Adv. Oper. Theory 3 (2018), no. 1, 117–122
http://doi.org/10.22034/aot.1702-1111
ISSN: 2538-225X (electronic)
http://aot-math.org

OPERATORS WITH COMPATIBLE RANGES IN AN ALGEBRA GENERATED BY TWO ORTHOGONAL PROJECTIONS

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Dedicated to the memory of Uffe Haagerup

Communicated by D. S. Djordjević

Abstract. The criterion is obtained for operators A from the algebra generated by two orthogonal projections P, Q to have a compatible range, i.e., coincide with A^* on the orthogonal complement to the sum of the kernels of A and A^*. In the particular case of A being a polynomial in P, Q, some easily verifiable conditions are derived.

1. Introduction and preliminaries

For a Hilbert space $\mathfrak{H}$, denote by $[\mathfrak{H}]$ the C^*-algebra of all bounded linear operators acting on $\mathfrak{H}$. Given $A \in [\mathfrak{H}]$, let $\mathcal{N}(A)$ and $\mathcal{R}(A)$ stand for its kernel and range, respectively. As in [2], we say that $A \in [\mathfrak{H}]$ is a *compatible range operator* (CoR for short) if A and its hermitian adjoint A^* coincide on $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(A^*)}$.

This requirement is satisfied vacuously if A is a DR operator, i.e., $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(A^*)} = \{0\}$, or equivalently $\mathcal{N}(A) + \mathcal{N}(A^*)$ is dense in $\mathfrak{H}$. On the other hand, EP operators ($A \in [\mathfrak{H}]$ for which $\mathcal{N}(A) = \mathcal{N}(A^*)$ and therefore $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^*)}$) are CoR if and only if $A = A^*$. In particular, normal operators with compatible ranges are hermitian.

It was also observed in [2], among other things, that for any orthogonal projections $P, Q \in [\mathfrak{H}]$ the products $P, PQ, PQP, PQPQ, \dots$ are all CoR. This is

Date: Received: Feb. 2, 2017; Accepted: Apr. 25, 2017.
2010 *Mathematics Subject Classification.* Primary 47A05; Secondary 47C15, 47L30.
Key words and phrases. Hermitian operators, orthogonal projections, von Neumann algebras.

not surprising: when the number of factors is odd, the respective product is hermitian, while for an even number $n = 2k$ of factors it is $A = (PQ)^k$. Then $\overline{\mathcal{R}(A)} \subset \overline{\mathcal{R}(P)}$, $\mathcal{R}(A^*) \subset \mathcal{R}(Q)$, and so the restrictions of both A and A^* onto $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(A^*)}$ are nothing but the identity operator.

It seems natural to ask a more general question: which operators from the algebra generated by P and Q have compatible range. The aim of this short note is to address this question.

The author would like to thank the anonymous referee for the suggestions which helped to improve the exposition, and in particular prompted the addition of Section 4.

The main tool in our considerations is the classical canonical representation of the pair of orthogonal projections P, Q going back to Halmos [4], and the resulting description of the von Neumann algebra $\mathfrak{A}(P,Q)$ generated by such a pair [3]. Namely, up to a unitary similarity operators $A \in \mathfrak{A}(P,Q)$ are as follows:

$$A = \left(\oplus_{(i,j)\in\Lambda} a_{ij} I_{\mathfrak{M}_{ij}}\right) \oplus \begin{bmatrix} \phi_{00}(H) & \phi_{01}(H) \\ \phi_{10}(H) & \phi_{11}(H) \end{bmatrix}. \tag{1.1}$$

Here

$$\begin{aligned} \mathfrak{M}_{00} &= \mathcal{R}(P) \cap \mathcal{R}(Q), \quad \mathfrak{M}_{01} = \mathcal{R}(P) \cap \mathcal{N}(Q), \\ \mathfrak{M}_{10} &= \mathcal{N}(P) \cap \mathcal{N}(Q), \quad \mathfrak{M}_{11} = \mathcal{N}(P) \cap \mathcal{R}(Q), \end{aligned} \tag{1.2}$$

Λ is the set of pairs (i,j) for which $\dim \mathfrak{M}_{ij} > 0$, H is the compression of Q onto the subspace

$$\mathfrak{M} = \mathcal{R}(P) \ominus (\mathfrak{M}_{00} \oplus \mathfrak{M}_{01}),$$

$a_{ij} \in \mathbb{C}$, and ϕ_{ij} are Borel-measurable and essentially bounded functions on $[0,1]$. In particular,

$$P = I_{\mathfrak{M}_{00}} \oplus I_{\mathfrak{M}_{01}} \oplus 0_{\mathfrak{M}_{10}} \oplus 0_{\mathfrak{M}_{11}} \oplus \begin{bmatrix} I_{\mathfrak{M}} & 0 \\ 0 & 0 \end{bmatrix},$$

while

$$Q = I_{\mathfrak{M}_{00}} \oplus 0_{\mathfrak{M}_{01}} \oplus 0_{\mathfrak{M}_{10}} \oplus I_{\mathfrak{M}_{11}} \oplus \begin{bmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{bmatrix}.$$

Note that H is a positive semidefinite contraction, with 0 and 1 not lying in its point spectrum.

We refer the interested reader to [1] for a detailed survey of these and other results on “two projections theory”.

2. CoR criterion

Theorem 2.1. *Let $A \in \mathfrak{A}(P,Q)$. Then for A to have compatible range it is necessary and sufficient that in its representation* (1.1)*:*

(i) *$a_{ij} \in \mathbb{R}$ for $(i,j) \in \Lambda$, and*

(ii) *for almost all $t \in \sigma(H)$, the matrix $\Phi(t) := \begin{bmatrix} \phi_{00}(t) & \phi_{01}(t) \\ \phi_{01}(t) & \phi_{11}(t) \end{bmatrix}$ is either*

(a) *hermitian, or* (b) *singular but not normal.*

The null sets here and in what follows are understood in the sense of the spectral measure E of H.

Proof. The CoR property is preserved under unitary similarities, and so without loss of generality we may suppose that A is in the form (1.1). Let us rewrite it as $A = \oplus_{j=0}^{3} A_j$, where $A_0 = \oplus_{(i,j)\in\Lambda} a_{ij} I_{\mathfrak{M}_{ij}}$ and $A_j = \Phi(H_j)$ with H_j equal the restriction of H onto its invariant subspace $\mathfrak{M}_j$ corresponding to the spectral subset Δ_j, $j = 1, 2, 3$. Here

$$\Delta_1 = \{t \in \sigma(H) \colon \omega(t) \neq 0\},$$
$$\Delta_2 = \{t \in \sigma(H) \colon \omega(t) = 0 \text{ and } \Phi(t) \text{ is not normal}\},$$
$$\Delta_3 = \{t \in \sigma(H) \colon \omega(t) = 0 \text{ and } \Phi(t) \text{ is normal}\},$$

with $\omega := \det \Phi = \phi_{00}\phi_{11} - \phi_{01}\phi_{10}$. Condition (ii) in these terms amounts to $\Phi(t)$ being hermitian on $\Delta_1 \cup \Delta_3$.

A direct sum of operators acting on mutually orthogonal subspaces has CoR property only simultaneously with all of its direct summands. So, it suffices to consider each of the operators A_j separately.

Operators A_0 and A_3 are normal, and so CoR if and only if they are hermitian. For A_0, this is equivalent to (i), while for A_3 corresponds to $\Phi(t)$ being hermitian on Δ_3.

For A_1 we have $\mathcal{N}(A_1) = \mathcal{N}(A_1^*) = \{0\}$ by [5, Theorem 2.1], implying that A_1 is an EP operator. As such, it also has CoR property if and only if it is hermitian, that is, if $\Phi(t)$ is hermitian on Δ_1.

To complete the proof we therefore only need to show that A_2 is CoR, with no conditions imposed on $\Phi(t)$ for $t \in \Delta_2$. We will establish this by proving that A_2 is a DR operator.

Note that $\Phi(t)$ has rank one for $t \in \Delta_2$. Invoking the pertinent part of [5, Theorem 2.1], we have

$$\mathcal{N}(A_2) = \begin{bmatrix} u\chi_1 \\ -\chi_0 \end{bmatrix} (H)\mathfrak{M}_2,$$

where

$$\chi_0 = \sqrt{\frac{|\phi_{00}|^2 + |\phi_{10}|^2}{|\phi_{00}|^2 + |\phi_{01}|^2 + |\phi_{10}|^2 + |\phi_{11}|^2}}, \chi_1 = \sqrt{\frac{|\phi_{01}|^2 + |\phi_{11}|^2}{|\phi_{00}|^2 + |\phi_{01}|^2 + |\phi_{10}|^2 + |\phi_{11}|^2}},$$

and

$$u = \operatorname{sgn}(\phi_{01}\overline{\phi_{00}} + \phi_{11}\overline{\phi_{10}}).$$

A simple change of notation yields

$$\mathcal{N}(A_2^*) = \begin{bmatrix} v\psi_1 \\ -\psi_0 \end{bmatrix} (H)\mathfrak{M}_2,$$

where

$$\psi_0 = \sqrt{\frac{|\phi_{00}|^2 + |\phi_{01}|^2}{|\phi_{00}|^2 + |\phi_{01}|^2 + |\phi_{10}|^2 + |\phi_{11}|^2}}, \psi_1 = \sqrt{\frac{|\phi_{10}|^2 + |\phi_{11}|^2}{|\phi_{00}|^2 + |\phi_{01}|^2 + |\phi_{10}|^2 + |\phi_{11}|^2}},$$

and

$$v = \operatorname{sgn}(\phi_{01}\overline{\phi_{11}} + \phi_{00}\overline{\phi_{10}}).$$

From the equality

$$\begin{bmatrix} u\chi_1 \\ -\chi_0 \end{bmatrix} \psi_0 - \begin{bmatrix} v\psi_1 \\ -\psi_0 \end{bmatrix} \chi_0 = \begin{bmatrix} g \\ 0 \end{bmatrix}, \tag{2.1}$$

where $g = u\chi_1\psi_0 - v\chi_0\psi_1$, it therefore follows that the sum $\mathcal{N}(A_2) + \mathcal{N}(A_2^*)$ contains the linear manifold $\mathcal{R}(g(H_2)) \oplus \{0\}$ of $\mathfrak{M}_2 \oplus \mathfrak{M}_2$.

Observe that the function g does not vanish on Δ_2. Indeed, for $t \in \Delta_2$ at which $|\phi_{01}| \neq |\phi_{10}|$ this is true because $|u\chi_1\psi_0| = |\chi_1\psi_0| \neq |\chi_0\psi_1| = |v\chi_0\psi_1|$. On the other hand, the equality $|\phi_{01}| = |\phi_{10}|$ implies that $\phi_{01}, \phi_{10} \neq 0$ (since otherwise Φ would be normal). Condition $\omega = 0$ in its turn implies that $\phi_{00}, \phi_{11} \neq 0$, and also that

$$\phi_{01}\overline{\phi_{00}} + \phi_{11}\overline{\phi_{10}} = \phi_{01}(|\phi_{00}|^2 + |\phi_{10}|^2)/\phi_{00},$$
$$\phi_{01}\overline{\phi_{11}} + \phi_{00}\overline{\phi_{10}} = \phi_{01}(|\phi_{11}|^2 + |\phi_{10}|^2)/\phi_{11}.$$

So, $u = \operatorname{sgn}\phi_{01}/\phi_{00}$ and $v = \operatorname{sgn}\phi_{01}/\phi_{11}$. If $u = v$, then $\arg\phi_{00} = \arg\phi_{11}$, which along with $\omega = 0$ implies

$$\arg\phi_{00} = \arg\phi_{11} = (\arg\phi_{01} + \arg\phi_{10})/2 \mod \pi,$$

and once again would mean the normality of Φ. Consequently, $u \neq v$, the arguments of $u\chi_1\psi_0$ and $v\chi_0\psi_1$ are different, and their difference $g = u\chi_1\psi_0 - v\chi_0\psi_1$is therefore non-zero.

So, the (normal) operator $g(H_2)$ is injective, its range is dense in $\mathfrak{M}_2$, and thus the closure of $\mathcal{N}(A_2) + \mathcal{N}(A_2^*)$ contains $\mathfrak{M}_2 \oplus \{0\}$.

Using

$$\begin{bmatrix} u\chi_1 \\ -\chi_0 \end{bmatrix} v\psi_1 - \begin{bmatrix} v\psi_1 \\ -\psi_0 \end{bmatrix} u\chi_1 = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

in place of (2.1), we by the same token arrive at the conclusion that the closure of $\mathcal{N}(A_2) + \mathcal{N}(A_2^*)$ contains $\{0\} \oplus \mathfrak{M}_2$. Consequently, $\mathcal{N}(A_2) + \mathcal{N}(A_2^*)$ is dense in $\mathfrak{M}_2 \oplus \mathfrak{M}_2$. This completes the proof. □

3. Polynomials in two projections

Consider now a particular case when $A \in \mathfrak{A}(P, Q)$ is just a polynomial in two variables P, Q. In other words,

$$f(P, Q) = \sum c_{m,i} P_{m,i} \tag{3.1}$$

with m assuming natural values and $i = 1, 2$. Here $c_{m,i} \in \mathbb{C}$ and $P_{m,i}$ stands for the alternating product of m multiples P, Q starting with P (Q) if $i = 1$ (resp., $i = 2$). Let us introduce scalar polynomials

$$\begin{aligned} f_1(t) &= \sum_k c_{2k+1,1} t^k, \quad f_2(t) = \sum_k c_{2k,1} t^{k-1}, \\ f_3(t) &= \sum_k c_{2k+1,2} t^k, \quad f_4(t) = \sum_k c_{2k,2} t^{k-1}. \end{aligned} \tag{3.2}$$

For $A = f(P,Q)$, a straightforward computation conducted in [6] shows that in (1.1)

$$\begin{aligned} \phi_{00} &= f_1 + t(f_2+f_3+f_4), \quad \phi_{01} = (f_2+f_3)\sqrt{t(1-t)}, \\ \phi_{10} &= (f_3+f_4)\sqrt{t(1-t)}, \quad \phi_{11} = f_3(1-t), \end{aligned} \tag{3.3}$$

while

$$a_{00} = \sum_{m,i} c_{m,i}, \;\; a_{01} = c_{11}, \;\; a_{10} = 0, \;\; a_{11} = c_{12}.$$

According to (3.3),

$$\det \Phi(t) = (1-t)(f_1 f_3 - t f_2 f_4)$$

is a polynomial in t. So, it is either identically equal to zero or is non-zero except for finitely many points. Respectively, we can state two results stemming from Theorem 2.1 in the polynomial setting.

Theorem 3.1. *Let the polynomial f be such that $f_1 f_3 - t f_2 f_4$ is not identically equal to zero. Then $f(P,Q)$ is CoR for any choice of orthogonal projections P, Q if and only if f is "formally" hermitian, i.e., for all admissible k:*

$$c_{2k+1,j} \in \mathbb{R} \quad (j = 1,2), \;\; \textit{while } c_{2k,1} = \overline{c_{2k,2}}. \tag{3.4}$$

Proof. Sufficiency. Conditions (3.4) mean that $f(P,Q)$ is a linear combination (with real coefficients) of hermitian operators $P(QP)^k$, $Q(PQ)^k$, and hermitian parts of $(PQ)^k$. Thus, it is hermitian.

Necessity. Due to part (ii) of Theorem 2.1, the matrix $\Phi(t)$ must be hermitian for all, except for possibly finitely many, points of $[0,1]$. Due to the continuity of the functions involved, the hermitian property thus extends to the whole interval $[0,1]$. In other words, ϕ_{00} and ϕ_{11} must be real valued on $[0,1]$, while ϕ_{01} and ϕ_{10} are complex conjugates of each other.

From the formula for ϕ_{11} in (3.3) we conclude that the polynomial f_3 is real valued on $[0,1]$, and so its coefficients are real. From here and the expressions for ϕ_{01}, ϕ_{10} we conclude that the values of f_2 and f_4 must be complex conjugate when the argument is in $[0,1]$, thus proving that their respective coefficients are complex conjugates of each other. In other words, the second part of (3.4) holds. Finally, since f_3 and $f_2 + f_4$ are real valued on $[0,1]$, due to the expression for ϕ_{00} from (3.3) the same is true for f_1, implying that its coefficients are also all real. ◻

Recall that the pair P, Q of orthogonal projections is in *generic position* if all fours subspaces (1.2) are trivial: $\dim \mathfrak{M}_{ij} = 0$, $i, j = 1, 2$.

Theorem 3.2. *Let $f_1 f_3 = t f_2 f_4$, while $|f_2 + f_3| \neq |f_3 + f_4|$ on $(0,1)$. Then $f(P,Q)$ is CoR for any pair of orthogonal projections P, Q in generic position.*

Proof. Condition (i) of Theorem 2.1 holds vacuously, since $\Lambda = \emptyset$. Also, the matrix $\Phi(t)$ is singular due to the equality $f_1 f_3 = t f_2 f_4$ and not normal because of $|f_2 + f_3| \neq |f_3 + f_4|$. So, condition (ii) holds as well. ◻

Example 3.3. Let $f(P,Q)$ be given by (3.1), (3.2) with $f_3 = f_4 = g$, $f_2 = cg$, and $f_1 = ctg$, where g is an arbitrary polynomial not vanishing on $(0,1)$, a constant

c is such that $|c+1| \neq 2$, and the projections P, Q are in generic position. Then $f(P,Q)$ has compatible range.

4. A side remark

According to the proof of Theorem 2.1, a CoR operator A from the algebra $\mathfrak{A}(P,Q)$ is the direct sum of a hermitian summand $A_0 \oplus A_1 \oplus A_3$ and a DR operator A_2. This is not a coincidence: in fact, the following result holds.

Proposition 4.1. *An operator $A \in [\mathfrak{H}]$ is CoR if and only if $\mathfrak{H}_0 = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(A^*)}$ is its reducing subspace, and the restriction A_0 of A onto $\mathfrak{H}_0$ is hermitian. If these conditions hold, then the restriction A_1 of A onto $\mathfrak{H}_1 = \mathfrak{H}_0^{\perp}$ is a DR operator.*

Proof. Necessity. Let us write A as

$$A = \begin{bmatrix} A_0 & B \\ C & A_1 \end{bmatrix}$$

with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. If A is a CoR operator, then directly from the definition it follows that A_0 is hermitian and $C = B^*$. Moreover, $\mathcal{N}(B) \supset \mathcal{N}(A)$ since $\mathcal{N}(A) \subset \mathfrak{H}_1$. Similary, $\mathcal{N}(C^*) \supset \mathcal{N}(A^*)$. But $C^* = B$, so in fact $\mathcal{N}(B) \supset \mathcal{N}(A) + \mathcal{N}(A^*)$. Since $\mathcal{N}(A) + \mathcal{N}(A^*)$ is dense in $\mathfrak{H}_1$, it follows that $B = 0$. Consequently, A is in fact of the form $A_0 \oplus A_1$, and $\mathfrak{H}_0$ is its reducing subspace.

Sufficiency. If $A = A_0 \oplus A_1$, then in particular $\mathcal{N}(A) = \mathcal{N}(A_0) \oplus \mathcal{N}(A_1)$. But by construction $\mathcal{N}(A) \subset \mathfrak{H}_1$, implying that $\mathcal{N}(A_0) = \{0\}$ and thus $\mathcal{N}(A) = \mathcal{N}(A_1)$. Similarly, $\mathcal{N}(A^*) = \mathcal{N}(A_1^*)$. So, $\mathcal{N}(A_1) + \mathcal{N}(A_1^*)$ is dense in the domain $\mathfrak{H}_1$ of A_1, thus proving that A_1 is a DR operator. Being a direct sum of a hermitian operator A_0 and a DR operator A_1, acting on orthogonal subspaces, the operator A is therfore CoR. ☐

Proposition 4.1 is a generalization of [2, Lemma 2.3] from the case of closed range operators.

Acknowledgments. The author was supported in part by Faculty Research funding from the Division of Science and Mathematics, New York University Abu Dhabi.

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