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PSEUDOSPECTRA OF ELEMENTS OF REDUCED BANACH ALGEBRAS

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ABSTRACT. Let A be a Banach algebra with identity 1 and $p \in A$ be a non-trivial idempotent. Then $q = 1-p$ is also an idempotent. The subalgebras pAp and qAq are Banach algebras, called reduced Banach algebras, with identities p and q respectively. For $a \in A$ and $\varepsilon > 0$, we examine the relationship between the ε-pseudospectrum $\Lambda_\varepsilon(A, a)$ of $a \in A$, and ε-pseudospectra of $pap \in pAp$ and $qaq \in qAq$. We also extend this study by considering a finite number of idempotents $p_1, \cdots, p_n$, as well as an arbitrary family of idempotents satisfying certain conditions.

1. INTRODUCTION

In this note, we study the decomposition of the spectrum and pseudospectrum of an element in a Banach algebra into the spectra and pseudospectra of its corresponding elements in reduced Banach algebras.

Suppose T is a bounded operator on a Hilbert space H that can be expressed as the direct sum of two closed subspaces H_1 and H_2. Let P_1 and $P_2 = I - P_1$ be the bounded linear projections onto H_1 and H_2 respectively. The operator T commutes with P_1 and P_2 if and only if H_1 and H_2 are invariant under T. In this case T can be expressed as the direct sum $T_1 \oplus T_2$, where $T_1 = P_1T \restriction_{H_1}$ and $T_2 = P_2T \restriction_{H_2}$. It is easily seen that $\sigma(B(H), T) = \sigma(B(H_1), T_1) \cup \sigma(B(H_2), T_2)$.

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This can be extended to a finite direct sum of operators. See Problem 98 in [6]. The case of infinite direct summands is a little different. See Theorem 2.3 in [4].

This can also be extended to the decomposition of the spectrum of an element in an arbitrary Banach algebra. If $p \in A$ is an idempotent in a Banach algebra, then $pAp = \{pap : a \in A\}$ is a closed subalgebra of A called a reduced Banach algebra. If $q = 1-p$, then q is an idempotent too, and thus qAq is also a closed subalgebra of A. Then for $a \in A$ that commutes with p, $\sigma(A, a) = \sigma(pAp, pa) \cup \sigma(qAq, qa)$. A similar result is true for the case of a finite number of idempotents $\{p_i\}_{i=1}^n$ satisfying $\sum_{i=1}^{n} p_i = 1$ (see Lemmas 3.1 and 3.2).

Further, we would like to examine if such a decomposition holds for the ε-pseudospectrum $\Lambda_\varepsilon(A, a)$ of $a \in A$. We show that if $p_1, \cdots, p_n$ are idempotents in A such that $\sum_{i=1}^{n} p_i = 1$, and $a \in A$ commutes with each p_i, then

$$\Lambda_{\frac{\varepsilon}{n}}(A, a) \subseteq \bigcup_{i=1}^{n} \Lambda_\varepsilon(p_i A p_i, p_i a) \subseteq \Lambda_{\max\limits_i \|p_i\| \varepsilon}(A, a).$$

See Theorem 3.3.

We then study specific cases in which

$$\Lambda_\varepsilon(A, a) = \bigcup_{i=1}^{n} \Lambda_\varepsilon(p_i A p_i, p_i a). \tag{1.1}$$

This happens, for instance, if a is a G_1 element or if each p_i is a self adjoint projection in a C^*-algebra. See Theorems 3.6 and 3.7. The decomposition (1.1) occurs precisely when $\|(\lambda - a)^{-1}\| = \max\limits_i \|p_i(\lambda - a)^{-1}\|$. We hence consider those Banach algebras and idempotents $\{p_i\}$ that satisfy $\|a\| = \max\limits_i \|p_i a\|$ for all $a \in A$. We use a result from [11] to show that for the Banach algebra of bounded linear operators on a Banach space X, and $P_i \in B(X)$ idempotent operators such that $P_i P_j = 0,\ i \neq j$ and $\sum_{i=1}^{n} P_i = I$, this property is satisfied precisely when the norm on X satisfies the following:

$$\|P_i x\| = \|P_i y\| \ \text{ for all } i \Rightarrow \|x\| = \|y\|.$$

Suppose A is a Banach algebra, $p_1, \cdots, p_n \in A$ are idempotents such that $\sum_{i=1}^{n} p_i = 1$ and $p_i p_j = 0,\ i \neq j$ and $\|a\| = \max\limits_i \|p_i a\|$ for all $a \in A$. Then we show that each p_i is Hermitian and (1.1) is satisfied. See Theorems 3.17 and 3.18. However, each p_i being Hermitian is not sufficient for (1.1) to occur as seen in Example 3.15.

In Section 4, we examine the infinite case. We consider a family of idempotent operators on a Hilbert space or a Banach space X, that converges strongly to the identity operator and consider the decomposition of the spectrum and pseudospectrum of $T \in B(X)$. The spectrum of $T \in B(X)$ is in general larger that the union of the spectra of $P_n T$, the difference in the two sets arising from those complex numbers λ for which $\sup\limits_n \|(\lambda P_n - P_n T)^{-1}\| = \infty$. Similarly, the pseudospectrum of T may properly contain the union of the pseudospectra of its reduced components. The difference in the two sets arises from those complex numbers λ for which $\|(\lambda - T)^{-1}\| = \frac{1}{\varepsilon}$. In special cases, the pseudospectrum of T is equal to the closure of the union of the pseudospectra of its reduced components. See Theorem 4.4 and Remark 4.5.

The primary objective of this note is to show that the pseudospectra of certain elements of Banach algebras can be decomposed into the pseudospectra of elements of certain reduced subalgebras. This could make it easier to compute the pseudospectra of certain operators. Examples 4.10 and 4.11 are simple illustrations of this.

2. Definitions

Let A be a complex Banach algebra with unit 1. For $\lambda \in \mathbb{C}$, $\lambda.1$ is identified with λ. Let $\mathrm{Inv}(A) = \{x \in A : x \text{ is invertible in } A\}$ and $\mathrm{Sing}(A) = \{x \in A : x \text{ is not invertible in } A\}$.

Definition 2.1. The *spectrum* of an element $a \in A$ is defined as:

$$\sigma(A, a) := \{\lambda \in \mathbb{C} : \lambda - a \in \mathrm{Sing}(A)\}.$$

Definition 2.2. The *spectral radius* of an element $a \in A$ is defined as:

$$r(A, a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Definition 2.3. Let $\varepsilon > 0$. The *ε-pseudospectrum* $\Lambda_\varepsilon(a)$ of $a \in A$ is defined by

$$\Lambda_\varepsilon(A, a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| \geq \varepsilon^{-1}\}$$

with the convention that $\|(\lambda - a)^{-1}\| = \infty$ if $\lambda - a$ is not invertible.

If the algebra A is fixed or is clear from the context, then we use simplified notations $\sigma(a), r(a)$ and $\Lambda_\varepsilon(a)$ in place of $\sigma(A, a), r(A, a)$ and $\Lambda_\varepsilon(A, a)$ respectively. The basic reference for pseudospectrum, especially for matrices, is the book [17]. The ε-pseudospectrum of an element of an arbitrary Banach algebra has been studied in [9].

Definition 2.4. Let A be a Banach algebra and $a \in A$. The *numerical range* (see Definition 1.10.1 in [2]) of a is defined by

$$V(a) := \{f(a) : f \in A', f(1) = 1, \|f\| = 1\},$$

where A' is the dual space of A.

Definition 2.5. Let A be a Banach algebra and $a \in A$. Then a is said to be *Hermitian* if $V(a) \subseteq \mathbb{R}$.

Definition 2.6. Let A be a Banach algebra and $p \in A$ be an idempotent. Then $pAp = \{pap : a \in A\}$ is a closed subalgebra of A called a reduced Banach algebra.

3. Reduced Banach Algebras

Let A be a Banach algebra and $p \in A$ be an idempotent element, that is, $p = p^2$. We shall always assume that p is non-trivial, that is, $p \neq 0$ and $p \neq 1$. Let $q = 1 - p$. Now, suppose $a \in A$ is such that $ap = pa$. Then observe that $qa = aq$. We observe that $\|p\|, \|q\| \geq 1$. We examine the relationships between the spectrum and pseudospectrum of $a \in A$ and the spectra and pseudospectra of $pa = pap \in pAp$ and $qa = qaq \in qAq$.

We show that $\sigma(A, a) = \sigma(pAp, pa) \cup \sigma(qAq, qa)$. This is an elementary result and a version of it in the case of Hilbert space operators can be found in Problem 98 of [6]. We provide the proof here for the sake of completeness.

Lemma 3.1. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $a \in A$ be such that $ap = pa$. Then*

$$\sigma(A, a) = \sigma(pAp, pa) \cup \sigma(qAq, qa).$$

Thus, $r(a) = \max\{r(pAp, pa), r(qAq, qa)\}$.

Proof. Without loss of generality, we assume that the element considered in the spectrum or union of spectra is $\lambda = 0$.

If a is invertible in A, then pa is invertible in pAp with inverse pa^{-1} and qa is invertible in qAq with inverse qa^{-1}. Hence

$$\sigma(pAp, pa) \cup \sigma(qAq, qa) \subseteq \sigma(A, a).$$

Conversely, suppose pa is invertible in pAp with inverse pbp and qa is invertible in qAq with inverse qcq. Then

$$pa(pbp) = pabp = p = pbap = (pbp)ap, \tag{3.1}$$

and

$$qacq = q = qcaq. \tag{3.2}$$

Adding (3.1) and (3.2) gives

$$pabp + qacq = p + q = pbap + qcaq.$$

Simplifying, we get

$$a(pbp + qcq) = 1 = (pbp + qcq)a.$$

Hence $\sigma(A, a) \subseteq \sigma(pAp, pa) \cup \sigma(qAq, qa)$. We also observe that the inverses of $\lambda p - pa$ and $\lambda q - qa$, when they exist, in pAp and qAq respectively are $(\lambda - a)^{-1}p$ and $(\lambda - a)^{-1}q$ respectively. □

Next, suppose $p_1, \cdots, p_n \in A$ such that $p_i^2 = p_i$ for all i and $\sum_{i=1}^{n} p_i = 1$. Suppose $ap_i = p_i a$ for all i. Then just as in the case of two idempotents, it can be easily shown that $\sigma(A, a) = \bigcup_{i=1}^{n} \sigma(p_i A p_i, p_i a)$. The same is mentioned in the case of Hilbert space operators in Problem 98 of [6].

Lemma 3.2. *Let A be a unital Banach algebra and $p_1, \cdots, p_n \in A$ such that $p_i^2 = p_i$ for all i and $\sum_{i=1}^{n} p_i = 1$. Suppose $ap_i = p_i a$ for all i. Then*

$$\sigma(A, a) = \bigcup_{i=1}^{n} \sigma(p_i A p_i, p_i a),$$

and

$$r(A, a) = \max_i r(p_i A p_i, p_i a).$$

We now consider the decomposition of the pseudospectrum into the pseudospectra of the corresponding elements of the reduced Banach algebras.

Theorem 3.3. *Let A be a unital Banach algebra and $p_1, \cdots, p_n \in A$ such that $p_i^2 = p_i$ for all i and $\sum_{i=1}^{n} p_i = 1$. Suppose $ap_i = p_i a$ for all i. Let $\varepsilon > 0$ and $K = \max_i \|p_i\|$. Then*

$$\Lambda_{\frac{\varepsilon}{n}}(A, a) \subseteq \bigcup_{i=1}^{n} \Lambda_\varepsilon(p_i A p_i, p_i a) \subseteq \Lambda_{K\varepsilon}(A, a).$$

Proof. Let $\varepsilon > 0$ and $\lambda \in \bigcup_{i=1}^{n} \Lambda_\varepsilon(p_i A p_i, p_i a)$. If $\lambda \in \sigma(p_i A p_i, p_i a)$ for some i, then $\lambda \in \sigma(A, a) \subseteq \Lambda_\varepsilon(A, a) \subseteq \Lambda_{K\varepsilon}(A, a)$. On the other hand, if $\lambda p_i - p_i a$ is invertible for all i, then $\lambda - a$ is invertible. We have $\lambda \in \Lambda_\varepsilon(p_i A p_i, p_i a)$ for some i. Hence $\|p_i(\lambda - a)^{-1}\| = \|(\lambda p_i - p_i a)^{-1}\| \geq \frac{1}{\varepsilon}$. Hence $\|(\lambda - a)^{-1}\| \geq \frac{1}{\|p_i\|}\frac{1}{\varepsilon} \geq \frac{1}{K\varepsilon}$. Thus

$$\bigcup_{i=1}^{n} \Lambda_\varepsilon(p_i A p_i, p_i a) \subseteq \Lambda_{K\varepsilon}(a).$$

Next, suppose $\lambda \notin \Lambda_\varepsilon(p_i A p_i, p_i a)$ for all i. Then each $\lambda p_i - p_i a$ is invertible with inverse $p_i b_i p_i$, say, and $\|p_i b_i p_i\| < \frac{1}{\varepsilon}$. Then $\lambda - a$ is invertible with inverse $\sum_{i=1}^{n} p_i b_i p_i$, and further,

$$\begin{aligned}\|(\lambda - a)^{-1}\| &= \|\sum_{i=1}^{n} p_i b_i p_i\| \\ &< n\frac{1}{\varepsilon}.\end{aligned}$$

Hence, we get

$$\Lambda_{\frac{\varepsilon}{n}}(A, a) \subseteq \bigcup_{i=1}^{n} \Lambda_\varepsilon(p_i A p_i, p_i a) \subseteq \Lambda_{K\varepsilon}(A, a).$$

□

Now, going back to the case $n = 2$, suppose $\|p\| = \|q\| = 1$. For example, this happens if p is a non-trivial Hermitian idempotent.

Corollary 3.4. *Let A be a unital Banach algebra, $p \in A$ be an idempotent and $q = 1 - p$. Let $a \in A$ be such that $ap = pa$. Suppose $\|p\| = \|q\| = 1$. Then*

$$\Lambda_{\frac{\varepsilon}{2}}(A, a) \subseteq \Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa) \subseteq \Lambda_\varepsilon(A, a). \quad (3.3)$$

We examine the question of when equality occurs in the second inclusion of (3.3). It is easily observed that equality occurs precisely when

$$\|(\lambda - a)^{-1}\| = \max\{\|p(\lambda - a)^{-1}\|, \|q(\lambda - a)^{-1}\|\}.$$

Definition 3.5. An element $a \in A$ is said to be a G_1-element if it satisfies the following equality:

$$\|(z - a)^{-1}\| = \frac{1}{\mathrm{d}(z, \sigma(a))} = r((z - a)^{-1}) \quad \text{for all } z \in \mathbb{C} \setminus \sigma(a).$$

See [12].

Now, suppose $a \in A$ commutes with p, and in addition, a is a G_1 element. Then $\Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa) = \Lambda_\varepsilon(A, a)$

Theorem 3.6. *Let A be a unital Banach algebra, $p \in A$ be an idempotent and $q = 1 - p$. Suppose $\|p\| = \|q\| = 1$. Let $a \in A$ be such that $ap = pa$, and suppose a is a G_1-element. Then*

$$\Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa) = \Lambda_\varepsilon(A, a).$$

Proof. The proof follows since

$$\begin{aligned}
\|(\lambda - a)^{-1}\| &= r((\lambda - a)^{-1}) \\
&= \max\{r(pAp, p(\lambda - a)^{-1}), r(qAq, q(\lambda - a)^{-1})\} \\
&\leq \max\{\|p(\lambda - a)^{-1}\|, \|q(\lambda - a)^{-1}\|\} \\
&\leq \|(\lambda - a)^{-1}\|.
\end{aligned}$$

□

We next consider the case that A is a C^*- algebra and $p \in A$ is a self-adjoint projection. We observe that the notion of a Hermitian idempotent coincides with that of a self-adjoint projection in C^*-algebras.

Theorem 3.7. *Let A be a unital C^*-algebra and $p \in A$ be a self-adjoint projection. Let $q = 1 - p$. Let $a \in A$ be such that $ap = pa$. Then*

$$\Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa) = \Lambda_\varepsilon(A, a).$$

Proof. Without loss of generality, assume that $\lambda = 0$. Then

$$\begin{aligned}
\|a^{-1}\|^2 &= \|(a^{-1})^* a^{-1}\| \\
&= r(A, (a^{-1})^* a^{-1}) \\
&= \max\{r(pAp, p(a^{-1})^* a^{-1} p), r(qAq, q(a^{-1})^* a^{-1} q)\} \\
&= \max\{r(pAp, (a^{-1}p)^*(a^{-1}p)), r(qAq, (a^{-1}q)^*(a^{-1}q))\} \\
&\leq \max\{\|(a^{-1}p)^*(a^{-1}p)\|, \|(a^{-1}q)^*(a^{-1}q)\|\} \\
&= \max\{\|pa^{-1}\|^2, \|qa^{-1}\|^2\}
\end{aligned}$$

□

Remark 3.8. Theorem 3.7 is stated for the finite dimensional case in (iv) of Theorem 2.4 of [17].

It is easy to see from Lemma 3.2 that Theorems 3.6 and 3.7 can be generalized to the case of finitely many idempotents of norm one satisfying $\sum_{i=1}^{n} p_i = 1$.

Example 3.9. We give an example in which $\|p\| = \|q\| = 1$, but equality does not occur in the second inclusion in (3.3). Let $A = (\mathbb{C}^{2\times 2}, \|\cdot\|_1)$ and $p = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Let $q = 1 - p$. Then $p^2 = p$ and $\|p\| = 1 = \|q\|$. Now, if $a \in A$ commutes with p, it is necessarily of the form $\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, and $pap = \frac{1}{2}(\alpha+\beta)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

If pap is invertible in pAp, then its inverse is equal to $\frac{1}{2(\alpha+\beta)}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Similarly, if qaq is invertible in qAq, then its inverse is equal to $\frac{1}{2(\alpha-\beta)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

The following can be computed: $\sigma(a) = \{\alpha+\beta, \alpha-\beta\}$, $\|(\lambda-a)^{-1}\| = \frac{|\beta|+|\lambda-\alpha|}{|(\lambda-\alpha)^2-\beta^2|}$, $\|p(\lambda-\alpha)^{-1}\| = \frac{1}{|\lambda-\alpha-\beta|}$, and $\|q(\lambda-a)^{-1}\| = \frac{1}{|\lambda-\alpha+\beta|}$.

Choosing $\alpha = 0, \beta = 1$, we get $\|(\lambda-a)^{-1}\| = \frac{1+|\lambda|}{|\lambda^2-1|}$, $\|p(\lambda-a)^{-1}\| = \frac{1}{|\lambda-1|}$, and $\|q(\lambda-a)^{-1}\| = \frac{1}{|\lambda+1|}$.

Now, let $\varepsilon = 1$. Then it can be verified that $\lambda = i \in \Lambda_\varepsilon(A,a)$ but $i \notin \Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa)$. Hence in this example, $\Lambda_\varepsilon(A,a)$ is strictly larger than $\Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa)$.

Remark 3.10. Let X be a Banach space and $P \in B(X)$ be an idempotent operator. Let $Q = I - P$. Then $X = \text{Range } P \oplus \text{Range } Q$. If $T \in B(X)$ is an operator such that $PT = TP$, then both Range P and Range Q are invariant under T. Hence $T = T_1 \oplus T_2$, where $T_1 = TP \restriction_{\text{Range } P}$ and $T_2 = TQ \restriction_{\text{Range } Q}$. Similarly, $(\lambda - T)^{-1} = (\lambda - T)^{-1}P \restriction_{\text{Range } P} \oplus (\lambda - T)^{-1}Q \restriction_{\text{Range } Q}$. We use the following theorem due to Lancaster and Farahat (See Theorem 2 of [11]):

Let $X = X_1 \oplus \cdots \oplus X_n$ be a direct sum of n Banach spaces. Let $T = T_1 \oplus \cdots \oplus T_n$ be a direct sum of operators acting on X. Then

$$\|T\| = \max_i \|T_i\|$$

if and only if the norm on X is absolute, i.e.

$$\|x_i\| = \|y_i\| \text{ for all } i \implies \|x\| = \|y\|,$$

where $x = (x_i)$ and $y = (y_i)$.

Hence, in order to check whether $\|(\lambda - T)^{-1}\| = \max\{\|(\lambda - T)^{-1}P\|, \|(\lambda - T)^{-1}Q\|\}$, it suffices to check whether $\|x\|$, the norm of $x \in X$ is a function of $\|Px\|$ and $\|Qx\|$.

We see some examples of idempotent operators which satisfy the above norm condition.

Example 3.11. Let $A = (\mathbb{C}^{n\times n}, \|\cdot\|_r)$, $1 \le r \le \infty$, $r \ne 2$. Here, $\|(x_n)\|_r = (\sum_{i=1}^n |x_n|^r)^{\frac{1}{r}}$ for $r \ne \infty$ and $\|(x_n)\|_\infty = \max_n |x_n|$. The Hermitian idempotent elements of A can be shown to be diagonal matrices with diagonal entries equal to 0 or 1. Without any change in the norm, we may assume that k 1-s appear first as diagonal entries, followed by $(n-k)$ 0-s. Let P be such a Hermitian idempotent matrix. If $T \in A$ commutes with P, it must be a block diagonal matrix.

It can be checked that $\|\cdot\|_r$ is an absolute norm on the decomposition of $\mathbb{C}^n$ into the ranges of P and Q in this case. Thus the induced operator norm has the maximum property. This yields $\|(\lambda - T)^{-1}\| = \max\{\|P(\lambda - T)^{-1}\|, \|Q(\lambda - T)^{-1}\|\}$, $\lambda \notin \sigma(A, T)$. Hence $\Lambda_\varepsilon(A, T) = \Lambda_\varepsilon(PAP, PT) \cup \Lambda_\varepsilon(QAQ, QT)$.

Example 3.12. Let Ω be a compact Hausdorff space and $A = C(\Omega)$, the space of continuous functions on X. Then an operator $P \in B(A)$ is Hermitian if and only if it is of the form:

$$Pf = hf,\ f \in C(\Omega),\ h \text{ a real valued function.}$$

(See Theorem 6.29.3 of [3].) Further, if we want h to be an idempotent continuous function, it must necessarily be a characteristic function of a connected component of Ω, say Δ. Hence Range $P = \{\chi_\Delta f : f \in C(\Omega)\}$ and Range $Q =$ Range $(I - P) = \{\chi_{\Delta^c} f : f \in C(\Omega)\}$. Clearly, for $f \in C(\Omega)$, $\|f\|_\infty = \sup_{\omega\in\Omega} |f(\omega)| = \max\{\sup_{\omega\in\Delta} |f(\omega)|, \sup_{\omega\in\Delta^c} |f(\omega)|\} = \max\{\|\chi_\Delta f\|_\infty, \|\chi_{\Delta^c} f\|_\infty\} = \max\{\|Pf\|, \|Qf\|\}$.

Thus if $\|Pf\| = \|Pg\|$ and $\|Qf\| = \|Qg\|$, then $\|f\| = \|g\|$. Hence the induced operator norm has the required maximum property. Note that if such a P exists, Ω must be disconnected.

We observe next that if the norm $\|x\|$ on the Banach space X is a function of $\|Px\|$ and $\|Qx\|$, then P (and thus Q) must necessarily be Hermitian.

Theorem 3.13. *Let X be a Banach space and $P \in B(X)$ be an idempotent operator. Let $Q = I - P$. Suppose there exists $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|x\| = f(\|Px\|, \|Qx\|) \text{ for all } x \in X.$$

Then P is Hermitian. Further, if $T \in B(X)$ such that $PT = TP$, then for $\varepsilon > 0$,

$$\Lambda_\varepsilon(A, T) = \Lambda_\varepsilon(PAP, PT) \cup \Lambda_\varepsilon(QAQ, QT).$$

Proof. By Corollary 1.10.13 of [2], it suffices to show that $\|e^{itP}\| = 1$ for all $t \in \mathbb{R}$. Now, $\|e^{itP}x\| = \|(I + P(e^{it} - 1))x\|$ since P is an idempotent. By the hypothesis we get

$$\begin{aligned}\|e^{itP}x\| &= f(\|P(I + P(e^{it} - 1))x)\|, \|Q(I + P(e^{it} - 1))x\|) \\ &= f(\|(Pe^{it})x\|, \|Qx\|) \\ &= f(\|Px\|, \|Qx\|) \\ &= \|x\|\end{aligned}$$

for all $t \in \mathbb{R}$. Hence, in fact, we get that e^{itP} is an isometry for every real t, and thus P is Hermitian. Hence, Q is also Hermitian and $\|P\| = \|Q\| = 1$. It then follows by the theorem of Lancaster and Farahat ([11]) that

$$\|(\lambda - T)^{-1}\| = \max\{\|(\lambda - T)^{-1}P\|, \|(\lambda - T)^{-1}Q\|\}.$$

Hence $\Lambda_\varepsilon(A, T) = \Lambda_\varepsilon(PAP, PT) \cup \Lambda_\varepsilon(QAQ, QT)$. Examples of idempotent operators that satisfy this condition are M- projections and L-projections (See [7]). □

Theorem 3.14. *Let A be a Banach algebra and $p \in A$ be an idempotent element. Let $q = 1 - p$. Suppose there exists $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|a\| = f(\|pa\|, \|qa\|) \text{ for all } a \in A.$$

Then p is Hermitian and

$$f(\lambda, \mu) = \max\{\lambda, \mu\} \; \lambda, \mu \in \mathbb{R}^+.$$

Further, if $a \in A$ commutes with p, then for $\varepsilon > 0$, $\Lambda_\varepsilon(A, a) = \Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa)$.

Proof. We first show that p must be a Hermitian element. By Corollary 1.10.13 of [2], It suffices to show that $\|e^{itp}\| = 1$ for all $t \in \mathbb{R}$. Since p is idempotent and by the hypothesis, we have

$$\begin{aligned}
\|e^{itp}\| &= \|1 + p(e^{it} - 1)\| \\
&= f(\|p(1 + p(e^{it} - 1))\|, \|q(1 + p(e^{it} - 1))\|) \\
&= f(\|p\|, \|q\|) \\
&= \|1\| \\
&= 1
\end{aligned}$$

for all $t \in \mathbb{R}$. Hence p is Hermitian, and thus $\|p\| = 1 = \|q\|$.

Now, let $\lambda, \mu > 0$. Let $x = \lambda p + \mu q$. Then $px = \lambda p$ and $qx = \mu q$. We have

$$\begin{aligned}
f(\lambda, \mu) &= f(\|px\|, \|qx\|) \\
&= \|x\| \\
&= \|(\lambda - \mu)p + \mu\| \\
&= r((\lambda - \mu)p + \mu) \\
&= \max\{\lambda, \mu\}.
\end{aligned}$$

The equality of spectral radius and norm of $(\lambda - \mu)p + \mu$ follows because p is Hermitian (See Proposition 2 of [16]). It then follows that $\|(\lambda - a)^{-1}\| = \max\{\|p(\lambda - a)^{-1}\|, \|q(\lambda - a)^{-1}\|\}$. By this observation and by Lemma 3.1, it follows that if $ap = pa$, then $\Lambda_\varepsilon(A, a) = \Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa)$. □

However, in general it is not true that if P is a Hermitian operator on a Banach space, then $\|Px\| = \|Py\|$ and $\|Qx\| = \|Qy\| \implies \|x\| = \|y\|$. We give an example of a Hermitian idempotent operator on a C^* algebra, for which the hypothesis of 3.13 does not hold.

Example 3.15. Let $X = \mathbb{C}^{2\times 2}$, the C^* algebra endowed with the norm induced by the Euclidean norm on $\mathbb{C}^2$ and $p \in X$ be the Hermitian idempotent matrix $p = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $q = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Let $L_p \in B(X)$ be the left multiplication operator by p on A. Then L_p is a Hermitian idempotent operator on X. Let L_q be the left multiplication by q on X. Let $x = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $y = \frac{1}{2}\begin{bmatrix} 5 & 5 \\ 3 & 1 \end{bmatrix}$. Then one can check that $\|L_p x\| = \|L_p y\| = \frac{5}{\sqrt{2}}$ and $\|L_q x\| = \|L_q y\| = \sqrt{\frac{5}{2}}$, but $\|x\| = \sqrt{\frac{(3+\sqrt{5})5}{2}} \neq \sqrt{\frac{15+\sqrt{200}}{2}} = \|y\|$.

We now examine the question of equality of pseudospectrum of $a \in A$ and the union of the pseudospectra of $pa \in pAp$ and $qa \in qAq$ in Banach algebras with a special property.

Theorem 3.16. *Let A be a Banach algebra which satisfies the following:*

$$\Lambda_\varepsilon(A, a) = \bigcup_{\|b\| \le \varepsilon} \sigma(A, a + b) \quad \text{for all } a \in A.$$

This is equivalent to:

$$\text{for all } a \in Inv(A), \text{ there exists } b \in Sing(A) \text{ such that } \|a - b\| = \frac{1}{\|a^{-1}\|}.$$

(See Corollary 2.7 of [9].)

Let $p \in A$ be a central idempotent element such that $\|p\| = \|q\| = 1$, where $q = 1 - p$. Then $pAp = pA$ and

$$\Lambda_\varepsilon(A, a) = \Lambda_\varepsilon(pA, pa) \cup \Lambda_\varepsilon(qA, qa) \quad \text{for all } a \in A.$$

Proof. Let $a \in A$. Then

$$\begin{aligned}
\Lambda_\varepsilon(A, a) &= \bigcup_{\|b\| \le \varepsilon} \sigma(A, a + b) \\
&= \bigcup_{\|b\| \le \varepsilon} \sigma(pA, p(a + b)) \cup \bigcup_{\|b\| \le \varepsilon} \sigma(qA, q(a + b)) \\
&\subseteq \bigcup_{\|pb\| \le \varepsilon} \sigma(pA, p(a + b)) \cup \bigcup_{\|qb\| \le \varepsilon} \sigma(qA, q(a + b)) \\
&\subseteq \Lambda_\varepsilon(pA, pa) \cup \Lambda_\varepsilon(qA, qa).
\end{aligned}$$

The reverse inclusion is already true, hence the theorem follows. It is easy to see from Lemma 3.2 that Theorem 3.16 can be generalised to the case in which there exists finitely many central idempotents of norm one satisfying $\sum_{i=1}^{n} p_i = 1$. □

We now extend some of the above results for a finite family of idempotents in a Banach algebra.

Theorem 3.17. *Let X be a Banach space and $\{P_i\}_{i=1}^n$ be a finite family of idempotents in $A = B(X)$ such that $P_i P_j = 0$ if $i \neq j$. Suppose there exists $f : (\mathbb{R}^+)^n \to \mathbb{R}$ such that*

$$\|x\| = f((\|P_1 x\|, \cdots, \|P_n x\|)) \quad \text{for all } x \in X.$$

Then each P_i is Hermitian. Further, if $T \in B(X)$ such that $P_iT = TP_i$ for all i, then for $\varepsilon > 0$,

$$\Lambda_\varepsilon(A,T) = \bigcup_{i=1}^{n} \Lambda_\varepsilon(P_iAP_i, P_iT).$$

Proof. For $i = 1$, say and $t \in \mathbb{R}$,

$$\begin{aligned}\|e^{itP_1}x\| &= \|(1 + P_1(e^{it} - 1))x\| \\ &= f(\|(P_1e^{it})x\|, \cdots, \|(P_nx\|) \\ &= f(\|P_1x\|, \cdots, \|P_nx\|) \\ &= \|x\|.\end{aligned}$$

Similary each e^{itP_i} is an isometry and hence P_i is Hermitian, with $\|P_i\| = 1$. It follows by the theorem of Lancaster and Farahat ([11]) that

$$\|(\lambda - T)^{-1}\| = \max_i\{\|(\lambda - T)^{-1}P_i\|\}.$$

□

Theorem 3.18. *Let A be a Banach algebra and $\{p_i\}_{i=1}^n$ be a finite family of idempotent elements of A such that $p_ip_j = 0$ if $i \neq j$. Suppose there exists $f : (\mathbb{R}^+)^n \to \mathbb{R}$ such that*

$$\|a\| = f((\|p_1a\|, \cdots, \|p_na\|)) \text{ for all } a \in A.$$

Then each p_i is Hermitian and

$$f(\lambda_1, \cdots, \lambda_n) = \max\{\lambda_1, \cdots, \lambda_n\}, \ \lambda_i \in \mathbb{R}^+.$$

Further, if $a \in A$ commutes with each p_i, then

$$\Lambda_\varepsilon(A,a) = \bigcup_{i=1}^{n} \Lambda_\varepsilon(p_iAp_i, p_ia).$$

Proof. For $i = 1$, say, and $t \in \mathbb{R}$,

$$\begin{aligned}\|e^{itp_1}\| &= \|1 + p_1(e^{it} - a)\| \\ &= f((\|(p_1e^{it})\|, \cdots, \|p_n\|)) \\ &= f((\|p_1\|, \cdots, \|p_n\|)) \\ &= \|1\| \\ &= 1.\end{aligned}$$

Similarly, each p_i is Hermitian and $\|p_i\| = 1$. Next, let $x = \sum_{i=1}^{n} \lambda_i p_i,\ \lambda_i > 0$. Then $p_i x = \lambda_i p_i$ for all i. Then

$$\begin{aligned}\lambda_i &= \|\lambda_i p_i\| \\ &= \|p_i x\| \\ &\leq \|x\| \\ &= \|\sum_{i=1}^{n} \lambda_i p_i\| \\ &= f(\|\lambda_1 p_1\|, \cdots, \|\lambda_n p_n\|) \\ &= f(\lambda_1, \cdots, \lambda_n)\end{aligned}$$

for all i. On the other hand,

$$\begin{aligned}f(\lambda_1, \cdots, \lambda_n) &= \|\sum_{i=1}^{n} \lambda_i p_i\| \\ &= r(\sum_{i=1}^{n} \lambda_i p_i) \\ &\leq \max_i \{\lambda_i\},\end{aligned}$$

the last inequality following because $\sigma(\sum_{i=1}^{n} \lambda_i p_i) \subseteq \{\lambda_1, \cdots, \lambda_n\}$, due to the fact that $p_i p_j = 0,\ i \neq j$. Hence $\|(\lambda - a)^{-1}\| = \max_i \|p_i(\lambda - a)^{-1}\|$ and thus $\Lambda_\varepsilon(A, a) = \bigcup_{i=1}^{n} \Lambda_\varepsilon(p_i A p_i, p_i a)$. □

4. The Infinite Case

We now examine the above results for an arbitrary family of idempotents $\{p_\alpha\}_{\alpha \in I}$ in a Banach algebra. We first consider the case $A = B(H)$ for a Hilbert space H. The following theorem is elementary and has been discussed in Theorem 2.3 of [4]. We provide the proof here for the convenience of the reader.

Theorem 4.1. *Let H be a Hilbert space and $\{P_n\}_{n \in \mathbb{N}}$ be a family of self-adjoint idempotent operators in $A = B(H)$, such that $(\sum_{n \in \mathbb{N}} P_n)h = h$ for all $h \in H$, that is, the convergence of the series is in the strong operator topology. Suppose $P_n P_m = 0$ if $n \neq m$. Then H can be written as the direct sum $\oplus_{n \in \mathbb{N}} P_n H$. Let $T \in B(H)$ be an operator that commutes with each P_n. Let $H_n = P_n H$ and $T_n = T P_n \restriction_{H_n} \in B(H_n)$. Then*

$$\sigma(A, T) = \bigcup_{n \in \mathbb{N}} \sigma(P_n A P_n, T_n) \cup \{\lambda \in \mathbb{C} : \sup_{n \in \mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \infty\}.$$

Proof. We recall that for an operator $T = \oplus_n T_n$ on a direct sum of Hilbert spaces, $\|T\| = \sup_n \|T_n\|$ (See Exercise II.1.12 of [5]). Suppose $\lambda - T$ is invertible in A. Then clearly, $\lambda P_n - T_n$ is invertible in $P_n A P_n$ for all n with inverses

$P_n(\lambda - T)^{-1}$, and $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \|(\lambda - T)^{-1}\| < \infty$. Conversely, suppose $(\lambda P_n - T_n)$ is invertible in P_nAP_n for all n with inverses $(\lambda P_n - T_n)^{-1}$, and $\sup_n \|(\lambda P_n - T_n)^{-1}\| < \infty$, then $\oplus_n(\lambda P_n - T_n)^{-1}$ is the inverse of $\lambda - T$ in A. $\square$

We now look at $B(X)$ for a Banach space X. As in the Hilbert space case, let $\{P_n\}_{n\in\mathbb{N}}$ be a family of idempotent operators in $B(X)$, such that $(\sum_{n\in\mathbb{N}} P_n)x = x$ for all $x \in X$ (that is, the series $\sum_{n\in\mathbb{N}} P_n$ converges to the identity operator in the strong operator topology) and $P_nP_m = 0$ if $n \neq m$. Let $X_n = P_nX$, and $x_n = P_nx$ for $x \in X$. For a direct sum of X_n to make sense, we must specify the subspace of $\prod_n X_n$ and the norm on it that we intend to use (See page 72 of [5]). Most often, we use either the r-norm $(1 \leq r < \infty)$ or the supremum norm. That is, $\oplus_n X_n = \{(x_n) : \sum_{n=1}^{\infty} \|x_n\|^r < \infty\}$ and $\|x\| = \|(x_n)\| = (\sum_{n=1}^{\infty} \|x_n\|^r)^{1/r},\ 1 \leq r < \infty$ and $\oplus_n X_n = \{(x_n) : \sup_n \|x_n\| < \infty\}$ and $\|x\| = \|(x_n)\| = \sup_n \|x_n\|$. We note that in each of these cases, if $T_n \in B(X_n)$, the operator $T = \oplus T_n \in B(X)$ if and only if $\sup_n \|T_n\| < \infty$. For $T \in B(X)$, let $T_n = TP_n \restriction_{X_n} \in B(X_n)$. Examples of such Banach spaces are l_p spaces or more generally, l_p direct sums of Banach spaces (See [15]).

The following result is an elementary generalisation of Theorem 4.1 for operators on direct sums of Banach spaces.

Theorem 4.2. *Let X be a Banach space and $\{P_n\}_{n\in\mathbb{N}}$ be a family of idempotent operators in $A = B(X)$ satisfying the above conditions. Let $T \in B(X)$ be an operator that commutes with each P_n. Suppose the norm of $x \in X$ is a function f of $(\|x_n\|)$ and satisfies the following monotone condition: $\|x_n\| \leq \|y_n\|$ for all $n \implies \|x\| \leq \|y\|,\ x, y \in X$ and $x_n = P_nx, y_n = P_ny$. Let $X_n = P_nX$ and $T_n = TP_n \restriction_{X_n}$. Then each P_n is Hermitian and*

$$\sigma(A, T) = \bigcup_{n\in\mathbb{N}} \sigma(P_nAP_n, T_n) \cup \{\lambda \in \mathbb{C} : \sup_{n\in\mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \infty\}.$$

Proof. Let $x = (x_n) \in X$. Let $t \in \mathbb{R}$. For $i = 1$, say

$$\begin{aligned}
\|e^{itP_1}x\| &= \|(I + P_1(e^{it} - T))x\| \\
&= f((\|(P_1e^{it})x\|, \|P_2x\|, \cdots)) \\
&= f((\|P_1x\|, \|P_2x\|, \cdots)) \\
&= \|x\|.
\end{aligned}$$

Similarly, each e^{itP_i} is an isometry whence each P_i is Hermitian and $\|P_i\| = 1$ for all i. Now,

$$\begin{aligned}\|Tx\| &= \|(T_n x_n)\| \\ &\leq \|(\|T_n\| x_n)\| \\ &\leq \|(\sup_n \|T_n\| x_n)\| \\ &= \|(\sup_n \|T_n\|)(x_n)\| \\ &= \sup_n \|T_n\| \|x\|.\end{aligned}$$

Clearly, $\|T_n\| \leq \|T\|$ for all n. Hence $\|T\| = \sup_n \|T_n\|$, and the theorem then follows as in Theorem 4.1. □

We next move on to the pseudospectra decomposition. We recall that some authors (such as in [1] and [17]) have defined the following set as the ε-pseudospectrum of a:

$$\Lambda^*_\varepsilon(a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| > \varepsilon^{-1}\}.$$

We first give a decomposition for the ε- pseudospectrum defined this way.

Theorem 4.3. *Let X be a Banach space and $\{P_n\}_{n\in\mathbb{N}}$ be a family of idempotent operators in $A = B(X)$ as above. Let $T \in B(X)$ be an operator that commutes with each P_n. Suppose the norm on X satisfies the following monotone condition: $\|x_n\| \leq \|y_n\|$ for all $n \implies \|x\| \leq \|y\|$, $x, y \in X$ and $x_n = P_n x, y_n = P_n y$ as before. Let $X_n = P_n X$ and $T_n = TP_n \restriction_{X_n}$. Let $\varepsilon > 0$. Then*

$$\Lambda^*_\varepsilon(A, T) = \bigcup_{n\in\mathbb{N}} \Lambda^*_\varepsilon(P_n A P_n, T_n).$$

Proof. Let $\lambda \in \Lambda^*_\varepsilon(A, T)$. If $\lambda \in \sigma(A, T)$, then either $\lambda \in \bigcup_n \sigma(P_n A P_n, T_n)$ or $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \infty$. If $\lambda \in \sigma(P_{n_0} A P_{n_0}, T_{n_0})$ for some n_0, then $\lambda \in \bigcup_n \Lambda^*_\varepsilon(P_n A P_n, T_n)$. If $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \infty$, then there exists n_0 such that $\|(\lambda P_{n_0} - T_{n_0})^{-1}\| > \frac{1}{\varepsilon}$, hence $\lambda \in \bigcup_n \Lambda^*_\varepsilon(P_n A P_n, T_n)$. If $\lambda - T$ is invertible and $\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}$, then $\|(\lambda P_{n_0} - T_{n_0})^{-1}\| > \frac{1}{\varepsilon}$ for some n_0, hence $\lambda \in \Lambda^*_\varepsilon(P_{n_0} A P_{n_0}, T_{n_0})$.

For the other inclusion, let $\lambda \in \bigcup_n \Lambda^*_\varepsilon(P_n A P_n, T_n)$. If $\lambda \in \bigcup_n \sigma(P_n A P_n, T_n)$, then $\lambda \in \sigma(A, T) \subseteq \Lambda^*_\varepsilon(T)$. If $\lambda - T_n$ is invertible for all n and $\sup_{n\in\mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| < \infty$, then $\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}$. If $\sup_{n\in\mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \infty$, then $\lambda \in \sigma(A, T) \subseteq \Lambda^*_\varepsilon(A, T)$. □

Now, let us consider the decomposition of the pseudospectrum defined with the non-strict inequality.

Theorem 4.4. *Let X be a Banach space and $\{P_n\}_{n\in\mathbb{N}}$ be a family of idempotent operators in $A = B(X)$ as above. Let $T \in B(X)$ be an operator that commutes with each P_n. Suppose the norm on X satisfies the following monotone condition:*

$\|x_n\| \leq \|y_n\|$ for all $n \implies \|x\| \leq \|y\|$, *$x, y \in X$ and $x_n = P_n x, y_n = P_n y$ as before. Let $X_n = P_n X$ and $T_n = TP_n \restriction_{X_n}$. Let $\varepsilon > 0$. Then*

$$\Lambda_\varepsilon(A,T) = \bigcup_{n\in\mathbb{N}} \Lambda_\varepsilon(P_nAP_n, T_n) \cup \{\lambda \in \mathbb{C} : \sup_{n\in\mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \frac{1}{\varepsilon}\}.$$

Proof. Let $\lambda \in \Lambda_\varepsilon(A,T)$. If $\lambda \in \sigma(A,T)$, then either $\lambda \in \bigcup_n \sigma(P_nAP_n, T_n)$ or $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \infty$. If $\lambda \in \sigma(P_{n_0}AP_{n_0}, T_{n_0})$ for some n_0, then $\lambda \in \bigcup_n \Lambda_\varepsilon(P_nAP_n, T_n)$. If $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \infty$, then there exists n_0 such that $\|(\lambda P_{n_0} - T_{n_0})^{-1}\| \geq \frac{1}{\varepsilon}$, hence $\lambda \in \bigcup_n \Lambda_\varepsilon(P_nAP_n, T_n)$. If $\lambda - T$ is invertible and $\|(\lambda - T)^{-1}\| \geq \frac{1}{\varepsilon}$, then either $\|(\lambda P_{n_0} - T_{n_0})^{-1}\| > \frac{1}{\varepsilon}$ for some n_0 or $\|(\lambda - T)^{-1}\| = \frac{1}{\varepsilon}$.

For the other inclusion, let $\lambda \in \bigcup_n \Lambda_\varepsilon(P_nAP_n, T_n)$. If $\lambda \in \bigcup_n \sigma(P_nAP_n, T_n)$, then $\lambda \in \sigma(A,T) \subseteq \Lambda_\varepsilon(A,T)$. If $\lambda - T_n$ is invertible for all n and $\sup_{n\in\mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| < \infty$, then $\|(\lambda - T)^{-1}\| \geq \frac{1}{\varepsilon}$. If $\sup_{n\in\mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \infty$, then $\lambda \in \sigma(A,T) \subseteq \Lambda_\varepsilon(A,T)$.

□

Remark 4.5. In general, $\Lambda_\varepsilon(a)$ is not the closure of $\Lambda^*_\varepsilon(a)$. However, this is true in many cases. See [13] and [14]. In such cases, the above decomposition can be modified to:

$$\Lambda_\varepsilon(A,T) = \overline{\bigcup_{n\in\mathbb{N}} \Lambda_\varepsilon(P_nAP_n, T_n)}.$$

This follows because

$$\begin{aligned} \Lambda_\varepsilon(A,T) &= \overline{\Lambda^*_\varepsilon(A,T)} \\ &= \overline{\bigcup_{n\in\mathbb{N}} \Lambda^*_\varepsilon(P_nAP_n, T_n)} \\ &\subseteq \overline{\bigcup_{n\in\mathbb{N}} \Lambda_\varepsilon(P_nAP_n, T_n)}. \end{aligned}$$

Already, since $\bigcup_{n\in\mathbb{N}} \Lambda_\varepsilon(P_nAP_n, T_n) \subseteq \Lambda_\varepsilon(A,T)$, and since $\Lambda_\varepsilon(A,T)$ is closed, we have

$$\overline{\bigcup_{n\in\mathbb{N}} \Lambda_\varepsilon(P_nAP_n, T_n)} \subseteq \Lambda_\varepsilon(A,T).$$

Theorems 4.2, 4.3 and 4.4 are also true if the countable family $\{P_n\}_{n\in\mathbb{N}}$ is replaced by an arbitrary family $\{P_\alpha\}_{\alpha\in I}$. We write it here for the sake of completeness. The proof is identical to the countable case.

Theorem 4.6. *Let X be a Banach space and $\{P_\alpha\}_{\alpha\in I}$ be a family of idempotent operators in $A = B(X)$ such that $(\sum_{\alpha\in I} P_\alpha)x = x$ for all $x \in X$. Let $T \in B(X)$ be an operator that commutes with each P_α. Suppose the norm on X satisfies the following monotone condition:* $\|x_\alpha\| \leq \|y_\alpha\|$ for all $\alpha \in I \implies \|x\| \leq$

$\|y\|$, $x, y \in X$ *and* $x_\alpha = P_\alpha x, y_\alpha = P_\alpha y$. *Let* $X_\alpha = P_\alpha X$ *and* $T_\alpha = TP_\alpha \restriction_{X_\alpha}$. *Then each* P_α *is Hermitian and*

$$\sigma(A,T) = \bigcup_{\alpha\in I} \sigma(P_\alpha A P_\alpha, T_\alpha) \cup \{\lambda \in \mathbb{C} : \sup_{\alpha\in I} \|(\lambda P_\alpha - T_\alpha)^{-1}\| = \infty\}.$$

Further, if $\varepsilon > 0$, *then*

$$\Lambda^*_\varepsilon(A,T) = \bigcup_{\alpha\in I} \Lambda^*_\varepsilon(P_\alpha A P_\alpha, T_\alpha),$$

and

$$\Lambda_\varepsilon(A,T) = \bigcup_{\alpha\in I} \Lambda_\varepsilon(P_\alpha A P_\alpha, T_\alpha) \cup \{\lambda \in \mathbb{C} : \sup_{\alpha\in I} \|(\lambda P_\alpha - T_\alpha)^{-1}\| = \frac{1}{\varepsilon}\}.$$

Remark 4.7. In the case of infinite direct sums, we must assume that the convergence of the infinite series ($\sum\limits_{n=1}^{\infty} P_n = 1$) is in the strong operator topology, since a sequence of idempotents cannot converge to 0, and $\sum\limits_{n=1}^{\infty} P_n$ convergent implies $\|P_n\| \to 0$. Since there is no natural notion of this convergence in the case of an abstract Banach algebra, we do not have the equivalent of Theorem 3.18 for the case of infinitely many idempotents.

Remark 4.8. In the case of finite direct sums, it is sufficient to assume that the norm on $X = \oplus_{i=1}^n X_i$ is an absolute norm (that is, $\|x_n\| = \|y_n\|$ for all $n \implies \|x\| = \|y\|$), since every absolute norm on a finite direct sum is monotonic. This is not clear for an infinite direct sum and hence we have assumed the monotone condition on the norm in this case.

Remark 4.9. We also observe that if we have an uncountable family of idempotent operators $\{P_\alpha\}_{\alpha\in I}$ satisfying $(\sum_\alpha P_\alpha)x = x$ for all $x \in X$, then for each $x \in X$, all but countably many of the $P_\alpha x$ must be 0 (see Corollary 5.28 in [10]). Thus, in the case of a separable Banach space, we obtain $P_\alpha = 0$ for all but countably many α and it suffices to consider the countable case.

We give some examples to illustrate the above theorems.

Example 4.10. Let X_n be Banach spaces and $X = \oplus_n X_n$ with a suitable monotonic norm defined on the direct sum. Let $T_n = \frac{1}{n} I_n \in B(X_n)$, and $T = \oplus_n T_n \in B(X)$. Then $\sigma(T_n) = \{\frac{1}{n}\}$ for all n. Then it can be seen that $\|(\lambda I_n - T_n)^{-1}\| = \frac{1}{|\lambda - \frac{1}{n}|}$ and hence $\sigma(T) = \bigcup_n \{\frac{1}{n}\} \cup \{0\}$, and $\Lambda_\varepsilon(T_n) = D(\frac{1}{n}; \varepsilon)$ for all n. For $\lambda \notin \sigma(T)$,

$$\sup_n \|(\lambda - T_n)^{-1}\| = \sup_n \frac{1}{|\lambda - \frac{1}{n}|} = \begin{cases} \frac{1}{|\lambda|}, & \text{Re } \lambda < 0 \\ \frac{1}{|\lambda - 1|}, & \text{Re } \lambda > 1 \\ \frac{1}{|\lambda - \frac{1}{n_0}|} & \text{for some } n_0 \in \mathbb{N},\ 0 < \text{ Re } \lambda < 1. \end{cases}$$

Hence

$$\Lambda_\varepsilon(T) = \bigcup_{n\in\mathbb{N}} D(\frac{1}{n};\varepsilon) \cup D(0;\varepsilon) = \bigcup_{n\in\mathbb{N}} \Lambda_\varepsilon(T_n) \cup \{\lambda \in \mathbb{C} : \sup_{n\in\mathbb{N}} \|(\lambda I_n - T_n)^{-1}\| = \frac{1}{\varepsilon}\}.$$

The following example is based on the example given in Problem 98 in [6].

Example 4.11. Let $X = l^1(\mathbb{N})$ and let $T \in B(X)$ be the weighted unilateral shift operator with weights given by the sequence $\{1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0 \cdots\}$. l^1 can be expressed as the direct sum

$$\mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \cdots,$$

where each $\mathbb{C}^n$ is endowed with the 1-norm and the direct sum is also endowed with the 1-norm. Then T can be expressed as the direct sum

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus \cdots.$$

Let $T_n \in B(\mathbb{C}^{n+1})$ be the n-th term in the direct sum. Then the following can be computed: $\sigma(T_n) = \{0\}$ for all n, and $\|(\lambda - T_n)^{-1}\| = \sum_{k=1}^{n+1} \frac{1}{|\lambda|^k}$ for all n. Hence $\sigma(T) = D(0;1)$ and

$$\begin{aligned} \Lambda_\varepsilon(T_n) &= \{\lambda \in \mathbb{C} : |\lambda|^{n+1} - \varepsilon(|\lambda|^n + \cdots + |\lambda| + 1) \le 0\} \\ &= \{\lambda \in \mathbb{C} : |\lambda|^{n+1} \le \varepsilon(\frac{|\lambda|^{n+1} - 1}{|\lambda| - 1})\}. \end{aligned}$$

We also see that $\|(\lambda - T)^{-1}\| = \frac{1}{|\lambda|-1}$ for $|\lambda| > 1$, and hence $\Lambda_\varepsilon(T) = D(0; 1 + \varepsilon)$. For $|\lambda| > 1$,

$$\begin{aligned} \Lambda_\varepsilon(T_n) &= \{\lambda \in \mathbb{C} : |\lambda|^{n+1} \le \varepsilon(\frac{|\lambda|^{n+1} - 1}{|\lambda| - 1})\} \\ &= \{\lambda \in \mathbb{C} : |\lambda|^{n+2} - (1 + \varepsilon)|\lambda|^{n+1} + \varepsilon \le 0\}. \end{aligned}$$

Now, if $|\lambda| \ge 1+\varepsilon$, then $|\lambda|^{n+2} - (1+\varepsilon)|\lambda|^{n+1} + 1 \ge |\lambda|^{n+2} - |\lambda||\lambda|^{n+1} + \varepsilon = \varepsilon > 0$ for all n. Hence for each n, $\Lambda_\varepsilon(T_n) \subsetneq D(0; 1 + \varepsilon)$. Thus we get

$$\Lambda_\varepsilon(T) = \bigcup_{n\in\mathbb{N}} \Lambda_\varepsilon(T_n) \cup \{\lambda \in \mathbb{C} : \sup_{n\in\mathbb{N}} \|(\lambda I_n - T_n)^{-1}\| = \frac{1}{\varepsilon}\}.$$

In fact, in this case, it is true that

$$\Lambda_\varepsilon(T) = \overline{\bigcup_{n\in\mathbb{N}} \Lambda_\varepsilon(T_n)}.$$

This is because the space $l^1(\mathbb{N})$ is complex uniformly convex, hence $\Lambda_\varepsilon(T) = \overline{\Lambda_\varepsilon^*(T)}$ (see [13] and [14]).

Remark 4.12. Let u be a non-trivial idempotent in a Banach algebra A. Any $x \in A$ can be represented as

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_u$$

where $a = uxu, b = ux(1-u), c = (1-u)xu$ and $d = (1-u)x(1-u)$. The representation is in diagonal form precisely when $ux = xu$.

In [8], the author states that if an element x of a Banach algebra has the diagonal representation

$$x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u$$

with respect to the idempotent u, then $\|x\|$ can be defined as $\max\{\|a\|, \|b\|\}$. However, this norm is not equal to the original norm on A in general.

Consider Example 3.9. Here, $A = (\mathbb{C}^{2\times 2}, \|\cdot\|_1)$ and $u = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It can be seen that for a as in the example and $\lambda = i$, $\|(\lambda - a)^{-1}\| = 1$ which is strictly greater than the maximum of $\|u(\lambda - a)^{-1}\|$ and $|(1-u)(\lambda - a)^{-1}\|$ both of which are equal to $\frac{1}{\sqrt{2}}$.

The author of [8] discusses the the (p, q)-generalised outer inverse of $a \in A$ and further defines the $(p, q) - \varepsilon$ pseudospectrum of $a \in A$.

Let $a \in A$ and $p, q \in A$ be idempotent elements. An element $b \in A$ satisfying $bab = b$, $ba = p$ and $1 - ab = q$ will be called the (p, q)-generalised outer inverse of a and denoted by $a^{(2)}_{p,q}$. The set of elements with (p, q)-outer generalised inverses is denoted by $A^{(2)}_{p,q}$.

The $(p, q) - \varepsilon$ pseudospectrum of $a \in A$ is defined as

$$\Lambda_{(p,q)-\varepsilon}(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{(2)}_{p,q} \text{ or } \|(\lambda - a)^{(2)}_{p,q}\| \geq \frac{1}{\varepsilon}\}.$$

It is shown that for $u \in A$, an idempotent and $a \in A$ such that $ua = au$, if p_1, q_1 idempotents in uAu and p_2, q_2 are idempotents in $(1-u)A(1-u)$ and $p = p_1 + p_2, q = q_1 + q_2$,

$$\Lambda_{(p,q)-\varepsilon}(A, a) = \Lambda_{(p_1,q_1)-\varepsilon}(uAu, ua) \cup \Lambda_{(p_2,q_2)-\varepsilon}((1-u)A(1-u), (1-u)a). \quad (4.1)$$

However, this is not true in general. This decomposition occurs if and only if

$$\|(\lambda - a)^{(2)}_{p,q}\| = \max\{\|(\lambda u - ua)^{(2)}_{p_1,q_1}\|, \|(\lambda(1-u) - (1-u)a)^{(2)}_{p_2,q_2}\|\}.$$

For instance, if the hypotheses of Theorem 3.13 or Theorem 3.14 hold, then we obtain the decomposition (4.1).

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