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SOME RESULTS ABOUT FIXED POINTS IN THE COMPLETE METRIC SPACE OF ZERO AT INFINITY VARIETIES AND COMPLETE CONVEX METRIC SPACE OF VARIETIES

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Abstract. This paper aims to study fixed points in the complete metric space of varieties which are zero at infinity as a subspace of the complete metric space of all varieties. Also, the convex structure of the complete metric space of all varieties will be introduced.

1. Introduction and preliminaries

After the introduction of varieties of Banach algebras by P. G. Dixon, it was shown that the space of all varieties is a complete metric space. This paper seeks to introduce the space of varieties which are zero at infinity (we call them "zero at infinity varieties") and some of its properties too. Also, some results about its fixed points will be shown. Furthermore, the convex structure of complete metric space of varieties of Banach algebras will be introduced and fixed points of mappings on this space and some of its subspaces will be shown. Now we recall some of the notions and concepts which will be used in this paper.

A non-empty class of complex associative algebras V is a variety if and only if there is a set $\mathfrak{L}$ of polynomials such that,

$$V = \{\mathcal{A} : p(x_1, ..., x_n) = 0, (x_1, ..., x_n \in \mathcal{A}), \forall p \in \mathfrak{L}\}.$$

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Birkhoff proved a therom about varieties of algebras in 1935 [2]. Based on the Birkhoff theorem, a variety of algebras is a non-empty class V of complex associative algebras which is closed under taking subalgebras, quotient algebras, direct sum and isomorphic images. Dixon defined varieties of Banach algebras and proved an analogue of Birkhoff's theorem based on varieties of universal algebras, for Banach algebras[4].

Definition 1.1. [4] If $\mathcal{A}$ is a Banach algebra and $\mathcal{A}_1 = \{x \in \mathcal{A} : \|x\| \leq 1\}$ then we define $\|p\|_{\mathcal{A}} = \sup\{\|p(x_1, ..., x_n)\| : x_i \in \mathcal{A}_1, 1 \leq i \leq n\}$ where $p = p(X_1, ..., X_n)$ is a non-commuting polynomial without constant term.

By a law we mean the formal expression $\|p\| \leq K$, where $K \in \mathbb{R}$ and p is a polynomial. We say that $\mathcal{A}$ satisfies the above law if $\|p\|_{\mathcal{A}} \leq K$. Also, the law $\|p\| \leq K$ is homogeneous if p is a homogeneous polynomial.

Definition 1.2. [4] A non-empty class V of Banach algebras is said to be a variety if there exists a non-negative real-valued function, $p \longrightarrow f(p)$ on the set of all polynomials, such that V is precisely the class of Banach algebras $\mathcal{A}$ for which, $\|p\|_{\mathcal{A}} \leq f(p)$ for each $p = p(X_1, ..., X_n)$.

Theorem 1.3. *([4] theorem 2.3)If V is a non-empty class of Banach algebras then V is closed under taking closed subalgebras, quotient algebras, products(direct sums) and images under isometric isomorphisms if and only if V is a variety.*

Definition 1.4. [5] For each $n \in \mathbb{N}$ such that $n \neq 1$, the variety determined by the law $\|X_1...X_n\| = 0$, is called N_n. If $n = 2$ then N_2 is the variety determined by the law $\|X_1X_2\| = 0$. It is the smallest variety of Banach algebras.

The variety of all Banach algebras is denoted by 1 and this variety is the largest variety of Banach algebras.

Theorem 1.5. *(Teorem 3.3 [5]) If L be the class of all varieties Then N_2 is its minimum and $L \backslash \{N_2, 1\}$ has no maximum and minimum.*

Definition 1.6. [5] Let C be a class of Banach algebras and $V(C)$ the intersection of all varieties containing C. Then, $V(C)$ is a variety called the variety generated by C. If C consists of a single Banach algebra $\mathcal{A}$ then $V(C)$ is written as $V(\mathcal{A})$ and it is said to be singly generated.

Definition 1.7. [5] We define $|P|_V = \sup\{\|P\|_{\mathcal{A}} : \mathcal{A} \in V\}$ where V is a variety and p a polynomial.

The following theorem shows that this supremum is always obtained.

Theorem 1.8. *([5], Theorem 2.4) For each variety V, there is an $\mathcal{A} \in V$ such that, for all Polyomials $p \in P$, $|p|_V = \|p\|_{\mathcal{A}}$.*

Corollary 1.9. *([5], Corollary 2.5) All varieties of Banach algebras are singly generated.*

Corollary 1.10. *([5], Corollary 2.6) Take V_1 and V_2, are varieties. Then, $V_1 \subseteq V_2$ if and only if $|P|_{V_1} \leq |P|_{V_2}$ for all polynomials p. Note that the class of all varieties is a complete lattice by inclusion.*

Definition 1.11. [6] Let L be the lattice of all varieties and L_H the lattice of all H-varieties. Let P be the set of all polynomials, and P_H the set of all homogeneous polynomials. Let P_{NH} be the set of all non-homogeneous polynomials. We define

$$P_1 = \{p \in P : |p| < 1\}$$

$$P_{H1} = \{p \in P_H : |p| < 1\}$$

$$P_{NH1} = \{p \in P_{NH} : |p| < 1\}.$$

If $V \in L$ then we define $\phi_V : P_1 \longrightarrow \mathbb{C}$ such that

$$\phi_V(p) = |p|_V.$$

It is easy to show that the mapping $\phi : L \longrightarrow L^\infty(P_1)$ with $\phi(V) = \phi_V$ is one to one. So, the metric $d_L(V,W) = d(\phi_V, \phi_W) = \|\phi_V - \phi_W\|_\infty = \sup_{p\in P_1} |\phi_V(p) - \phi_W(p)| = \sup_{p\in P_1} \big||p|_V - |p|_W\big|$ on $L^\infty(P_1)$ is induced by L. Therefore, (L, d_L) is a metric space. Similarly, (L_H, d_H) is also a metric space with this metric $d_H(V,W) = \sup_{p\in P_{H1}} \big||p|_V - |p|_W\big|$. Also, (L, d_{NH}) is a metric space with $d_{NH}(V,W) = \sup_{p\in P_{NH1}} \big||p|_V - |p|_W\big|$.

Definition 1.12. If $x \in \mathbb{R}^{\geq 0}$ and $V \neq N_2$ be a variety then V_x is the variety that is determined by these laws $\|p\| \leq x|p|_V$ where p is a homogeneous polynomial.

2. Zero at infinity varieties and fixed points

Definition 2.1. [6] A variety V is said to be a "zero at infinity variety" if for each $\epsilon > 0$ there exists $N > 0$ such that for all $p \in P_{H1}$, if $\deg(p) > N$ then $|p|_V \leq \epsilon$. We show the set of all zero at infinity varieties by L^0.

Definition 2.2. Suppose L is the complete metric space of varieties of Banach algebras. We define $\mathcal{L} = \{V_x : x \in \mathbb{R}^{\geq 0}, V \in L\} = \bigcup_{V\in L}\{V_x : x \in \mathbb{R}^{\geq 0}\}$ and $\mathcal{L}_H = \{V_x : x \in \mathbb{R}^{\geq 0}, V \in L_H\} = \bigcup_{V\in L_H}\{V_x : x \in \mathbb{R}^{\geq 0}\}$. Also, we define $\mathcal{L}^0 = \{V_x : x \in [0,1], V \in L^0\} = \bigcup_{V\in L^0}\{V_x : x \in [0,1]\}$.

Lemma 2.3. *If $\mathcal{L}$, $\mathcal{L}_H$ and $\mathcal{L}^0$ are defined as above then we have $L = \mathcal{L}$, $L_H = \mathcal{L}_H$ and $L^0 = \mathcal{L}^0$.*

Proof. We show that $L = \mathcal{L}$, other items are proved similarly. Let V be a variety of Banach algebras, it is clear that V_x for all $x \in \mathbb{R}^{\geq 0}$ is a variety of Banach algebras (by Definition 1.11), then $\mathcal{L} \subseteq L$. On the other hand, for each $V \in L$ it is clear that $V = V_1$. □

Theorem 2.4. *If V is a variety such that $V \neq N_2$ then the mapping $x \longrightarrow V_x$ from $[0,1]$ into (L_H, d_H) is a strictly increasing injective continuous mapping.*

Proof. If $x = 0$ then there is nothing to prove.
Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}_x$ the Banach algebra $\mathcal{A}$ with a new norm that defined as $x^{-\frac{1}{i}}$ times the norm of $\mathcal{A}$, where $i = \deg(p)$ for all homogeneous polynomial P in P_{H1}. Suppose $V = V(\mathcal{A})$ for some Banach algebra $\mathcal{A}$, obviously $V_x = V(\mathcal{A}_x)$ and

$$\|p\|_{\mathcal{A}_x} = \sup\{\|p(x_1, ..., X_n)\| : x_j \in \mathcal{A} \;\; (1 \leq j \leq n), x^{-\frac{1}{i}}\|x_j\| \leq 1\}$$

$$= \sup\{\|p(x_1, ..., X_n)\| : x_j \in \mathcal{A} \ \ (1 \le j \le n), \|x^{-\frac{1}{i}} x_j\| \le 1\}$$
$$= \sup\{x\|p(x^{-\frac{1}{i}} x_1, ..., x^{-\frac{1}{i}} X_n)\| : x_j \in \mathcal{A} \ \ (1 \le j \le n), \|x^{-\frac{1}{i}} x_j\| \le 1\}$$
$$= x \sup\{\|p(y_1, ..., y_n)\| : y_j \in \mathcal{A} \ \ (1 \le j \le n), \|y_j\| \le 1\}$$
$$= x\|p\|_{\mathcal{A}} \qquad .$$

Thus for any homogeneous polynomial p of degree i, we have $|p|_{V_x} = x|p|_V$. Moreover, if $0 < x_1 < x_2 < 1$ then for any homogeneous p with $|p|_V \neq 0$. If $\deg(p) > 1$ then $|p|_{V_{x_1}} < |p|_{V_{x_2}}$. So, the mapping $x \longrightarrow V_x$ of $[0, 1]$ into the H-varieties is an injective. Now, take $0 < a < 1$, $x_n \in (0, 1)$ and $x_n \longrightarrow a$. Then:

$$\begin{aligned} d_H(V_{x_n}, V_a) &= \sup_{p \in P_{H1}} \big||p|_{V_{x_n}} - |p|_{V_a}\big| \\ &= \sup_{p \in P_{H1}} |x_n - a||p|_V \\ &\le |x_n - a|. \end{aligned}$$

Thus, $V_{x_n} \longrightarrow V_a$. Now, take $x_n \in [0, 1]$ and $x_n \longrightarrow 0$. Put $p \in P_{H1}$ and $\deg(p) > 1$,then:

$$\begin{aligned} \big||p|_{V_{x_n}} - |p|_{N_2}\big| &= |p|_{V_{x_n}} - |p|_{N_2} \\ &\le |p|_{V_{x_n}} \\ &= x_n|p|_V \le x_n \end{aligned}$$

Since $V_0 = N_2$, the mapping is continuous. Let $x_n \in [0, 1]$ and $x_n \longrightarrow 1$ and V be a variety. Hence,$|p|_V$ is bounded for all $p \in P_{H1}$. Also, there exists $N > 0$ such that, if $n > N$ then $|x_n - 1| < \frac{\epsilon}{|p|_V}$.

On the other hand $\big||p|_{V_{x_n}} - |p|_{H(V)}\big| = |p|_V|xn - 1|$. So we have $|p|_V|x_n - 1| \le \epsilon$. Thus, for all $p \in P_{H1}$ and all $n > N$, we have $\big||p|_{V_{x_n}} - |p|_V\big| < \epsilon$. Hence, $V_{x_n} \longrightarrow H(V)$, because $H(V) = V_1$. □

Note that the following results are in [6].

Theorem 2.5. *(Theorem 4.2) The set of all zero at infinity H-varieties L^0 with metric d_H is a closed subspace of (L, d_L).*

Definition 2.6. If V is a zero at infinity variety then we define $[N_2, V] = \{W : N_2 \subseteq W \subseteq V, W \in L\}$. Obviously, each member of $[N_2, V]$ is a zero at infinity varieties.

Theorem 2.7. *(Theorem 4.5) For each zero at infinity variety V the $[N_2, v]$ is a closed set.*

Corollary 2.8. *suppose $V \in L^0$. Then, $[N_2, V]$ for all $n \in \mathbb{N}$ and $n > 2$ is a complete metric subspace.*

Theorem 2.9. *(Theorem 5.2) Let $\alpha \in I$ and $V_a^\alpha \in L^0$. If $C_\alpha = \{V_a^\alpha : 0 \le a \le 1\}$ then $\bigcup_{\alpha \in I} C_\alpha$ is connected.*

Corollary 2.10. *(Corollary 5.3) The set of all zero at infinity varieties is a connected set.*

The next theorem is proved in [6] (Theorem 5.1). Now, we intend to prove it with some changes in its limitations.

Theorem 2.11. *Let V be zero at infinity variety. Then, $\{V_x : 0 \leq x \leq 1\}$ is a path-connected subspace of the metric space (L_H, d_H).*

Proof. Take $0 \leq a < a' \leq 1$. Also, V_a and $V_{a'}$ are two zero at infinity varieties. We define the mapping $f : [0,1] \longrightarrow \{V_a : 0 \leq a \leq 1\}$ as follows:

$$f(t) = V_{a+t(a'-a)}.$$

Then, we have $f(0) = V_a$ and $f(1) = V_{a'}$. Now, we prove that f is a continuous mapping of $[0,1]$ onto $\{V_a : 0 \leq a \leq 1\}$. For all $n \in \mathbb{N}$, if $t_n, t \in [0,1]$ and $t_n \longrightarrow t$ Then:

$$\begin{aligned} d_H(f(t_n), f(t)) &= \sup_{p \in P_{H1}} \big| |p|_{V_{a+t_n(a'-a)}} - |p|_{V_{a+t(a'-a)}} \big| \\ &= \sup_{p \in P_{H1}} \big| [a + t_n(a'-a)]|p|_V - [a + t(a'-a)]|p|_V \big| \\ &= \sup_{p \in P_{H1}} \big| (a + t_n(a'-a)) - (a + t(a'-a)) \big| |p|_V \\ &= \sup_{p \in P_{H1}} |(t_n - t)(a'-a)||p|_V \\ &= \sup_{p \in P_{H1}} |t_n - t||a'-a||p|_V \end{aligned}$$

Since $|a'-a|$ is bounded and $\{|t_n - t|\}_{n=1}^{\infty}$ is convergent to 0, then $d_H(f(t_n), f(t)) \longrightarrow 0$. □

Now, we know that, the class of all zero at infinity varieties is a complete, compact, and connected subspace of L.

Theorem 2.12. *If $\{x_n\}$ is a sequence in $[0,1]$ and $V \in L^0$ then $\{x_n\}$ is a convergent sequence if and only if $\{V_{x_n}\}$ is a convergent sequence in L^0. If $\{x_n\}$ convergs to x then the sequence $\{V_{x_n}\}$ is convergent to V_x.*

Proof. It is obvious. □

Proposition 2.13. *Let (L^0, d_H) be the complete metric space of zero at infinity varieties. If $\{x_n\} \subseteq [0,1]$ be a sequence and $T : L^0 \longrightarrow L^0$ is defined as $T(V_{x_n}) = V_{x_{n+1}}$ for all $V_{x_n}, V_{x_{n+1}} \in L^0$, Then T is a Lipschitzian map.*

Proof. Suppose that $V_{x_n}, V_{y_n} \in L^0$ where $\{x_n\}, \{y_n\}$ are sequences in $[0,1]$. Then, we have:

$$\begin{aligned} d_H(T^k V_{x_n}, T^k V_{y_n}) &= d_H(V_{x_{n+k}}, V_{y_{n+k}}) \\ &= \sup_{p \in P_{H1}} |x_{n+k} - y_{n+k}||p|_V \\ &\leq L \sup_{p \in P_{H1}} |x_n - y_n||p|_V \\ &= L d_L(V_{x_n}, V_{y_n}) \end{aligned}$$

By Archimedean property, existance of L is clear. □

Proposition 2.14. *Let (L^0, d_H) be the complete metric space of zero at infinity varieties and $T : L^0 \longrightarrow L^0$ be a Lipschitzian mapping with constant $k \neq 1$, defined as above. Also, let $\psi : L^0 \longrightarrow (0, \infty)$ defined as $\psi(V_x) = \frac{1}{1-k} d_H(V_x, TV_x)$ for all $V_x \in L^0$. If $\{V_{x_n}\}$ is a sequence in L^0 such that $V_{x_{n+1}} = TV_{x_n}$ and $\{x_n\} \subseteq [0, 1]$. Then, we have:*
i) ψ is continuous.
ii) $d_H(V_{x_n}, V_{x_{n+1}}) = \psi(V_{x_n}) - \psi(V_{x_{n+1}})$ for all $n \in \mathbb{N}_0$.
iii) For all $n \in \mathbb{N}_0$, $d_H(V_{x_n}, V_x) = \psi(V_{x_n}) - \psi(V_x)$.

Proof. i) Suppose $\{V_{x_n}\}$ is a sequence in L^0 such that $V_{x_n} \longrightarrow V_x$, so $d_H(V_{x_n}, V_x) \longrightarrow 0$. Then, we have:

$$\begin{aligned}
|\psi(V_{x_n}) - \psi(V_x)| &= |\frac{1}{1-k} d_H(V_{x_n}, TV_{x_n}) - \frac{1}{1-k} d_H(V_x, TV_x)| \\
&= |\frac{1}{1-k}|\big| \sup_{p \in P_{H1}} ||p|_{V_{x_n}} - |p|_{TV_{x_n}}| - \sup_{p \in P_{H1}} ||p|_{V_x} - |p|_{TV_x}|\big| \\
&\leq |\frac{1}{1-k}| \sup_{p \in P_{H1}} \big||p|_{V_{x_n}} - |p|_{TV_{x_n}} - |p|_{V_x} + |p|_{TV_x}\big| \\
&\leq |\frac{1}{1-k}|(\sup_{p \in P_{H1}} \big||p|_{V_{x_n}} - |p|_{V_x}\big| + \sup_{p \in P_{H1}} \big||p|_{TV_{x_n}} - |p|_{TV_x}\big|) \\
&= \big|\frac{1}{1-k}\big|(d_H(V_{x_n}, V_x) + d_H(TV_{x_n}, TV_x)) \\
&= \big|\frac{1+k}{1-k}\big| d_H(V_{x_n}, V_x).
\end{aligned}$$

Therefore, $\psi(V_{x_n}) \longrightarrow \psi(V_x)$ and consequently ψ is continuous.
ii) If $V_{x_n}, V_{x_{n+1}} \in L^0$ then:

$$\begin{aligned}
\psi(V_{x_n}) - \psi(V_{x_{n+1}}) &= \frac{1}{1-k} d_H(V_{x_n}, TV_{x_n}) - \frac{1}{1-k} d_H(V_{x_{n+1}}, TV_{x_{n+1}}) \\
&= \frac{1}{1-k}(d_H(V_{x_n}, TV_{x_n}) - d_H(V_{x_{n+1}} - TV_{x_{n+1}})) \\
&= \frac{1}{1-k}(d_H(V_{x_n}, TV_{x_n}) - d_H(TV_{x_n} - T(TV_{x_n}))) \\
&= \frac{1}{1-k}(d_H(V_{x_n}, TV_{x_n}) - k d_H(V_{x_n} - TV_{x_n})) \\
&= d_H(V_{x_n}, TV_{x_n}).
\end{aligned}$$

iii) According to Corollary 2.9, it is concluded that there exists a $V_x \in L^0$ such that $V_{x_n} \longrightarrow V_x$. In proof (i) it is proved that ψ is continuous, therefore:

$$\psi(V_{x_n}) \longrightarrow \psi(V_x).$$

Also, we have $d_H(V_{x_n}, V_x) = \psi(V_{x_n}) - \lim_{n\to\infty} \psi(V_{x_m}) = \psi(V_{x_n}) - \psi(V_x)$. □

Theorem 2.15. *Let L^0 be the complete metric space of zero at infinity varieties and $\psi : L^0 \longrightarrow (-\infty, \infty)$ is a proper, bounded below and lower semicontinuous function. Suppose that for each $V_u \in L^0$ with $\inf_{V_x \in L^0} \psi(V_x) < \psi(V_u)$ there exists a $V_w \in L^0$ such that $V_w \neq V_u$ and $d_H(V_u, V_w) \leq \psi(V_u) - \psi(V_w)$. Then, there is a $V_{x_0} \in L^0$ such that $\psi(V_{x_0}) = \inf_{V_x \in L^0} \psi(V_x)$*

Proof. See theorem (2.1)' in [3]. □

Corollary 2.16. *Let L^0 be the complete metric space of zero at infinity varieties and $T : L^0 \longrightarrow L^0$ a Lipschitzian map with constant $k \neq 1$. Let $\psi : L^0 \longrightarrow (\infty, \infty]$ be defined as $\psi(V_x) = \frac{1}{1-k} d_H(V_x, TV_x)$ for all $V_x \in L^0$. Then, we have $\psi(V_{x_0}) = \inf_{V_x \in L^0} \psi(V_x)$ where $\psi(V_{x_0}) = \lim_{n\to\infty} \psi(T^n(V_{x_0})$*

Theorem 2.17. *Let L^0 be the complete metric space of zero at infinity varieties and $\psi : L^0 \longrightarrow (\infty, \infty]$ be a proper bounded below and lower semicontinuous function. Let $T : L^0 \longrightarrow L^0$ be a mapping such that, $d_H(V_x, TV_x) \leq \psi(V_x) - \psi(TV_x)$ for all $V_x \in L^0$. Then, there exists a $V_y \in L^0$ such that $V_y = TV_y$ and $\psi(V_y) < \infty$.*

Proof. It is similar to the proof of Theorem 4.1.3 in [1]. □

Remark 2.18. The fixed point of the mapping T in previous theorem does not need to be unique.

Example 2.19. Suppose L^0 is the complete metric space of zero at infinity varieties and $\psi : L^0 \longrightarrow (-\infty, \infty]$ be defined as $\psi(V_x) = f(x)$ where $f : [0, 1] \longrightarrow (-\infty, \infty]$ is a continuous bijection. Also, let $T : L^0 \longrightarrow L^0$ be defined as $TV_x = V_y$ such that $|x - y| < \frac{f(x)-f(y)}{\sup_{p\in P_{H1}} |p|_V}$ where $y \in [0, 1]$. Then, T has a fixed point. Because:

$$\begin{aligned} d_H(V_x, V_y) &= \sup_{p\in P_{H1}} |x - y||p|_V \\ &\leq \frac{f(x) - f(y)}{\sup_{p\in P_{H1}} |p|_V} \sup_{p\in P_{H1}} |p|_V \\ &\leq f(x) - f(y) \\ &= \psi(V_x) - \psi(V_y) \\ &= \psi(V_x) - \psi(TV_x). \end{aligned}$$

According to theorem 2.16, there is $V_{x_0} \in L^0$ such that $TV_{x_0} = V_{x_0}$.

Theorem 2.20. *Let L^0 be the complete metric space of zero at infinity varieties and $T : L^0 \longrightarrow L^0$ be defined as $TV_x = V_{\alpha(x)}$ where $\alpha : [0, 1] \longrightarrow [0, 1]$ is a function. Then, T has a fixed point if and only if α has a fixed point. Also, if $x_0 \in [0, 1]$ is a fixed point for α then V_{x_0} is a fixed point for T.*

Proof. If T has a fixed point as $V_x \in L^0$ then $TV_x = V_x$. Therefore, $V_x = V_{\alpha(x)}$ and $V_x \subseteq V_{\alpha(x)}$ so, $|p|_{V_x} \leq |p|_{V_{\alpha(x)}}$. As a result $\sup_{p\in P_{H1}} x|p|_V \leq \sup_{p\in P_{H1}} \alpha(x)|p|_V$ and $x \leq \alpha(x)$. Similarly, it is obtained that $\alpha(x) \leq x$. Hence $\alpha(x) = x$ and α has a fixed point. Inverse is obvious. □

Theorem 2.21. *Let L^0 be the complete metric space of zero at infinity varieties and $T : L^0 \longrightarrow L^0$ a contraction mapping with Lipschitzian constant $k \in (0, 1)$. Then, we have the followings:*
i) There exists a unique fixed point $V_x \in L^0$ for T.
ii) For arbitrary $V_x \in L^0$ the picard iteration process is defined by

$$V_{x_{n+1}} = TV_{x_n}$$

for all $n \in \mathbb{N}$ *converge to* V_x.
iii) For all $n \in \mathbb{N}$ *we have* $d_H(V_{x_n}, V_{x_0}) \leq \frac{k^n}{1-k} d_H(V_{x_1}, V_{x_0})$.

Proof. Its proof is similar to the proof of Theorem 4.1.5 in [1]. □

Lemma 2.22. *Let* $\{x_n\}$ *be a increasing (decreasing) sequence in* $\mathbb{R}$ *and for each* $n \in \mathbb{N}$ *we have* $x_n > \frac{x_{n+1}+x_{n-1}}{2}$ $(x_n < \frac{x_{n+1}+x_{n-1}}{2})$. *Then, for each* $m, l \in \mathbb{N}$ *we have,* $|x_{m+n} - x_{l+n}| < |x_m - x_l|$ *such that* $n \in \mathbb{N}$.

Proof. It is obvious by induction. □

Theorem 2.23. *Let* L^0 *be the complete metric space of zero at infinity varieties and* $\{x_n\} \subseteq [0,1]$ *be a decreasing sequence such that* $x_n < \frac{x_{n+1}+x_{n-1}}{2}$ *for each* $n \in \mathbb{N}$. *If* $T : L^0 \longrightarrow L^0$ *is defined as* $TV_{x_n} = V_{x_{n+1}}$ *where* $\{V_{x_n}\}$ *is a sequnce in* L^0 *then we have:*
i) T *is a contraction.*
ii) There exists a unique fixed point $V_x \in L^0$ *for* T *where* $x = \lim_{n\to\infty} x_n$.
iii) For arbitrary $V_{x_0} \in L^0$ *the picard iteration process is convergent to* V_x.
iv) For all $n \in \mathbb{N}$, *it is proved that* $d_H(V_{x_n}, V_x) \leq \frac{k^n}{1-k} d_H(V_{x_0}, V_{x_1})$ *where* $k \in (0,1)$ *is the Lipschitz constant of* T.

Proof. Part (i) must be proved, and the rest of the cases according to the previous theorem are obvious. Let $n \in \mathbb{N}$ and $V_{x_m}, V_{x_l} \in L^0$ where m, l are natural numbers.Then:

$$\begin{aligned} d_H(T^n V_{x_m}, T^n V_{x_l}) &= d_H(V_{x_{m+n}}, V_{x_{l+n}}) \\ &= \sup_{p \in P_{H1}} \big| |p|_{V_{x_{m+n}}} - |p|_{V_{x_{l+n}}} \big| \\ &= \sup_{p \in P_{H1}} |x_{m+n} - x_{l+n}| |p|_V \\ &< \sup_{p \in P_{H1}} |x_m - x_l| |p|_V \\ &= \sup_{p \in P_{H1}} \big| |p|_{V_{x_m}} - |p|_{V_{x_l}} \big| \\ &= d_H(V_{x_m}, V_{x_l}). \end{aligned}$$

So, there exists $k \in (0,1)$ such that

$$d_H(T^n V_{x_m}, T^n V_{x_l}) \leq k d_H(V_{x_m}, V_{x_l}).$$

Therefore, T is a contraction mapping. □

Example 2.24. Let L^0 be the complete metric space of zero at infinity varieties and $T : L^0 \longrightarrow L^0$ be a mapping defined as

$$TV_x = V_{\frac{x}{2}}.$$

Then, T is a contraction, because

$$\begin{aligned} d_H(TV_x, TV_y) &= d_H(V_{\frac{x}{2}}, V_{\frac{y}{2}}) \\ &= \sup_{p \in P_{H1}} \big| \frac{x}{2} |p|_V - \frac{y}{2} |p|_V \big| \\ &= \frac{1}{2} \sup_{p \in P_{H1}} \big| x|p|_V - y|p|_V \big| \end{aligned}$$

$$= \frac{1}{2} d_H(V_x, V_y).$$

Also, T is a uniformely Lipschitzian mapping. Hence, by previous theorem it has a fixed point in L^0.

Example 2.25. Let $T : L^0 \longrightarrow L^0$ be a mapping defined as

$$TV_x = V_{1-x^2}$$

for all $x \in [0, 1]$. Then, T is non-contraction but it has a fixed point. Because

$$\begin{aligned} d_H(TV_x, TV_y) &= d_H(V_{1-x^2}, V_{1-y^2}) \\ &= \sup_{p \in P_{H1}} |x^2 - y^2||p|_V \\ &= \sup_{p \in P_{H1}} |x - y||x + y||p|_V. \end{aligned}$$

If $x > \frac{1}{2}, y > \frac{1}{2}$ then:

$$\begin{aligned} d_H(TV_x, TV_y) &= \sup_{p \in P_{H1}} |x - y||x + y||p|_V \\ &> \sup_{p \in P_{H1}} |x - y||p|_V \end{aligned}$$

according to previous relations $d_H(TV_x, TV_y) > d_H(V_x, V_y)$. Therefore, T is non-contraction but it has a fixed point, because

$$TV_{\frac{\sqrt{5}-1}{2}} = V_{\frac{\sqrt{5}-1}{2}}.$$

3. CONVEX STRUCTURE OF L

In this section, we try to show that L is a convex metric space, also we introduce its convex structure.

Definition 3.1. [1] Let C be a non-empty subset of a metric space X and $T : C \longrightarrow C$ a mapping. Then, the sequence $\{x_n\}$ in C is said to be an approximating fixed point sequence of T if $\lim_{n \longrightarrow \infty} d(x_n, Tx_n) = 0$.

Henceforth, approximating fixed point sequence is denoted by AFPS, in short.

Example 3.2. Let $V \in L$ and $C = \{V_x : x \in \mathbb{R}^{\geq 0}\}$. If $\{x_n\}$ is a cauchy sequence in $\mathbb{R}^{\geq 0}$ and $T : C \longrightarrow C$ defined as $TV_{x_n} = V_{x_{n+1}}$ then $\{x_n\}$ is a AFPS of T.

Based on the Banach contraction principale theorem, every contraction mapping has an AFPS in a metric space. In fact, this AFPS is the picard iterative sequence of contraction mapping T. But the picard iterative sequence is not necessarily an AFPS of non-expansive mappings.

Example 3.3. Let $X = L$ and $T : L \longrightarrow L$ be a mapping defined by

$$TV_x = V_{1-x}$$

for all $V_x \in L$. Clearly, T is non-expansive mapping with $F(T) = \{V_{\frac{1}{2}}\}$. However, for all $V_{x_0} \neq V_{\frac{1}{2}}$ the iterative sequence of the Picard iteration process is

$$V_{x_{n+1}} = TV_{x_n} = V_{1-x_n}, \quad n \in \mathbb{N}.$$

Now,

$$\begin{aligned} d_L(V_{x_n}, TV_{x_n}) &= d_L(T^n V_{x_0}, T^{n+1} V_{x_0}) \\ &= \sup_{p\in P_1} |1 - x_0 - x_0||p|_V \\ &= |1 - 2x_0| \sup_{p\in P_1} |p|_V \nrightarrow 0, \quad n \longrightarrow \infty. \end{aligned}$$

Definition 3.4. [7] Suppose (X, d) is a metric space. A continuous mapping $W : X \times X \times [0, 1] \longrightarrow X$ is said to be a convex structure on X if for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$ the following condition is satisfied:

$$d\big(u, W(x, y; \lambda)\big) \leq \lambda d(x, y) + (1 - \lambda)d(x, y). \text{ for all } u \in X$$

A metric space X with convex structure is called a convex metric space.

Also, a subset C of a convex metric space X with convex structure W on it is said to be convex if $W(x, y; \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. A convex metric space X is said to have property (B) if $d(W(u, x; \lambda), W(u, y; \lambda)) = (1 - \lambda)d(x, y)$ for all $u, x, y \in X$ and $\lambda \in (0, 1)$.

Theorem 3.5. *The metric space $X = (L, d_L)$ is a convex metric space.*

Proof. Let $X = L$. For all $V_x, V_y \in X$ and $\lambda \in [0, 1]$, we define $W : X \times X \times [0, 1] \longrightarrow X$ such that ,

$$W(V_x, V_y; \lambda) = V_{\lambda x+(1-\lambda)y}.$$

Obviously W is continuous. For each $V_u \in X$ and $V_x, V_y \in X, \lambda \in [0, 1]$, we will have:

$$\begin{aligned} d_L(V_u, W(V_x, V_y; \lambda)) &= d_L(V_u, V_{\lambda x+(1-\lambda)y}) \\ &= \sup_{p\in P_1} |u - (\lambda x + (1 - \lambda)y)||p|_V \\ &\leq \lambda \sup_{p\in P_1} |u - x||p|_V + (1 - \lambda) \sup_{p\in P_1} |u - y||p|_V \\ &= \lambda d_L(V_u, V_x) + (1 - \lambda)d_L(V_u, V_y) \end{aligned}$$

□

Corollary 3.6. *Subspaces (L_H, d_H) and (L^0, d_H) of cmplete convex metric space (L, d_L) are convex.*

Proof. Let $X = L_H(or L^0)$, if $V_x, V_y \in L_H(or L^0)$ and $\lambda \in [0, 1]$ then we define $W : X \times X \times [0, 1] \longrightarrow X$ such that $W(V_x, V_y; \lambda) = V_{\lambda x+(1-\lambda)y}$. Obviously W is the convex structure of $L_H(or L^0)$. □

Proposition 3.7. *Let $X = L$. The following are established:*
i) For all $V_x, V_y \in L$ we have $d_L(V_x, V_y) = d_L(V_{|x-y|}, N_2)$.
ii) For all $V_x, V_y \in L$ and $\lambda \in [0, 1]$ it is concluded that:

$$d_L(V_{\lambda x+(1-\lambda)y}, N_2) = \lambda d_L(V_x, N_2) + (1 - \lambda)d_L(V_y, N_2).$$

Proof. i) Take $V_x, V_y \in L$. Then:

$$\begin{aligned} d_L(V_x, V_y) &= \sup_{p\in P_1} \big| |p|_{V_x} - |p|_{V_y} \big| \\ &= \sup_{p\in P_1} |x-y||p|_V \\ &= \sup_{p\in P_1} \big| |x-y| - 0 \big| |p|_V \\ &= \sup_{p\in P_1} \big| |p|_{|x-y|} - |p|_{V_0} \big| \\ &= d_L(V_{|x-y|}, N_2). \end{aligned}$$

ii) For each $V_x, V_y \in L$ and $\lambda \in [0,1]$, we have:

$$\begin{aligned} d_L(V_{\lambda x+(1-\lambda)y}, N_2) &= \sup_{p\in P_1} \big| |p|_{V_{\lambda x+(1-\lambda)y}} - |p|_{N_2} \big| \\ &= \sup_{p\in P_1} |\lambda x + (1-\lambda)y - 0||P|_V \\ &= \lambda \sup_{p\in P_1} x|P|_V + (1-\lambda) \sup_{p\in P_1} y|P|_V \\ &= \lambda d_L(V_x, N_2) + (1-\lambda) d_L(V_y, N_2). \end{aligned}$$

□

Definition 3.8. Suppose $X = L$. For all $V_x \in L$ the open ball $\mathcal{B}_r(V_x)$ and the closed ball $\mathcal{B}_r[V_x]$ of L are defined as follows:

$$\mathcal{B}_r(V_x) = \{V_y \in L : d_L(V_x, V_y) < r\}$$

and

$$\mathcal{B}_r[V_x] = \{V_y \in L : d_L(V_x, V_y) \leq r\}$$

Theorem 3.9. *Open and closed balls in L are its convex subsets.*

Proof. If $V_u, V_w \in \mathcal{B}_r(V_x)$ and $\lambda \in [0,1]$ then:

$$\begin{aligned} d_L(V_x, W(V_u, V_w; \lambda)) &\leq \lambda d_L(V_x, V_u) + (1-\lambda) d_L(V_x, V_w) \\ &< \lambda r + (1-\lambda) r = r. \end{aligned}$$

Therefore, $W(V_u, V_w) \in \mathcal{B}_r(V_x)$ it means that $\mathcal{B}_r(V_x)$ is convex. Similarly, $\mathcal{B}_r[V_x]$ is convex. □

Theorem 3.10. *If $X = L$ then L has (B) property.*

Proof. Let $W : X \times X \times [0,1] \longrightarrow X$ be determined by $W(V_x, V_y; \lambda) = V_{\lambda x+(1-\lambda)y}$. For each $V_x, V_y, V_u \in L$ and $\lambda \in (0,1)$ it is concluded that:

$$\begin{aligned} d_L(W(V_u, V_x; \lambda), W(V_u, V_y; \lambda) &= d_L(V_{\lambda u+(1-\lambda)x}, V_{\lambda u+(1-\lambda)y}) \\ &= \sup_{p\in P_1} |(\lambda u + (1-\lambda)x) - (\lambda u + (1-\lambda)y)||p|_V \\ &= (1-\lambda) \sup_{p\in P_1} |x-y||p|_V \\ &= (1-\lambda) d_L(V_x, V_y). \end{aligned}$$

□

Proposition 3.11. *Let $X = L$. Then, for all $V_x, V_y \in L$ and $\lambda \in [0,1]$ we have:*

$$d_L(V_x, V_y) = d_L(V_x, W(V_x, V_y; \lambda)) + d_L(W(V_x, V_y; \lambda), V_y).$$

Proof. Refer to Proposition 3 in [7]. $\square$

Definition 3.12. For each subset C of the convex metric space (L, d_L), we define

$$diam(C) = \sup\{d_L(V_x, V_y) : V_x, V_y \in C\}.$$

Proposition 3.13. *Suppose $X = L$ and C is a non-empty closed convex subset of L, also $T : C \longrightarrow C$ be a non-expansive mapping. Then:*
i) take $V_u \in C$ and $t \in (0,1)$. There exists exactly one point $V_{x_t} \in C$ such that $V_{x_t} = W(V_u, TV_{x_t}; 1-t)$.
ii) If C is bounded then $d_L(V_{x_t}, TV_{x_t}) \to 0$ as $t \to 1$; namely, T has an AFPS.

Proof. i) Let $t \in (0,1)$ and $T_t : C \longrightarrow C$ is defined as $T_tV_x = W(V_u, TV_x; 1-t)$. Regarding to Theorem 3.10, we have:

$$d_L(T_tV_x, T_tV_y) = td_L(TV_x, TV_y) \leq td_L(V_x, V_y).$$

Therefore, T_t is a contraction and by Banach contraction principle theorem, it has a unique fixed point as V_{x_t} in C. So $V_{x_t} = W(V_u, TV_{x_t}; 1-t)$.
ii)According to boundedness of C, we have:

$$\begin{aligned} d_L(V_{x_t}, TV_{x_t}) &= d_L(TV_{x_t}, W(V_u, TV_{x_t}; 1-t)) \\ &\leq (1-t)d_L(TV_{x_t}, V_u) \\ &\leq (1-t)diam(C) \to 0 \quad as \quad t \to 1. \end{aligned}$$

$\square$

Theorem 3.14. *Take $X = L$ and C as a non-empty complete convex subset of L. If $T : C \longrightarrow C$ is a non-expansive mapping then T has a fixed point in C.*

Proof. Based on Proposition 3.13, there exists an AFPS as $\{V_{x_n}\}$ in C. By compactness of C, we have a subsequnce of $\{V_{x_n}\}$ as $\{V_{x_{n_k}}\}$ such that $V_{x_{n_k}} \longrightarrow V \in C$. therefore, $V = TV$. $\square$

Example 3.15. If $V \in L^0$ and $C = \{V_x : x \in [a,b]\}$ then C is a non-empty complete convex subset of L^0 and each non-expansive mapping as $T : C \longrightarrow C$ has a fixed point on C.

Definition 3.16. [1] Let X be a metric space and C a subset of X. Then, for $x \in C$ we define:

$$\begin{aligned} r_x(C) &= \sup\{d(x,y) : y \in C\}. \\ r(C) &= \inf\{r_x(C) : x \in C\}. \\ \mathcal{Z}_C &= \{x \in C : r_x(C) = r(C)\}. \end{aligned}$$

A point $x_0 \in C$ is said to be a diametral point of C if $\sup\{d(x_0, y) : y \in C\} = diam(C)$.

Lemma 3.17. *Take $C \subseteq L$. if $\Psi : C \longrightarrow \mathbb{R}^{\geq 0}$ is defined as $\Psi(V_x) = x$ then Ψ is a homeomorphism between C and $\Psi(C)$.*

Proposition 3.18. *Let $C \subseteq L$ and $C_0 = \Psi(C)$ where $\Psi : C \longrightarrow \mathbb{R}^{\geq 0}$ is determined by $\Psi(V_x) = x$. The followings are satisfied:*
i) For each $V_x \in C$ we have $r_{V_x}(C) = r_x(C_0). \sup_{p\in P_1} |p|_V$.
ii) $r(C) = r(C_0). \sup_{p\in P_1} |p|_V$.
iii) $\mathcal{Z}_C = \{V_x : x \in \mathcal{Z}_{C_0}\}$.

Proof. i) If $V_x \in C$ then:

$$\begin{aligned} r_{V_x}(C) &= \sup\{d_L(V_x, V_y) : V_y \in C\} \\ &= \sup\{\sup_{p\in P_1} |x-y||p|_V : y \in C_0\} \\ &= \sup_{p\in P_1} |p|_V (\sup\{d(x,y) : y \in C_0\}) \\ &= \sup_{p\in P_1} |p|_V . r_x(C_0) \end{aligned}$$

where $x \in C_0$.
ii) The proof is similar to that of (i).
iii) It is concluded from (i) and (ii). □

Proposition 3.19. *Suppose $C \subseteq L$. Then, $V_{x_0} \in C$ is a diametral point of C if and only if x_0 is a diametral point of C_0.*

Proof. Each point of C as V_{x_0} is a diametral point iff $r_{V_{x_0}}(C) = diam(C)$ iff $r_{x_0}(C). \sup_{p\in P_1} |p|_V = diam(C_0). \sup_{p\in P_1} |p|_V$ iff $r_{x_0}(C_0) = diam(C_0)$. iff x_0 is a diametral point of C_0. □

Definition 3.20. [7] A convex metric space X is said to have normal structure if for each closed convex bounded subset C of X that contains at least two points, there exists $x_0 \in C$ such that, it is not a diametral point of C.

Theorem 3.21. *the metric space of varieties of Banach algebras L has normal structure.*

Proof. Suppose that L does not have normal structure. Then, there exists a closed convex bounded subset C of L with at least two points such that every point of it is diametral. So, for each $V_x \in C$, it is a diametral point of C.
By Proposition 3.19, x is a diametral point of C_0 where $C_0 = \Psi(C)$ and $\Psi : C \longrightarrow \mathbb{R}^{\geq 0}$ is determined by $\Psi(V_x) = x$. We know that Ψ is a homeomorphism between C and C_0, it means that C_0 is a closed convex bounded subset of $\mathbb{R}^{\geq 0}$ such that, it has at least two points. Therefore, $C_0 = [a, b]$ where $a, b \in \mathbb{R}^{\geq 0}$ and $a \neq b$. However $\sup\{d(x, y) : y \in C_0\} = diam(C_0)$. So

$$\sup_{y\in C_0} |x-y| = \sup_{z,t\in C_0} |z-t| \tag{3.1}$$

for each $x \in C_0$. If $x = \frac{b+a}{2}$ then $\sup_{y\in[a,b]} |x-y| = \sup_{y\in[a,b]} |\frac{b+a}{2} - y| = \frac{b-a}{2}$ and

$$\sup_{z,t\in[a,b]} |z-t| = b-a. \tag{3.2}$$

According to (3.1) and (3.2), we have $b - a = \frac{b-a}{2}$ and it is a contradiction. □

Proposition 3.22. *([7] proposition 5) Let C be a non-empty compact subset of a convex metric space X and D be the least closed convex set containing C. If $diam(C) > 0$ then there exists an element $X_0 \in D$ such that*

$$\sup\{d(x, x_0) : x \in C\} < diam(C).$$

Theorem 3.23. *Every compact convex metric space has normal structure.*

Proof. Suppose X is a compact convex metric space and C is an arbitrary closed convex bounded subset of X contains at least two points. Take $x, y \in C$, So $diam(C) > d(x, y) \geq 0$. However, C is a compact subset of X and according to previous proposition C has a point as x_0 such that $diam(C) > \sup\{d(x, x_0); x \in C\}$. □

Example 3.24. It is clear that L^0 is a compact convex metric subspace of L, so by previous theorem it has normal structure.

Definition 3.25. [7] A convex metric space X is said to have property(C) if every bounded decreasing net of non-empty closed convex subset of X has a non-empty intersection.

Example 3.26. It is clear that L, L^0 have property(C), Since all bounded decreasing net of non-empty closed convex subsets of L or L^0 as $\{C_\alpha\}_{\alpha\in\Lambda}$ such that $\bigcap_{\alpha\in\Lambda} C_\alpha = \phi$. Define $\Psi : L \longrightarrow \mathbb{R}^{\geq 0}$ with $\Psi(V_x) = x$. Then $\{\Psi(C_\alpha)\}_{\alpha\in\Lambda}$ is a bounded decreasing net of non-empty closed convex subsets of $\mathbb{R}^{\geq 0}$ and we have $\bigcap_{\alpha\in\Lambda} \Psi(C_\alpha) = \Psi(\bigcap_{\alpha\in\Lambda} C_\alpha) = \Psi(\phi) = \phi$ and it is a contradiction.

Proposition 3.27. *([7] proposition 4) If convex metric space X has property(C) then $\mathcal{Z}_C$ is a non-empty closed and convex.*

Example 3.28. Regarding L, we have:

$$\begin{aligned}\mathcal{Z}_L &= \{V_x : r_x(\Psi(L)) = r(\Psi(L))\}\\ &= \{V_x : r_x(\mathbb{R}^{\geq 0}) = \infty\}.\end{aligned}$$

Therefore, $\mathcal{Z}_L = L$.
Also for L^0 we have:

$$\mathcal{Z}_{L^0} = \{V_x : r_x(\Psi(L^0)) = r(\Psi(L^0))\}.$$

On the other hand,

$$\begin{aligned}r(\Psi(L^0)) &= r([0, 1])\\ &= \inf\{r_x([0, 1]); x \in [0, 1]\}\\ &= \inf_{x\in[0,1]}(\sup_{y\in[0,1]} |x - y|)\\ &= \inf_{x\in[0,1]}[\frac{1}{2}, 1] = \frac{1}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{Z}_{L^0} &= \{V_x : r_x([0, 1]) = \frac{1}{2}\}\\ &= \{V_x : \sup_{y\in[0,1]} |x - y| = \frac{1}{2}\}\end{aligned}$$

$$= \{V_x : \sup_{y\in[0,1]} |x-y| = \frac{1}{2}\} = \{V_{\frac{1}{2}}\}.$$

References

1. R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Vol. 6 Springer Dordrecht, Heidelberg, London, 2009.
2. G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge. Philos. Soc **31** (1935), 433–454.
3. J. Caristi, *Fixed point theorems for mappings satisfying inwardnes conditions*, Trans. Amer. Math. Soc **215** (1976), 241–252.
4. P. G. Dixon, *Varieties of Banach algebras*, Quart. J. Math. Oxford, Ser,(2) **27** (1976), 481-487.
5. M. H. Faroughi, *Uncountable chains and anti chains of varieties of Banach algebras*, J. Math. Anal. Appl. **168** (1992), no. 1, 184–194.
6. G. Khalilzadeh and M. H. Faroughi, *The Compact Metric Space of the Lattice of Varieties and *-varieties of C^*-algebras*, Turkish J. Math. **31** (2007), no. 4, 379–393.
7. W. Takahashi, *A convexity in metric space and nonexpansive mappings*, I. Kōdai Math. Sem. Rep. **22** (1970), 142–149.

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