and a unique function* $u\colon (0,T) \to X, u \in C((0,T), X_A) \cap C^2((0,T), X)$ *which satisfies* (2)*.*

Proof. For $t \in (0,T)$, define a mapping

$$\begin{aligned} (Ku)(t) := C(t)u_0 + S(t)u_1 &+ \int_0^t S(t-s)f\Big(s, u(s), {}^c D^{\alpha}u(s)\Big)\mathrm{d}s \\ &+ \int_0^t S(t-s)\int_0^s g\Big(s, \tau, u(\tau), {}^c D^{\beta}u(\tau)\Big)\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{9}$$

It follows from $u_0 \in X_A$ and $AC(t)u_0 = C(t)Au_0$ that $C(t)u_0 \in X_A$. Clearly, $S(t)u_1 \in X_A$ because $u_1 \in E$ and $S(t)E \subset X_A$ (see (ii) of Lemma 2.2). Moreover, by Lemma 2.3, (H2) and (H3), we know that both integral terms in (9) are in X_A.

Therefore, $Ku \in C((0,T), X_A)$. By Lemma 2.3, we have

$$
\begin{aligned}
(AKu)(t) = {} & C(t)Au_0 + AS(t)u_1 + \int_0^t C(t-s) f'\Big(s, u(s), {}^c D^\alpha u(s)\Big) \mathrm{d}s \\
& + C(t) f(0, u_0, {}^c D^\alpha u_0) - f\Big(t, u(t), {}^c D^\alpha u(t)\Big) \\
& + \int_0^t C(t-s) \Big[\int_0^s g_1\Big(s, \tau, u(\tau), {}^c D^\beta u(\tau)\Big) \mathrm{d}\tau \\
& + g\Big(s, s, u(s), {}^c D^\beta u(s)\Big) \Big] \mathrm{d}s \\
& - \int_0^t g\Big(t, \tau, u(\tau), {}^c D^\beta u(\tau)\Big) \mathrm{d}\tau, \qquad t \in (0,T).
\end{aligned} \tag{10}
$$

Differentiating (9), we get

$$
\begin{aligned}
(Ku)'(t) = {} & AS(t)u_0 + C(t)u_1 + \int_0^t C(t-s) f\Big(s, u(s), {}^c D^\alpha u(s)\Big) \mathrm{d}s \\
& + \int_0^t C(t-s) \int_0^s g\Big(s, \tau, u(\tau), {}^c D^\beta u(\tau)\Big) \mathrm{d}\tau \mathrm{d}s, \qquad t \in (0,T).
\end{aligned} \tag{11}
$$

Hence, $Ku \in C^1((0,T), X)$ and K maps C^1 into C^1.

It is claimed that K is a contraction on C^1 endowed with the metric

$$
\rho(u,v) := \sup_{0 \le t \le T} \big(\|u(t) - v(t)\| + \|A(u(t) - v(t))\| + \|u'(t) - v'(t)\| \big). \tag{12}
$$

For $u, v \in C^1$, it can be derived that

$$
\begin{aligned}
& \|(Ku)(t) - (Kv)(t)\| \\
& \le \int_0^t |S(t-s)| \Big[L_f\big(\|u(s) - v(s)\|_A + \|{}^c D^\alpha u(s) - {}^c D^\alpha v(s)\|\big) \\
& \qquad + \int_0^s L_g\big(\|u(\tau) - v(\tau)\|_A + \|{}^c D^\beta u(\tau) - {}^c D^\beta v(\tau)\|\big) \mathrm{d}\tau \Big] \mathrm{d}s \\
& \le \int_0^t M \int_0^{t-s} \mathrm{e}^{\omega\tau} \, \mathrm{d}\tau \Big[L_f\big(\|u(s) - v(s)\|_A \\
& \qquad + \frac{1}{\Gamma(1-\alpha)} \int_0^s (s-\tau)^{-\alpha} \|u'(\tau) - v'(\tau)\| \mathrm{d}\tau\big) \\
& \qquad + \int_0^s L_g\big(\|u(\tau) - v(\tau)\|_A \\
& \qquad + \frac{1}{\Gamma(1-\beta)} \int_0^\tau (\tau - \sigma)^{-\beta} \|u'(\sigma) - v'(\sigma)\| d\sigma\big) \mathrm{d}\tau \Big] \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le M\int_0^T \mathrm{e}^{\omega\tau}\,\mathrm{d}\tau \int_0^t \Bigg[L_f\big(\|u(s)-v(s)\|_A \\
&\qquad\qquad + \frac{s^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\le t\le T}\|u'(t)-v'(t)\|\big) \\
&\quad + \int_0^s L_g\big(\|u(\tau)-v(\tau)\|_A + \frac{\tau^{1-\beta}}{\Gamma(2-\beta)}\sup_{0\le t\le T}\|u'(t)-v'(t)\|\big)\mathrm{d}\tau\Bigg]\mathrm{d}s \\
&\le M\int_0^T \mathrm{e}^{\omega\tau}\,\mathrm{d}\tau \int_0^t \Bigg[L_f\max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big\}\rho(u,v) \\
&\qquad\qquad + L_g\max\Big\{1,\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}\rho(u,v)s\Bigg]\mathrm{d}s \\
&\le M\int_0^T \mathrm{e}^{\omega\tau}\,\mathrm{d}\tau \max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)},\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}(L_f+L_gT/2)T\rho(u,v),
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\|(AKu)(t)-(AKv)(t)\| \\
&\quad\le \int_0^t M\,\mathrm{e}^{\omega(t-s)}\,L_{f'}\big(\|u(s)-v(s)\|_A + \|{}^cD^\alpha u(s) - {}^cD^\alpha v(s)\|\big)\mathrm{d}s \\
&\quad\quad + L_f\big(\|u(t)-v(t)\|_A + \|{}^cD^\alpha u(t) - {}^cD^\alpha v(t)\|\big) \\
&\quad\quad + \int_0^t M\,\mathrm{e}^{\omega(t-s)}\Big[\int_0^s L_{g_1}\big(\|u(\tau)-v(\tau)\|_A + \|{}^cD^\beta u(\tau) - {}^cD^\beta v(\tau)\|\big)\mathrm{d}\tau \\
&\quad\quad + L_g\big(\|u(s)-v(s)\|_A + \|{}^cD^\beta u(s) - {}^cD^\beta v(s)\|\big)\Bigg]\mathrm{d}s \\
&\quad\quad + \int_0^t L_g\big(\|u(\tau)-v(\tau)\|_A + \|{}^cD^\beta u(\tau) - {}^cD^\beta v(\tau)\|\big)\mathrm{d}\tau \\
&\quad\le \int_0^T M\,\mathrm{e}^{\omega(T-s)}\,\mathrm{d}s L_{f'}\max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big\}\rho(u,v) \\
&\quad\quad + L_f\max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big\}\rho(u,v) \\
&\quad\quad + \int_0^T M\,\mathrm{e}^{\omega(T-s)}\,\mathrm{d}s\Bigg[L_{g_1}\max\Big\{1,\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}T \\
&\quad\quad + L_g\max\Big\{1,\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}\Bigg]\rho(u,v) + L_g\max\Big\{1,\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}T\rho(u,v) \\
&\quad\le \max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)},\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\} \\
&\quad\quad \cdot\Big[\int_0^T M\,\mathrm{e}^{\omega(T-s)}\,\mathrm{d}s\Big(L_{f'} + L_{g_1}T + L_g\Big) + L_gT + L_f\Big]\rho(u,v),
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
&\|(Ku)'(t)-(Kv)'(t)\| \\
&\quad\le \int_0^t M\,\mathrm{e}^{\omega(t-s)}\Big[L_f\big(\|u(s)-v(s)\|_A+\|{}^cD^\alpha u(s)-{}^c\,D^\alpha v(s)\|\big)\mathrm{d}s \\
&\qquad + \int_0^s L_g\big(\|u(\tau)-v(\tau)\|_A+\|{}^cD^\beta u(\tau)-{}^c\,D^\beta v(\tau)\|\big)\mathrm{d}\tau\Big]\mathrm{d}s \\
&\quad\le \int_0^T M\,\mathrm{e}^{\omega(T-s)}\,\mathrm{d}s\max\left\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)},\frac{T^{1-\beta}}{\Gamma(2-\beta)}\right\}\big(L_f+L_gT\big)\rho(u,v),
\end{aligned}
\tag{15}
$$

The above three relations (13)–(15) and condition $L_f<1$ guarantee that for sufficiently small T, K is a contraction on C^1. Therefore, there exists a unique mild solution $u\in C^1$. Clearly, $u\in C^2((0,T),X)$ and satisfies the problem (2). This completes the proof. □

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Zhenyu Guo, School of Sciences, Liaoning Shihua University Fushun, Liaoning 113001, China, *e-mail*: guozy@163.com

Min Liu, School of Sciences, Liaoning Shihua University Fushun, Liaoning 113001, China, *e-mail*: min_liu@yeah.net