Adv. Oper. Theory 3 (2018), no. 1, 231–246
http://doi.org/10.22034/aot.1702-1115
ISSN: 2538-225X (electronic)
http://aot-math.org

ON A CLASS OF BANACH ALGEBRAS ASSOCIATED TO HARMONIC ANALYSIS ON LOCALLY COMPACT GROUPS AND SEMIGROUPS

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This paper is dedicated to Professor Uffe Haagerup for his many tremendous and deep contributions in mathematics.

Communicated by M. Mathieu

Abstract. The purpose of this paper is to present some old and recent results for the class of F-algebras which include most classes of Banach algebras that are important in abstract harmonic analysis. We also introduce a subclass of the class of F-algebras, called normal F-algebras, that captures better the measure algebras and the (reduced) Fourier–Stieltjes algebras, and use this to give new characterisations the reduced Fourier–Stieltjes algebras of discrete groups.

1. Introduction

From each locally compact group G, several Banach algebras could be constructed that encode essential information about G. The most important ones are the group algebra $L^1(G)$ and the Fourier algebra $A(G)$. The group algebra $L^1(G)$ is the space (of equivalence classes) of integrable functions on G with convolution product; note that $L^1(G)$ is naturally identified with the predual of the commutative von Neumman algebra $L^\infty(G)$. While the Fourier algebra $A(G)$, introduced by Eymard in [8], is a subalgebra of $\mathcal{C}_0(G)$ the algebra of continuous

Date: Received: Feb. 11, 2017; Accepted: Jun. 29, 2017.
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2010 *Mathematics Subject Classification.* Primary 43A30, 22D25; Secondary 46J05, 22D10.
Key words and phrases. Fourier algebra, locally compact group, group algebra, Fourier–Stieltjes algebra, F-algebra.

functions on G vanishing at the infinity, that consists of all continuous functions on G of the form $\overline{k} * \check{h}$, where $k, h \in L^2(G)$, $\overline{k}(s) = \overline{k(s)}$ and $\check{h}(s) = h(s^{-1})$. The norm on $A(G)$ comes from its natural identification with the predual of the group von Neumann algebra $VN(G)$, the weak*-closed subalgebra of $\mathcal{B}(L^2(G))$ generated by the left regular representation $\lambda_G : G \to \mathcal{U}(G)$: each $\varphi \in A(G)$ is considered as a normal functional on $VN(G)$ by

$$\varphi(T) = (Th|k) \qquad (T \in VN(G)),$$

where k, h are any pair of elements of $L^2(G)$ such that $\varphi = \overline{k} * \check{h}$.

The two Banach algebras $L^1(G)$ and $A(G)$ are important topological algebraic objects in the study of harmonic analysis on locally compact groups. They have been shown to have deep relation with the structure of the underlying group G (see Wendel [41] and Walter [39]). They also share a crucial common property: each of them is the predual of a W^*-algebra and the identity of the W^*-algebra is in the spectrum of the Banach algebra.

In [20], a class of Banach algebras called F-algebras (also known as Lau algebras, see [31]) was introduced which not only includes the group algebra $L^1(G)$ and the Foureir algebra $A(G)$ of a locally compact group G but also many other classes of algebras that are studied in abstract harmonic analysis. It is the purpose of this paper to give an updated account on some recent work on this class of Banach algebras.

Definition 1.1. By an *F-algebra*, we shall mean a pair (A, M) such that A is a complex Banach algebra and M is a W^*-algebra such that $A = M_*$ (isometrically), where M_* is the predual of M, and the identity of M is a character of A (i.e. a nonzero multiplicative linear functional on A). If there is no confusion, we shall simply say that A is an F-algebra and we shall identify A' with M. The identity of A' will be denoted by e. Also, $P(A) := \{\varphi \in A \colon \varphi \geq 0\}$ and $P_1(A) := \{\varphi \in P(A) \colon \varphi(e) = 1\}$, the set of normal states of A'.

Note that the W^*-algebraic structure on the dual A' of an F-algebra A needs not be unique. In fact, let M be a W^*-algebra such that the reversed or opposite algebra M^o (see [5]), where the only modification is the product $a \cdot^o b := ba$, is not W^*-isomorphic to M. Define on $A := M_*$ the following product: $\varphi \cdot \psi := \varphi(e)\psi$ for any $\varphi, \psi \in A$. Then both (A, M) and (A, M^o) are F-algebras. However, we have the following [20] as a consequence of Kadison's theorem on isometries of W^*-algebras:

Proposition 1.2. *If (A, M_1) and (A, M_2) are F-algebras such that M_1 and M_2 have the same set of positive functionals. Then there exists central projections $z_i \in M_i$, $i = 1, 2$, such that $M_1 z_1$ is W^*-isomorphic to $M_2 z_2$ and $M_1(e_1 - z_1)$ is W^*-isomorphic to the reversed algebra of $M_2(e_2 - z_2)$. In particular, if M_1 is commutative, then M_1 and M_2 are W^*-isomorphic.*

Besides the group algebras $L^1(G)$ and the Fourier algebras $A(G)$ of locally compact groups G, the class of F-algebras also includes the Fourier–Stieltjes algebras $B(G)$ of locally compact groups G, the measure algebra $M(S)$ of a locally compact semigroup S or locally compact hypergroups. It also includes the class of convolution measure algebras studied by Taylor [36, 37, 38], the class of L-algebras (for which the identity of the dual algebra is in the spectrum of the L-algebra) considered by McKilligan and White [29], and the class of semi-convos [17].

Many of the examples mentioned above are subclasses of the class of predual algebras of Hopf von Neumann algebras [35]; in fact, its subclass of quantum group algebras [19] already includes the group algebras $L^1(G)$ and the Fourier algebras $A(G)$ of locally compact groups G. We shall not define the quantum group algebras here, but let us recall that a Hopf von Neumann algebra is a pair $(\mathfrak{M}, \Gamma)$ where $\mathfrak{M}$ is a W^*-algebra and $\Gamma : \mathfrak{M} \to \mathfrak{M}\overline{\otimes}\mathfrak{M}$ is a co-multiplication, that is, a normal unital $*$-homomorphism that satisfies the co-associativity:

$$(\mathrm{id} \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \mathrm{id}) \circ \Gamma\,.$$

Here, $\mathfrak{M}\overline{\otimes}\mathfrak{M}$ denotes the von Neumann algebra tensor product of $\mathfrak{M}$ with itself. Since Γ is normal, it has the pre-adjoint $\Gamma_* : \mathfrak{M}_*\widehat{\otimes}\mathfrak{M}_* \to \mathfrak{M}_*$, which is associative due to the coassociativity of Γ. This makes $\mathfrak{M}_*$ an F-algebra.

If G is a locally compact group, then we have seen that $L^1(G)$ is an F-algebra. But in fact, it is even the predual algebra of the Hopf von Neumann algebra $(L^\infty(G), \Gamma_G)$, where $\Gamma_G : L^\infty(G) \to L^\infty(G)\overline{\otimes}L^\infty(G) = L^\infty(G \times G)$ is defined by

$$\Gamma_G(f)(s,t) = f(st) \qquad (f \in L^\infty(G),\ s,t \in G)\,.$$

On the other hand, the Fourier algebra $A(G)$ is the predual algebra of a Hopf von Neumann algebra, if the group von Neumann algebra $VN(G)$ is given the comulitiplication that maps $\lambda(s) \mapsto \lambda(s) \otimes \lambda(s)$ for every $s \in G$.

The classes of group algebras and of measure algebras of locally compact groups are also subclasses of the following constructions for hypergroups. A locally compact space H is a hypergroup if there is a convolution product, denoted by $*$, defined on $M(H)$, the space of bounded Radon measures on H, for which several general conditions are satisfied. We refer to [1] for the precise definition of a locally compact hypergroup. With the convolution product, $M(H)$ becomes a Banach algebra, it is in fact an F-algebra. When a locally compact hypergroup H has a left invariant Haar measure λ, the convolution product on $L^1(H) := L^1(H, \lambda)$ is then naturally defined to make it a Banach algebra, called the hypergroup algebra of H. Again, $L^1(H)$ is an F-algebra, although, in general, $L^1(H)$ cannot be a predual algebra of a Hopf von Neumann algebra unless H is already a locally compact group [42, Theorem 5.2.2] (see also [43, Remark 5.3]).

In the following, we shall first review three topics in the theory of F-algebras:

(i) left-amenability and fixed point properties,
(ii) finite-dimensional invariant subspace properties, and
(iii) characterisations of the Fourier algebras of locally compact groups.

A new notion of normal F-algebras is then introduced in section 5. This is a specialisation to those F-algebras whose W^*-algebraic dual is actually the bidual of a C^*-algebra. We use this to give new characterisations of the reduced Fourier–Stieltjes algebras of discrete groups. The final section 6 ends the paper with some open problems relating to the above topics.

2. Left-amenable F-algebras

Let A be a Banach algebra, and let X be a Banach A-bimodule. A *derivation* $D : A \to X$ is a linear map such that

$$D(\varphi\psi) = \varphi D(\psi) + D(\varphi)\psi \qquad (\varphi, \psi \in A)\,.$$

It is easy to see that $D_{x_0} : A \to X\,,\ \varphi \mapsto \varphi x_0 - x_0\varphi\,,$ is a bounded derivation for every $x_0 \in X$. Such a derivation is called *an inner derivation.* It is often desirable to know whether or not every bounded derivation $D : A \to X$ is inner. However, we are often more interested in this question when the bimodule X is the dual of another bimodule.

Given a Banach A-bimodule X. Its Banach space dual X' has a natural structure of a Banach A-bimodule, where the left and right module products are defined as

$$\langle \varphi \cdot f, x\rangle = \langle f, x \cdot \varphi\rangle \quad \text{and} \quad \langle f \cdot \varphi, x\rangle = \langle f, \varphi \cdot x\rangle$$

for all $f \in X'$, $\varphi \in A$, and $x \in X$. A Banach algebra A is *amenable* if, for any Banach A-bimodule X, every bounded derivation $A \to X'$ is inner. B. Johnson proved in [18, Theorem 2.5] that a locally compact group G is amenable if and only if $L^1(G)$ is amenable. The class of amenable Banach algebras has been studied extensively, see the books [6], [31], and [34] for examples.

However, Johnson's theorem is no longer valid for semigroups. In fact, the semigroup $\mathbb{N}$ of positive integers with addition is amenable but the Banach algebra $\ell^1(\mathbb{N})$ is not [2, p. 244]. Is there a similar theorem that works for semigroups? For that we have to restrict to the class of F-algebras. But let us recall some terminologies first.

Recall that a *semitopological semigroup* is a semigroup S with a Hausdorff topology such that the product on S is separately continuous; it is called *a topological semigroup* if the product is jointly continuous. Let S be a semitopological semigroup. Denote by $\mathcal{C}^b(S)$ the commutative C^*-algebra of all bounded continuous complex-valued functions on S, and by $LUC(S)$ its C^*-subalgebra consisting of all left uniformly continuous functions, i.e. those $f \in \mathcal{C}^b(S)$ such that the mapping $s \mapsto l_s(f), S \to \mathcal{C}^b(S)$, is continuous, where $l_s(f)(t) = f(st)$ for all $s, t \in S$. Evidently, $LUC(S)$ is translation-invariant and contains the constant functions.

A semitopological semigroup S is *left-amenable* if $LUC(S)$ has a left-invariant mean, i.e. an element $m \in LUC(S)'$ such that

$$\|m\| = m(1) = 1, \quad m(l_s(f)) = m(f) \quad (f \in LUC(S), s \in S)\,.$$

S is *extremely left-amenable* if if there is a left-invariant mean m which is multiplicative, i.e. such that furthermore

$$m(fg) = m(f)m(g) \qquad (f, g \in LUC(S))\,.$$

Let A be an F-algebra. A *topological left invariant mean* (abbreviated as TLIM) on A' is an element $m \in P_1(A'')$ such that

$$m(x \cdot \varphi) = m(x) \quad \text{for each } \varphi \in P_1(A), x \in A'\,.$$

The set of TLIM on A' will be denoted by TLIM(A'). The notion of TLIM has been considered for various special cases of A by many authors; see for examples [7], [16], [32], and [44].

An F-algebra A is said to be *left-amenable* if every bounded derivation $A \to X'$ is inner whenever X is a Banach A-bimodule whose left multiplication is given by $\varphi \cdot x = \varphi(e)x$ for all $\varphi \in A$ and $x \in X$. This notion was introduced in [20] where the following theorem was proved.

Theorem 2.1. *Let A be an F-algebra. Then A' has a TLIM if and only if A is left-amenable.*

The following is an analogue of Johnson's theorem [18, Theorem 2.5]:

Corollary 2.2. *A semigroup S is left-amenable if and only if the F-algebra $\ell^1(S)$ is left-amenable.*

Corollary 2.3. *Let G be a locally compact group. The following are equivalent:*

(i) *G is amenable.*
(ii) *The group algebra $L^1(G)$ is left-amenable.*
(iii) *The measure algebra $M(G)$ is left-amenable.*

Proof. This follows from Theorem 2.1 and [14, Theorem 2.2.1] and [44, Theorem 3.3]. $\square$

Remark 2.4.

(i) Any commutative F-algebras are left- (and right-) amenable.
(ii) Let M be a W^*-algebra, and $A = M_*$. Then:
  (a) A is left-amenable if the product on A is defined by $\varphi \cdot \psi := \varphi(e)\psi$;
  (b) A is right-amenable if the product on A is defined by $\varphi \cdot \psi := \psi(e)\varphi$;
  (c) A is both left- and right-amenable (but not amenable) if the product on A is defined by $\varphi \cdot \psi := \varphi(e)\psi(e)\theta$, where $\theta \in P_1(A)$ is fixed.
(iii) Let A_1, A_2 be F-algebras. Then $A_1 \oplus A_2$ is left-amenable if and only if A_1 is left-amenable. In particular, $\mathbb{C} \oplus A$ is left-amenable for any F-algebra A.
(iv) An F-algebra A is left-amenable if and only if A'', with the first Arens product, is left-amenable.

Theorem 2.5. *Let A be an F-algebra. Then the left-amenability of A is equivalent to each of the following:*

(i) *There exists a net $\varphi_\alpha \in P_1(A)$ such that $\|\varphi \cdot \varphi_\alpha - \varphi_\alpha\| \to 0$ for each $\varphi \in P_1(A)$.*
(ii) *There exists a net $\varphi_\alpha \in P_1(A)$ such that $\|\psi \cdot \varphi_\alpha\| = |\psi(e)|$ for each $\psi \in A$.*
(iii) *For each $x \in A'$, the set $\overline{K(x)}^\sigma$ contains λe for some $\lambda \in \mathbb{C}$, where $K(x) = \{\varphi \cdot x \colon \varphi \in P_1(A)\}$ and the closure is in the weak*-topology of A'.*

  In this case, $\lambda e \in \overline{K(x)}^\sigma$ if and only if there exists a TLIM on A' with $m(x) = \lambda$.
(iv) *For each $\psi \in A$ with $\psi(e) = 0$ and each $\varepsilon > 0$, there exists a $\varphi \in P_1(A)$ such that $\|\psi \cdot \varphi\| < \varepsilon$.*

(v) *The Mazur distance* $d(I_1, I_2) = 0$ *for any two right-ideals* I_1, I_2 *of the semigroup* $P_1(A)$.

(vi) $N(A)$ *is closed under addition, where* $N(A)$ *denotes the set of all* $x \in A'$ *such that* $\inf \{\|\varphi \cdot x\| : \varphi \in P_1(A)\} = 0$.

We refer the reader to [20, p. 172] for more details.

A semigroup S is *left-reversible* if any two right ideals in S has nonempty intersection. Commutative semigroups (or more generally left amenable semigroups) and groups are left-reversible.

Corollary 2.6. *An* F*-algebra* A *is left-amenable if* $P_1(A)$ *is left-reversible.*

Let S be a semigroup and let Ω be a Hausdorff topological space. We say that $\mathcal{S} = \{T_s \colon s \in S\}$ is a *representation* of S on Ω if each T_s is a continuous mapping $\Omega \to \Omega$ and $T_{st} = T_s \circ T_t$ $(s, t \in T)$. Sometimes, we simply write sx instead of $T_s(x)$ if there is no ambiguity. When S is a semitopological semigroup, we say that the representation is *seperately continuous* if the mapping $(s, x) \mapsto T_s(x), S \times \Omega \to \Omega$ is separately continuous. The representation is *(jointly) continuous* if the mapping $(s, x) \mapsto T_s(x), S \times \Omega \to \Omega$ is (jointly) continuous.

Mitchell [30] showed that a semitopological semigroup S is extremely left-amenable if and only if it has the following fixed point property:

(F_E): Every jointly continuous representation of S on a compact Hausdorff space Ω has a common fixed point in Ω.

For an F-algebra, it is pleasing that the left-amenability of A is equivalent to the extreme left-amenability of $P_1(A)$ as shown by the following theorem, proved in [28].

Theorem 2.7. *Let* A *be an* F*-algebra. Then* A *is left-amenable if and only if* $P_1(A)$ *has the fixed point property* (F_E).

Corollary 2.8. *A locally compact group* G *is amenable if and only if the semigroup* $P_1(G) := P_1(L^1(G))$ *has the fixed point property* (F_E).

Corollary 2.9. *A semigroup* S *is left-amenable if and only if the semigroup* $P_1(S) := P_1(\ell^1(S))$ *has the fixed point property* (F_E).

3. Finite-dimensional invariant subspace properties

Let E be a locally convex vector space, and X a subset of E. Given an $n \in \mathbb{N}$, we denote by $\mathcal{L}_n(X)$ the collection of all n-dimensional subspaces of E that are included in X. Let S be a semigroup and $\mathcal{S} = \{T_s \colon s \in S\}$ a linear representation of S on E. We say that X is n-consistent with respect to S if $\mathcal{L}_n(X) \neq \emptyset$ and $\mathcal{L}_n(X)$ is S-invariant, that is, $T_s(L) \in \mathcal{L}_n(X)$ for all $s \in S$ whenever $L \in \mathcal{L}_n(X)$.

In [10], see also [9, 21, 25, 27], Ky Fan proved the following remarkable "Invariant Subspace Theorem" for left-amenable semigroups:

Our standing assumption will be that the topology of a locally convex space is always Hausdorff, a property sometimes called *separated* in the literature.

Theorem 3.1. *Let S be a left-amenable semigroup, and let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as continuous linear operators on a locally convex space E. Then the following holds:*

(KF): *If X is a subset of E that is n-consistent with respect to S, and if there exists a closed $\mathcal{S}$-invariant subspace H in E of codimension n with the property that $(x + H) \cap X$ is compact convex for each $x \in E$, then there exists an $L_0 \in \mathcal{L}_n(X)$ such that $T_s(L_0) = L_0$ for all $s \in S$.*

A similar result for F-algebras is established by Lau and Zhang in [28]. For convenient, let us say that a linear representation $\mathcal{S}$ of a semigroup S on a locally convex vector space$E$ is jointly continuous on compact sets if the following is true: For each compact set $K \subset E$, if the nets $s_\alpha \to s$ in S , $x_\alpha \to x$ in K, and $T_{s_\alpha}(x_\alpha) \in K$ for all α, then $T_{s_\alpha}(x_\alpha) \to T_s(x)$. Obviously, if the mapping $(s, x) \mapsto T_s(x) : S \times E \to E$ is continuous, then S is jointly continuous on compact sets.

The main theorem of [28] says:

Theorem 3.2. *Let A be an F-algebra, and consider the semigroup $S := P_1(A)$. Then the left-amenability of A is equivalent to each of the following n-dimensional invariant subspace properties where $n \in \mathbb{N}$:*

(F_n): *Let $\mathcal{S} = \{T_s : s \in S\}$ be a linear representation of S on a locally convex space E such that the mapping $s \mapsto T_s(x)$ is continuous for each fixed $x \in E$ and $\mathcal{S}$ is jointly continuous on compact subsets of E. If X is a subset of E that is n-consistent with respect to S, and if there exists a closed $\mathcal{S}$-invariant subspace H in E of codimension n with the property that $(x + H) \cap X$ is compact for each $x \in E$, then there exists an $L_0 \in \mathcal{L}_n(X)$ such that $T_s(L_0) = L_0$ for all $s \in S$.*

Since the left-amenability of a semigroup S is equivalent to the left-amenability of the F-algebra $\ell^1(S)$, the theorem implies that a semigroup S is left-amenable if and only if $\ell^1(S)$ satisfies each (and hence all) of the invariant subpace properties (F_n).

For another application of the above theorem, let us consider a locally compact group G. Then, combining Corollary 2.3 and Theorem 3.2, we obtain that for each $n \in \mathbb{N}$, G is amenable if and only if the group algebra $L^1(G)$ satisfies (F_n) if and only if the measure algebra $M(G)$ satisfies (F_n).

4. Characterisation of Fourier algebras

In this section, the notion of F-algebras is used to give axiomatic characterisations of Fourier algebras on locally compact groups. As Fourier algebras are commutative, let us first recall some concepts from the theory of commutative Banach algebras.

Let Ω be a topological space that is not necessarily Hausdorff nor locally compact. We denote by $\mathcal{C}(\Omega)$ the algebra of complex-valued continuous functions on Ω, and by $\mathcal{C}_c(\Omega)$ its subalgebra consisting of those functions with compact supports in Ω, while by $\mathcal{C}_0(\Omega)$ the subalgebra of $\mathcal{C}(\Omega)$ consisting of those functions that vanish at the infinity. The latter is actually a commutative C^*-algebra whose natural norm is the uniform norm (and whose spectrum is Ω in the case where Ω is a locally compact Hausdorff space).

Let A be a commutative Banach algebra. Recall that the *spectrum* of A, denoted by $\sigma(A)$, is a locally compact Hausdorff space with respect to the relative weak*-topology of A', and the *Gelfand transform* $f \mapsto \hat{f}$ is a homomorphism from A into $\mathcal{C}_0(\sigma(A))$. Thus, the image of A via the Gelfand transform, which is denoted as $\widehat{A}$, is a subalgebra of $\mathcal{C}_0(\sigma(A))$ and a Banach algebra under the norm induced from A. This allows us to call A *conjugation-closed* if $\widehat{A}$ is, and call an element f of A *a real element* if $\hat{f}$ is a real function.

In particular, A *contains a nontrivial real element* if $\widehat{A}$ *contains a nontrivial real function*, i.e. not the zero function. In [26, Definition 2.2], a more general notion of *approximate containment of a nontrivial real element/function* is introduced that would also include the following situations as special cases: A subalgebra $A \subseteq \mathcal{C}(\Omega)$ approximately contains a nontrivial real function if one of the following conditions hold:

(i) if $A \subseteq \mathcal{C}_0(\Omega)$ and the closure of A in the uniform topology on Ω contains a nonzero real function ϕ; or
(ii) more generally, if there is a real continuous bounded function ϕ with a compact level set that, on each compact subset of Ω, is the pointwise limit of a bounded sequence in A; or
(iii) if Ω is locally compact, A separates points of Ω, and there is a compact subset K of Ω with a nonempty interior and with the property that for any neighbourhood W of K there exists a function $f \in A$ such that $f = 1$ on K, $f = 0$ on $\Omega \setminus W$, and $|f| \le 1$ on $W \setminus K$.

In the statement of the following results, we shall stick with the more special notion of containment of a nontrivial real element/function, but the reader could generalise it to the approximate version without changing the result. For more details, we refer the reader to [26].

A *Tauberian subalgebra of* $\mathcal{C}(\Omega)$ is a subalgebra of $\mathcal{C}(\Omega)$ that is also a Banach algebra under some norm such that $A \cap \mathcal{C}_c(\Omega)$ is dense in A. In this case, we also say that A is *Tauberian on* Ω. For a general commutative Banach algebra A, we say that A is *Tauberian* if $\widehat{A}$ is Tauberian on the spectrum $\sigma(A)$ of A.

The following characterisation of the Fourier algebra was recently proved in [26]:

Theorem 4.1. *Let A be an F-algebra that is also a Tauberian subalgebra of $\mathcal{C}(\Omega)$ that contains a nontrivial real function, for some topological space Ω. Suppose that:*

(i) *every character of A is implemented by some element of Ω;*
(ii) *Ω is a group and A is left translation-invariant;*
(iii) *$\|\sum_{i=1}^{m} \alpha_i l_{s_i} : A \to A\| \le 1$ whenever $\alpha_i \in \mathbb{C}$ and $s_i \in \Omega$ with*

$$\left|\sum_{i=1}^{m} \alpha_i f(s_i)\right| \le \|f\| \quad (f \in A)\,.$$

Then $A \cong A(G)$, i.e. A is isometrically isomorphic to $A(G)$, for some locally compact group G.

Remark 4.2. We remark that Ω is not required a priori to be a topological group and the topology on Ω is not required to be locally compact nor Hausdorff.

Remark 4.3. We also remark that condition (i) implies that

(i.a) there is an $u \in \Omega$ such that $\langle e, f\rangle = f(u)$ for every $f \in A$, and

(i.b) for each $s \in \Omega$ there is a $t \in \Omega$ such that $f^*(s) = \overline{f(t)}$ for all $f \in A$;

where e is the identity of A'. However, if in the hypothesis of the previous theorem, instead of condition (i), we assume only (i.a) and (i.b), then it is no longer true that $A = A(G)$; the closure of polynomials in $A(\mathbb{T})$ provides an example. In this case, in order to still conclude that $A = A(G)$, we could replace the assumption that A (approximately) contains a nonzero real function by the stronger condition that A is conjugation-closed.

The following generalises [4, Theorem 3.2.12].

Corollary 4.4. *Let A be a commutative semisimple Banach algebra that is also Tauberian and contains a nontrivial real element. Suppose that A' is a W^*-algebra and that $\sigma(A)$, under the multiplication of A', is a group. Then $A \cong A(G)$ for some locally compact group G.*

We have the following characterisation of when the predual of a Hopf–von Neumann algebra is the Fourier algebra of a locally compact group.

Corollary 4.5. *Let A be the predual of a Hopf–von Neumann algebra such that A is commutative, semisimple, Tauberian, and contains a nontrivial real element, and that $\sigma(A)$ has at most one positive element. Then $A \cong A(G)$ for some locally compact group G.*

Note that we cannot omit from the above corollary the assumption that $\sigma(A)$ has at most one positive element, see [26][Example 4.7].

The identity element of a unital semisimple commutative Banach algebra A is always a (nonzero) real element, and moreover, in this case, A is automatically Tauberian. Thus we have the following characterisations of $A(G)$ for compact G.

Corollary 4.6. *Let A be a unital F-algebra that is also a subalgebra of $\mathcal{C}(\Omega)$, for some topological space Ω. Suppose also that:*

(i) *every character of A is implemented by some element of Ω;*

(ii) *Ω is a group and A is left translation-invariant;*

(iii) *$\|\sum_{i=1}^m \alpha_i l_{s_i} : A \to A\| \le 1$ whenever $\alpha_i \in \mathbb{C}$ and $s_i \in \Omega$ with*

$$\left|\sum_{i=1}^m \alpha_i f(s_i)\right| \le \|f\| \quad (f \in A)\,.$$

Then $A \cong A(G)$ for some compact group G.

Corollary 4.7. *Let A be the predual of a Hopf–von Neumann algebra such that A is unital, commutative, and semisimple, and that $\sigma(A)$ has at most one positive element. Then $A \cong A(G)$ for some compact group G.*

To state our the next characterisations of Fourier algebras from [26], the following concepts will play an important role.

Definition 4.8. Let A be a Banach algebra. An automorphism T of A is *a dual automorphism* if it is isometric and satisfies

$$\left\|f - \mathrm{e}^{i\theta}Tf\right\|^2 + \left\|f + \mathrm{e}^{i\theta}Tf\right\|^2 \leq 4\left\|f\right\|^2 \qquad (f \in A,\ \theta \in \mathbb{R})\,.$$

This concept is suggested by Walter's notion of a dual group of a Banach algebra in [40, Definition 5], which we shall recall and extend below.

Theorem 4.9. *Let A be a Banach algebra that is also a conjugation-closed Tauberian algebra on some topological space Ω. Suppose that*

(i) *A' is a W^*-algebra whose identity is implemented by some element of Ω;*
(ii) *Ω is a group, A is left translation-invariant, and, for each $s \in \Omega$, the automorphism l_s is dual for A.*

Then $A \cong A(G)$ for some locally compact group G.

Note that condition (i) implies immediately that A is an F-algebra.

It is possible to replace condition (ii) above by [40, axiom (ii) on page 155] as in the following. For this, we need to recall and extend the concept of *a dual group* of a Banach algebra introduced in [40, Definition 5]: we shall relax Walter's notion slightly by not requiring such groups to be maximal, and more importantly, we shall even relax the requirement of being a group to just a semigroup.

Definition 4.10. Let A be a Banach algebra. *A dual [semi]group of A* is a [semi]group of dual automorphisms of A. Note that no topology is imposed on any dual semigroup.

Remark 4.11. Let A be a Banach algebra.

(i) The trivial group $\{\mathrm{id}_A\}$ is always a dual group of A, and the union of any chain of dual [semi]groups is again a dual [semi]group. It then follows from Zorn's lemma that *maximal* dual [semi]groups of A always exist; although even maximal dual [semi]group may not be unique.
(ii) Any dual semigroup of A acts naturally on $\sigma(A)$, as any semigroup of automorphisms of A would act on $\sigma(A)$ by transposition.

Theorem 4.12. *Let A be a commutative semisimple F-algebra that is Tauberian, contains a nontrivial real element, and possesses a dual semigroup that acts transitively on $\sigma(A)$. Then $A \cong A(G)$ for a locally compact group G.*

We note below the particularly (even more) simple formulation of the above result for the classes of locally compact abelian groups and of compact groups.

Corollary 4.13. *Let A be a commutative semisimple F-algebra that possesses an abelian dual semigroup that acts transitively on $\sigma(A)$. Then $A \cong L^1(\Gamma)$ for a locally compact abelian group Γ.*

Corollary 4.14. *Let A be a unital commutative semisimple F-algebra that possesses a dual semigroup that acts transitively on $\sigma(A)$. Then $A \cong A(G)$ for some compact group G.* □

These results and their proofs can be found in [26], where another characterisation of Fourier algebras is presented, generalising the well-known characterisation of group algebras of locally compact abelian groups by Rieffel [33]. In this characterisation, no group or semigroup structure is involved, but at the cost of requiring that the given commutative semisimple Banach algebra A has sufficiently many F-algebraic structures to make each of its character the identity of the W^*-algebra associated with one such structure. There are also similar characterisations of Fourier–Stieltjes algebras, albeit at a slightly less satisfactory level. For more details, we refer the reader to [26]. However, in the next section, we shall introduce a subclass of the class of F-algebras, that captures the Fourier–Stieltjes algebras (and measure algebras) better, and use it to give a new characterisation of the reduced Fourier–Stieltjes algebras of discrete groups.

5. Normal F-algebras

Recall that the measure algebra $M(G)$ of a locally compact group G is not only an F-algebra, but it is actually the dual of $\mathcal{C}_0(G)$ – a (commutative) C^*-algebra. This motivates us to give the following definition.

Definition 5.1. A Banach algebra B is called *a normal F-algebra* if there is a C^*-algebra B_* such that $B = B_*'$ isometrically and the identity element $e \in B' = (B_*)''$ is a character on B.

Since the second dual of a C^*-algebra has a natural structure of a W^*-algebra, a normal F-algebra is necessarily an F-algebra. Besides measure algebras, important examples of normal F-algebras are the Fourier–Stieltjes algebra $B(G)$, whose natural predual is the group C^*-algebra $\mathcal{C}^*(G)$. In fact, this is a special case of the following more general construction.

For each continuous unitary representation π of G, the *group C^*-algebra associated with* π, denote by $\mathcal{C}^*_\pi(G)$, is defined to be the norm closure of $\{\pi(f) :\ f \in L^1(G)\}$ in $\mathcal{B}(\mathcal{H}_\pi)$, where $\mathcal{H}_\pi$ is the Hilbert space associated with π. Then $\mathcal{C}^*_\pi(G)$ is naturally a quotient of the *full group C^*-algebra* $\mathcal{C}^*(G)$ of G, constructed as the group C^*-algebra associated with the universal representation of G.

The dual of $\mathcal{C}^*_\pi(G)$, denoted by $B_\pi(G)$, is identified naturally with the linear space of functions on G spanned by the coefficient functions of all continuous unitary representations ρ that are weakly contained in π (i.e. those unitary representations ρ whose group C^*-algebras are naturally a quotient of $\mathcal{C}^*_\pi(G)$). The dual of $\mathcal{C}^*(G)$ is simply denoted as $B(G)$ and it is the *Fourier–Stieltjes algebra* of G discussed above.

A particular important representation of G is the *left regular representation* λ_G, which is mentioned in our introduction of the *Fourier algebra* $A(G)$ and the *group von Neumann algebra* $VN(G)$ of G earlier. The group C^*-algebra associated with λ_G, denoted by $\mathcal{C}^*_r(G)$, is called the *reduced group C^*-algebra* of G, and its dual, denoted by $B_r(G)$, is called the *reduced Fourier–Stieltjes algebra* of G. Thus $VN(G)$ is the weak*-closure of $\mathcal{C}^*_r(G)$ in $\mathcal{B}(L^2(G))$.

The reduced Fourier–Stieltjes algebras of locally compact groups are another important class of normal F-algebras.

Theorem 5.2. *Let B be a subalgebra of $\mathcal{C}(\Omega)$, for some topological space Ω, that is also a normal F-algebra with predual B_*. Suppose also that:*

(i) *there is an $u \in \Omega$ such that $\langle e, f\rangle = f(u)$ for every $f \in B$;*
(ii) *for each $s \in \Omega$ there is a $t \in G$ such that $f^*(s) = \overline{f(t)}$ for all $f \in B$;*
(iii) *$B \cap \mathcal{C}_c(\Omega)$ is conjugation-closed, and weak*-dense in B;*
(iv) *Ω is a group and B is left translation-invariant;*
(v) *$l_s : B \to B$ is weak*-continuous for each $s \in \Omega$;*
(vi) *$\|\sum_{i=1}^m \alpha_i l_{s_i} : B \to B\| \le 1$ whenever $\alpha_i \in \mathbb{C}$ and $s_i \in \Omega$ with*

$$\left|\sum_{i=1}^m \alpha_i f(s_i)\right| \le \|f\| \quad (f \in B)\,;$$

(vii) *B_* is unital.*

Then there exists a discrete group G such that $B \cong B_r(G)$, and there is a natural identification of the duality pairs $\langle B, B_*\rangle$ and $\langle B_r(G), \mathcal{C}_r^*(G)\rangle$.

Remark 5.3. We remark that Ω is not required a priori to be a topological group and the topology on Ω is not required to be discrete nor even locally compact.

Proof. Since the Banach algebra B is a subalgebra of $\mathcal{C}(\Omega)$, there is a natural continuous map $\eta : \Omega \to \sigma(B)$. Then arguing as in the proof of [26, Theorem 4.1], we see that, after modifying the products of Ω and of $B' = B_*''$ if necessary, we obtain that η is a group homomorphism from Ω into the unitary group of B'; in fact, a significant part of that proof is to show that it is possible to make this additional assumption. Then the translations by elements of Ω are the same as module multiplication of $\eta(\Omega) \subseteq B'$ on the predual B of the von Neumann algebra B'. For example, for all $f \in B$ and $s, t \in \Omega$

$$f(st) = \langle f, \eta(st)\rangle = \langle f, \eta(s)\eta(t)\rangle = \langle f \cdot \eta(s), \eta(t)\rangle = (f \cdot \eta(s))(t) \tag{5.1}$$

where $f \cdot \eta(s)$ is the right module multiplication of $\eta(s) \in B'$ with $f \in B$. (In particular, B is also right translation-invariant in addition to being left translation-invariant as assumed.)

Note that with this new product of B', B_* is no longer a subalgebra of $B' = B_*''$ (although, under the natural identification, B_* is still a subspace of B'). However, B_* and B' still share the same identity element.

Set $A := \overline{B \cap \mathcal{C}_c(\Omega)}$. Then A is a closed subalgebra of B that is translation-invariant. Since the linear span of $\eta(\Omega)$ is weak*-dense in B', it follows that there is a central projection $z \in B'$ such that $A = B \cdot z$. Thus A' is naturally identified with zB', a W^*-algebra. Set $G := z\eta(\Omega)$. Then G is a topological group where the topology on G is the relative weak*-topology of zB'; i.e. G is a subspace of $\sigma(A)$. Moreover, since $A \cap \mathcal{C}_c(\Omega)$ is dense in A, it is easy to see that G is also locally compact, and the last part of the proof of [26, Theorem 4.1] (see also [26, Remark 4.3]) shows that $A \cong A(G)$.

Next, since $l_s : B \to B$ is weak*-continuous for each $s \in \Omega$, we see from (5.1) that $\eta(s)B_* \subseteq B_*$. Since B_* contains the identity element of B', we obtain that $\eta(\Omega) \subseteq B_*$. The Hahn–Banach theorem then shows that $B_* = \overline{\langle \eta(\Omega)\rangle}$. In particular, with the new product of B', B_* is a C^*-subalgebra of B'. So from now on, we shall consider B_* with this new product.

Consider the mapping $x \mapsto zx, B_* \to zB' \cong VN(G) \subseteq \mathcal{B}(L^2(G))$. This mapping is a $*$-homomorphism, which is injective since A is assumed to be weak*-dense in B. This and the previous paragraph implies that $z\eta(s) = z\eta(t)$ if and only if $f(s) = f(t)$ for all $f \in B$. Thus, passing to a quotient space if necessary, we shall consider Ω as the set G with a finer topology (i.e has possibly more open sets). Since $\overline{B \cap \mathcal{C}_c(\Omega)} = A = A(G)$, we see first that $A(G) \subseteq \mathcal{C}_0(\Omega)$, then that Ω is locally compact, and finally that Ω has the same topology as G.

Let us now consider λ_G but as a representation of G_d, the given group G with the discrete topology. This gives us a unitary representation $\delta : G_d \to \mathcal{U}(L^2(G))$, and since $zB_* = \mathcal{C}^*_\delta(G_d)$, we see that $B = B_\delta(G_d)$. But combining with the previous paragraph, it can then be seen that $B \subseteq B(G)$, and so $B = B_\pi(G)$ for some continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$. Since $A(G) \subseteq B$, it follows that $B_r(G) \subseteq B$, and so $\mathcal{C}^*_r(G)$ is a natural quotient of $\mathcal{C}^*_\pi(G) = zB_*$. Tracing how various maps are defined, we obtain the following commutative diagram

$$\begin{tikzcd} B_* \arrow[r, "x\mapsto zx"] \arrow[dr, "\text{natural quotient}"'] & zB' = VN(G) \\ & \mathcal{C}^*_r(G) \arrow[u, "\text{inclusion}"'] . \end{tikzcd}$$

Since the horizontal arrow is a $*$-isomorphism onto $zB_* = \mathcal{C}^*_\delta(G_d)$, we see that $\mathcal{C}^*_r(G) = \mathcal{C}^*_\delta(G)$. Hence, $B = B_r(G)$, where G must be discrete as $\mathcal{C}^*_r(G)$ is now seen to be unital. $\square$

Using the notion of dual automorphisms (Definition 4.8) in place of condition (vi) above, we have another characterisation as follows.

Theorem 5.4. *Let B be a subalgebra of $\mathcal{C}(\Omega)$, for some topological space Ω, that is also a normal F-algebra with predual B_*. Suppose also that:*

(i) *there is an $u \in \Omega$ such that $\langle e, f\rangle = f(u)$ for every $f \in B$;*
(ii) *$B \cap \mathcal{C}_c(\Omega)$ is conjugation-closed, and weak*-dense in B;*
(iii) *Ω is a group and B is left translation-invariant;*
(iv) *l_s is a weak*-continuous dual automorphism of B for each $s \in \Omega$;*
(v) *B_* is unital.*

Then there exists a discrete group G such that $B \cong B_r(G)$, and there is a natural identification of the duality pairs $\langle B, B_*\rangle$ and $\langle B_r(G), \mathcal{C}^*_r(G)\rangle$.

Proof. This follows from the previous theorem and the results of [26, §5]. Note that condition (ii) of Theorem 5.2 now is a consequence of Corollary 5.5 in [26]. $\square$

6. Remarks and open problems

Problem 6.1. Is the converse of Corollary 2.6 true? That is, can an F-algebra be left-amenable without $P_1(A)$ being left-reversible?

Problem 6.2. Let A be an F-algebra. Let (F'_n) denote the same property as (F_n) defined in Section 3 but with "joint continuity" replaced by "separate continuity" on compact subsets of E. Does (F'_n) implies the left-amenability of A?

Problem 6.3. Let A be a left-amenable F-algebra. What can we say about A if $\dim \langle \mathrm{TLIM}(A) \rangle$, the dimension of the linear span of $\mathrm{TLIM}(A)$, is finite? In general, what can we say about the dimension of $\langle \mathrm{TLIM}(A) \rangle$? In the case $A = L^1(G)$ for a locally compact group G, it is known that $\dim \langle \mathrm{TLIM}(A) \rangle = 1$ if and only if G is compact.

The study of the cardinality of the set of invariant means on a group was initiated by Day [7] and Granirer [12]. In 1976, Chou [3] showed that for a discrete infinite amenable group G, the cardinality of the set $\mathrm{LIM}(G)$ of all left-invariant means on $\ell^\infty(G)$ is $2^{2^{|G|}}$. For the more general class of locally compact groups, it is obvious that if G is a compact group, then $\mathrm{TLIM}(G)$ consists of only the normalized Haar measure on G; in fact, it was shown by Lau [22] that conversely if $\mathrm{TLIM}(G)$ is a singleton, then G must be compact. Later, Lau and Paterson [24] proved that if G is a noncompact amenable locally compact group, then the set $\mathrm{MTL}(G)$ of all topological left-invariant means on $L^\infty(G)$ has cardinality $2^{2^{d(G)}}$, where $d(G)$ is the smallest cardinality of a covering of G by compact sets.

When $A = A(G)$, Hu [15] showed that if G is a non-discrete locally compact group, then the cardinality of the set of topological invariant mean on $VN(G)$ is $2^{2^{b(G)}}$, where $b(G)$ is the smallest cardinality of an open basis at the unit element e of G.

Problem 6.4. When does a normal F-algebra have the common fixed point property for nonexpansive mapping? In the case of the Fourier–Stieltjes algebras on locally compact groups G, the necessary and sufficient condition for $B(G)$ to have the weak$*$ fixed point property for non-expansive mapping is that G is compact. The sufficient condition was proved in [23], while the necessary condition has only been proved recently in [11].

Problem 6.5. Generalise the results of §5 to a characterisation of when a normal F-algebra B with predual B_* is the (reduced) Fourier–Stieltjes algebra of a locally compact group. More specifically, is it possible to remove the condition that B_* being unital in Theorems 5.2 and 5.4 to obtain characterisations for general locally compact groups? If that is not possible, could this condition be replaced by some other (simple) one?

Acknowledgement. The first author is supported by an NSERC–grant ZC912. The authors would like to thank the anonymous referee for his/her helpful comments and corrections.

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