Characterization of a b-metric space completeness via the existence of a fixed point of Ciric-Suzuki type quasi-contractive multivalued operators and applications

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Abstract

The aim of this paper is to introduce Ciric-Suzuki type quasi-contractive multivalued operators and to obtain the existence of fixed points of such mappings in the framework of b-metric spaces. Some examples are presented to support the results proved herein. We establish a characterization of strong b-metric and b-metric spaces completeness. An asymptotic estimate of a Hausdorff distance between the fixed point sets of two Ciric-Suzuki type quasi-contractive multivalued operators is obtained. As an application of our results, existence and uniqueness of multivalued fractals in the framework of b-metric spaces is proved.

1 Introduction and preliminaries

Let \((X, d)\) be a metric space. Let \(CB(X) (P(X))\) be the family of nonempty closed and bounded (nonempty subsets of \(X\)). For \(A, B \in CB(X)\), let

\[H(A, B) = \max \{\delta(A, B), \delta(B, A)\}\]

where \(d(x, B) = \inf_{w \in B} d(x, w)\) and \(\delta(A, B) = \sup_{x \in A} d(x, B)\). The mapping \(H\) is said to be a Hausdorff metric on \(CB(X)\) induced by \(d\). The metric space

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(CB(X), H) is complete if (X, d) is complete. For f : X → X and T : X → P(X), the pair (f, T) is called a hybrid pair of mappings. The fixed point problem of T is to find an x ∈ X such that x ∈ Tx (fixed point inclusion). The solution of a fixed point inclusion problem of T is called a fixed point of T. The set F(T) denotes the set of fixed points of T. A point x ∈ X is a coincidence point (common fixed point) of (f, T) if fx ∈ Tx (x = fx ∈ Tx).

Denote C(f, T) and F(f, T) by the set of coincidence and common fixed point of (f, T), respectively. The hybrid pair (f, T) is w-compatible ([1]) if f(Tx) ⊆ T(fx) for all x ∈ C(f, T). A mapping f is T-weakly commuting at x ∈ X if f2(x) ∈ T(fx). The letters $\mathbb{R}^+$ and $\mathbb{N}^*$ will denote the set of nonnegative real numbers and the set of nonnegative integers, respectively.

A mapping T : X → CB(X) is called a multivalued weakly Picard operator (A MWP operator) ([34]), if for all x ∈ X and for some y ∈ Tx, there exists a sequence $\{x_n\}$ satisfying (a1) $x_0 = x$, $x_1 = y$, (a2) $x_{n+1} ∈ Tx_n$, $n ∈ \mathbb{N}^*$ (a3) $\{x_n\}$ converges to some $z ∈ F(T)$.

The sequence $\{x_n\}$ satisfying (a1) and (a2) is called a sequence of successive approximations (ssa at (x, y)) of T starting from (x, y).

If a single valued mapping T satisfies (a1) to (a4), then it is a Picard operator.

Let T : X → P(X) be a MWP operator. Define the mapping $T^∞ : G(T) → P(F(T))$ by

$$T^∞(x, y) = \{z : \text{there is an ssa at (x, y) of T converging to } z\}$$

where $G(T) = \{(x, y) : x ∈ X, y ∈ Tx\}$ is called graph of T. A mapping f : X → X is called a selection of T : X → P(X) if C(f, T) = X.

**Definition 1.1.** ([34]) Let (X, d) be a metric space and c > 0. A MWP operator $T : X → P(X)$ is called $c$-multivalued weakly Picard ($c$-MWP) operator if there exists a selection $t^∞$ of $T^∞$ such that $d(x, t^∞(x, y)) ≤ cd(x, y)$ for all $(x, y) ∈ G(T)$.

One of the main result dealing with $c$-MWP operators is the following.

**Theorem 1.2.** ([34]) Let (X, d) be a metric space and $T_1, T_2 : X → P(X)$. If $T_i$ is a $c_i$-MWP operator for each $i ∈ \{1, 2\}$ and there exists $λ > 0$ such that $H(T_1x, T_2x) ≤ λ$ for all $x ∈ X$. Then

$$H(F(T_1), F(T_2)) ≤ λ\max\{c_1, c_2\}.$$

Banach contraction principle (BCP) [7] states that if (X, d) is a complete metric space and $f : X → X$ satisfies

$$d(fx, fy) ≤ rd(x, y) \quad (1.1)$$
for all \( x, y \in X \) with \( r \in (0,1) \), then \( f \) has a unique fixed point.

Due to its applications in mathematics and other related disciplines, BCP has been generalized in many directions. Suzuki [39] proposed a contraction condition that does not imply the continuity of a mapping \( f \). Suzuki type fixed point theorems are remarkable in the sense that these results characterize the completeness of underlying metric spaces ([39, Theorem 3]) whereas BCP does not ([15]).

A mapping \( f : X \to X \) is called quasi-contraction [12, Theorem 1] if
\[
d(f(x), f(y)) \leq r \max \{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}
\] (1.2)
for all \( x, y \in X \) with \( r \in [0,1) \).

Nadler [31] proved a multivalued version of BCP as follows.

**Theorem 1.3.** Let \((X, d)\) be a complete metric space and \( T : X \to CB(X) \). If for all \( x, y \in X \),
\[
H(Tx, Ty) \leq rd(x, y)
\]
holds for some \( r \in [0,1) \), then \( F(T) \) is nonempty.


**Theorem 1.4.** [2] Let \((X, d)\) be a complete metric space and \( T : X \to CB(X) \). If for all \( x, y \in X \),
\[
H(Tx, Ty) \leq r \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}
\] (1.3)
holds for some \( r \in \left[0, \frac{1}{2}\right) \). Then \( F(T) \) is nonempty.

Define the mapping \( \xi_1 : [0,1) \to \left(\frac{1}{2}, 1\right] \) by \( \xi_1(r) = \frac{1}{1+r} \).

Kikkawa and Suzuki [28] obtained an interesting generalization of Theorem 1.3 as follows.

**Theorem 1.5.** [28] Let \((X, d)\) be a complete metric space and \( T : X \to CB(X) \). If there exists an \( r \in [0,1) \) such that
\[
\xi_1(r)d(x, Tx) \leq d(x, y) \quad \text{implies that} \quad H(Tx, Ty) \leq rd(x, y).
\] (1.4)
for all \( x, y \in X \). Then \( F(T) \) is nonempty.

The mapping satisfying (1.4) is called \( r-KS \) multivalued operator.

Using axioms of choice, Haghi et al. [21] proved the following lemma.

**Lemma 1.6.** [21] For a nonempty set \( X \) and \( f : X \to X \), there exists a subset \( E \subseteq X \) such that \( f(E) = f(X) \) and \( f : E \to X \) is one-to-one.
Euclidean distance is an important measure of "nearness" between two real or complex numbers. This notion has been generalized further in one to many directions (see [3]). Among which one of the most important generalization is the concept of a b-metric initiated by Czerwik [17]. The reader interested in fixed point results in setup of b-metric spaces is referred to ([3, 9, 14, 16, 17, 18, 22, 29, 35]).

**Definition 1.7.** [16] Let $X$ be a nonempty set. A mapping $d: X \times X \to [0, \infty)$ is said to be a b-metric on $X$ if there exists some real constant $b \geq 1$ such that for any $x, y, z \in X$, the following condition hold:

- $(b_1)$ $d(x, y) = 0$ if and only if $x = y$,
- $(b_2)$ $d(x, y) = d(y, x)$,
- $(b_3)$ $d(x, y) \leq bd(x, z) + bd(z, y)$.

The pair $(X, d)$ is termed as b-metric space with b-metric constant $b$. If $(b_3)$ is replaced by

- $(b_4)$ $d(x, y) \leq d(x, z) + bd(z, y)$

then $(X, d)$ is called a strong b-metric space (Kirk and Shahzad [26]) with strong b-metric constant $b \geq 1$.

If $b = 1$, then strong b-metric space is a metric space. Every metric is a strong b-metric and every strong b-metric is b-metric but converse does not hold in general ([4, 5, 13, 16, 35]).

Consistent with [16, 17, 18, 35], the following (definitions and lemmas) will be needed in the sequel.

**Lemma 1.8.** [16, 17, 18, 35] Let $(X, d)$ be a b-metric space, $x, y \in X$ and $A, B \in CB(X)$. The following statements hold:

- $(c_1)$ $(CB(X), H)$ is a b-metric space.
- $(c_2)$ $d(x, B) \leq H(A, B)$ for all $x \in A$.
- $(c_3)$ $d(x, A) \leq bd(x, y) + bd(y, A)$.
- $(c_4)$ For $h > 1$ and $z \in A$, there is a $w \in B$ such that $d(z, w) \leq hH(A, B)$.
- $(c_5)$ For every $h > 0$ and $z \in A$, there is a $w \in B$ such that $d(z, w) \leq H(A, B) + h$.
- $(c_6)$ $d(w, A) = 0$ if and only if $w \in A = A$.
- $(c_7)$ For $\{x_n\} \subseteq X$, $d(x_0, x_n) \leq bd(x_0, x_1) + \ldots + b^{n-1}d(x_{n-2}, x_{n-1}) + b^{n-1}d(x_{n-1}, x_n)$.

**Definition 1.9.** Let $(X, d)$ be a b-metric space. A sequence $\{x_n\}$ in $X$ is called:
Lemma 1.10. \cite{36} If a sequence \( \{u_n\} \) in a b-metric \((X,d)\) satisfies 
\[ d(u_{n+1}, u_{n+2}) \leq h d(u_n, u_{n+1}) \]
for all \( n \in \mathbb{N} \) and for some \( 0 \leq h < 1 \), then it is a Cauchy sequence in \( X \) provided that \( h b < 1 \).

Equivalently, a sequence \( \{x_n\} \) in b-metric space \( X \) is Cauchy if and only if 
\[ \lim_{n \to \infty} d(x_n, x_{n+p}) = 0 \]
for all \( p \in \mathbb{N} \). A sequence \( \{x_n\} \) is convergent to \( x \in X \) if and only if 
\[ \lim_{n \to \infty} d(x_n, x) = 0. \]

Lemma 1.11. Let \((X,d)\) be a b-metric space, \( A, B \in P(X) \). If there exists a \( \lambda > 0 \) such that 
(i) for each \( \tilde{a} \in A \), there exists a \( \tilde{b} \in B \) such that 
\[ d(\tilde{a}, \tilde{b}) \leq \lambda, \]
(ii) for each \( \tilde{b} \in B \), there exists an \( \tilde{a} \in A \) such that 
\[ d(\tilde{a}, \tilde{b}) \leq \lambda, \]
then \( H(A, B) \leq \lambda \).

A subset \( Y \subset X \) is closed if and only if for each sequence \( \{x_n\} \) in \( Y \) which converges to an element \( x \), we must have \( x \in Y \). A subset \( Y \subset X \) is bounded if \( \text{diam}(Y) \) is finite, where \( \text{diam}(Y) = \sup \{d(a, b), a, b \in Y\} \). A b-metric space \((X,d)\) is said to be complete if every Cauchy sequence in \( X \) is convergent in \( X \).

An et al. \cite{4} studied the topological properties of b-metric spaces. In a b-metric space \((X,d)\), \( d \) is not necessarily continuous in each variable. In a b-metric space \((X,d)\), if \( d \) is continuous in one variable, then \( d \) is continuous in other variable. A ball \( B_\varepsilon(x_0) = \{x : d(x, x_0) < \varepsilon\} \) in b-metric space \((X,d)\) is not necessarily an open set. A ball in a b-metric space \((X,d)\) is open if \( d \) is continuous in one variable (see \cite{4}).

In what follows we assume that a b-metric \( d \) is continuous in one variable.

Aydı et al. \cite{6} proved the following result as a generalization of Theorem 1.4 \cite[Theorem 1.4]{2}.

Theorem 1.12. \cite{6} Let \((X,d)\) be a complete b-metric space and \( T : X \to CB(X) \). If there exists some \( r \in [0,1) \) with 
\[ r < \frac{1}{b^2 + b} \]
such that 
\[ H(Tx, Ty) \leq r \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \]
holds for all \( x, y \in X \), then \( F(T) \) is nonempty.

Define the mapping \( \xi_2 : [0,1) \to \left[ \frac{1}{2}, 1 \right] \) by 
\[ \xi_2(r) = \frac{1}{1 + br}. \]
Kutbi et al. [29] obtained the following Suzuki type fixed point theorem result in the setup of b-metric spaces.

**Theorem 1.13.** [29] Let $(X, d)$ be a complete b-metric space and $T : X \to CB(X)$. If there exists some $r \in [0, 1)$ with $r < \frac{1}{b^2 + b}$ such that
\[
\xi_2(r)d(x, Tx) \leq bd(x, y)
\]
implies that
\[
H(Tx, Ty) \leq rd(x, y)
\]
for $x, y \in X$, then $F(T)$ is nonempty.

Let $(X, d)$ be a b-metric space, $f : X \to X$, $T : X \to CB(X)$ and $x, y \in X$. We use the notations
\[
M_f(x, y) = \max \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\},
\]
\[
M_T(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},
\]
\[
M_f^I(x, y) = \max \{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}.
\]
Define
\[
\Lambda = \{\xi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} : \xi(s, t) \leq \frac{s}{b} - t\}
\]
where $b$ is the b-metric constant. Note that $\xi(bt, t) \leq 0$ and $\xi\left(s, \frac{s}{b}\right) \leq 0$ for all $s \in \mathbb{R}^+$.

**Example 1.14.** For $i \in \{3, 4\}$, define $\xi_i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

1. $\xi_3(s, t) = \psi(s) - \varphi(t)$, where $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are functions satisfying $\psi(t) \leq \frac{t}{b}$, $t \leq \varphi(t)$, and $b \geq 1$.

2. $\xi_4(s, t) = \frac{s}{b} - \frac{\psi(s, t)}{\varphi(s, t)}t$, where $\psi, \varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are functions satisfying $\varphi(s, t) \leq \psi(s, t)$ for all $s, t \geq 0$.

**Definition 1.15.** Let $(X, d)$ be a b-metric space. A mapping $T : X \to CB(X)$ is called a Ciric-Suzuki type quasi-contractive multivalued operator if there exists an $r \in [0, 1)$ satisfying $r < \frac{1}{b^2 + b}$ such that
\[
\xi(d(x, Tx), d(x, y)) \leq 0
\]
implies that
\[
H(Tx, Ty) \leq rM_T(x, y)
\]
for all $x, y \in X$, where $\xi \in \Lambda$. 
If $CB(X) = \{\{x\} : x \in X\}$, then $T : X \to CB(X)$ is called a Ciric-Suzuki type quasi-contractive operator.

**Definition 1.16.** Let $(X, d)$ be a b-metric space, $f : X \to X$ and $T : X \to CB(X)$. A hybrid pair $(f, T)$ is said to be Ciric-Suzuki type quasi-contractive hybrid pair if there exists an $r \in [0, 1)$ satisfying $r < \frac{1}{b^2 + b}$ such that

$$
\xi(d(fx, Tx), d(fx, fy)) \leq 0
$$

implies that

$$
H(Tx, Ty) \leq r M^T_f(x, y)
$$

for all $x, y \in X$ and for some $\xi \in \Lambda$.

In this paper, we obtain fixed point results for Ciric-Suzuki type quasi-contractive multivalued operators in b-metric space. Further, completeness characterization of strong b-metric and b-metric spaces via the existence of fixed point of Ciric-Suzuki type quasi-contractive operators is obtained. Our results extend, unify and generalize the comparable results in [2, 6, 12, 27, 29, 31, 33, 39]. As applications of our results:

1. We prove the existence of coincidence and common fixed point of hybrid pair of Ciric-Suzuki type quasi-contractive single valued and multivalued operators.
2. We give an estimate of Hausdorff distance between the fixed point sets of two Ciric-Suzuki type quasi-contractive multivalued operators.
3. We show that for a uniformly convergent sequence of Ciric-Suzuki type quasi-contractive multivalued operators, the corresponding sequence of fixed points set is uniformly convergent.
4. We obtain a unique multivalued fractal with respect to iterated multifunction system of Ciric-Suzuki type quasi-contractive multivalued operators.

**2 Fixed points of Ciric-Suzuki type quasi-contractive multivalued operators**

In this section, we obtain some fixed point results of Ciric-Suzuki type quasi-contractive multivalued operators in the framework of complete b-metric spaces.

We start with the following result.

**Theorem 2.1.** Let $(X, d)$ be a complete b-metric space and $T : X \to CB(X)$ a Ciric-Suzuki type quasi-contractive multivalued operator. Then $T$ is a MWP operator.
\textbf{Proof.} Let \( u \) and \( v \) be given points in \( X \). If \( M_T(u, v) = 0 \), then \( u = v \in Tu \). Define a sequence \( \{u_n\} \) by \( u_n = u = v \), for all \( n \in \mathbb{N}^* \). Clearly, \( u_n \in Tu_n \) and \( \{u_n\} \) converges to \( u = v \in F(T) \). Hence \( T \) is a MWP operator.

Suppose that \( M_T(u, v) > 0 \) for all \( u, v \in X \). As \( r < \frac{1}{b^2 + b} \), there exist \( \alpha \in \mathbb{R}^+ \) such that \( \frac{r}{2} + \alpha = \frac{1}{2} \left( \frac{1}{b^2 + b} \right) \). Clearly,

\[ 0 < r + \alpha = \frac{1}{2} \left( \frac{1}{b^2 + b} + r \right) = \beta \ (\text{say}) < 1. \]

Let \( u_0 \) be any point in \( X \) and \( u_1 \in Tu_0 \). Note that

\[
\xi (d (u_0, Tu_0), d (u_0, u_1)) \leq \frac{1}{b} d (u_0, Tu_0) - d (u_0, u_1) \\
\leq d (u_0, Tu_0) - d (u_0, u_1) \\
\leq d (u_0, u_1) - d (u_0, u_1) = 0.
\]

As \( T \) is a Ciric-Suzuki type quasi-contractive multivalued operator, we obtain

\[
H(Tu_0, Tu_1) \leq r M_T(u_0, u_1). \tag{2.1}
\]

By Lemma 1.8, there exists an element \( u_2 \in Tu_1 \) such that

\[
d(u_1, u_2) \leq H(Tu_0, Tu_1) + \alpha M_T(u_0, u_1). \tag{2.2}
\]

From (2.1) and (2.2), we have

\[
d(u_1, u_2) \leq \frac{1}{b} d (u_0, Tu_1) - d (u_1, u_2) \\
\leq \frac{1}{b} d (u_0, Tu_1) + \alpha M_T(u_0, u_1) \\
= \beta M_T(u_0, u_1) \\
= \beta \max \{d (u_0, u_1), d (u_0, Tu_0), (u_1, Tu_1), d (u_0, Tu_1), d (u_1, Tu_0)\} \\
\leq \beta \max \{d (u_0, u_1), (u_1, u_2), d (u_0, u_2), d (u_1, u_1)\} \\
\leq \beta \max \{d (u_0, u_1), (u_1, u_2), b (d (u_0, u_1) + d (u_1, u_2))\} \\
= b \beta (d (u_0, u_1) + d (u_1, u_2)).
\]

That is

\[
d(u_1, u_2) \leq b \beta (d (u_0, u_1) + d (u_1, u_2)). \tag{2.3}
\]

As

\[
\xi (d (u_1, Tu_1), d (u_1, u_2)) \leq \frac{1}{b} d (u_1, Tu_1) - d (u_1, u_2) \\
\leq d (u_1, Tu_1) - d (u_1, u_2) \\
\leq d (u_1, u_2) - d (u_1, u_2) = 0.
\]
We have
\[ H(Tu_1, Tu_2) \leq r M_T(u_1, u_2). \] (2.4)

Again by Lemma 1.8, there exists an element \( u_3 \in Tu_2 \) such that
\[ d(u_2, u_3) \leq H(Tu_1, Tu_2) + \alpha M_T(u_1, u_2). \] (2.5)

By (2.4) and (2.5), we obtain that
\[ d(u_2, u_3) \leq H(Tu_1, Tu_2) + \alpha M_T(u_1, u_2) \leq r M_T(u_1, u_2) + \alpha M_T(u_1, u_2) = \beta M_T(u_1, u_2) = \beta \max\{d(u_1, u_2), d(u_2, u_3), d(u_2, u_2)\}. \]

That is
\[ d(u_2, u_3) \leq b \beta (d(u_1, u_2) + d(u_2, u_3)). \] (2.6)

Continuing this way, we can obtain a sequence \( \{u_n\} \) in \( X \) such that \( u_{n+1} \in Tu_n \) and it satisfies:
\[ d(u_n, u_{n+1}) \leq b \beta (d(u_{n-1}, u_n) + d(u_n, u_{n+1})) \] (2.7)

for all \( n \in \mathbb{N}^* \). If \( \delta_n = d(u_n, u_{n+1}) \), then from (2.7), we have \( \delta_n \leq \gamma \delta_{n-1} \), where \( \gamma = \frac{b \beta}{1 - b \beta^2} \). Now by \( b \geq 1 \) and \( r < \frac{1}{b^2 + b} \), we have
\[ b \beta = \frac{b}{2} \left( \frac{1}{b^2 + b} + r \right) < \frac{1}{1 + b} \quad \text{and} \quad \gamma = \frac{b \beta}{1 - b \beta} < \frac{1}{b}. \]

That is \( b \gamma < 1 \). By Lemma 1.10, \( \{u_n\} \) is a Cauchy sequence and hence
\[ \lim_{n \to \infty} d(u_n, z) = 0 \] (2.8)

for some \( z \in X \). Now we claim that
\[ d(z, Tx) \leq r \max\{d(z, x), d(x, Tx)\} \] (2.9)

for all \( x \neq z \). As \( \lim_{n \to \infty} d(u_n, z) = 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( d(u_n, z) <
\[ \frac{1}{3b} d(z, x) \text{ for all } n \geq n_0 \text{ and } x \neq z. \]

Note that
\[
\xi \left( d(u_n, Tu_n), d(u_n, x) \right) \leq \frac{1}{b} d(u_n, Tu_n) - d(u_n, x)
\]
\[
\leq \frac{1}{b} d(u_n, u_{n+1}) - d(u_n, x)
\]
\[
\leq \frac{1}{b} \left( bd(u_n, z) + bd(z, u_{n+1}) \right) - d(u_n, x)
\]
\[
\leq \frac{2}{3b} d(z, x) - d(u_n, x)
\]
\[
= \frac{1}{b} \left( d(z, x) - \frac{1}{3} d(z, x) \right) - d(u_n, x)
\]
\[
\leq \frac{1}{b} \left( d(z, x) - bd(u_n, z) \right) - d(u_n, x)
\]
\[
\leq \frac{1}{b} \left( bd(u_n, x) \right) - d(u_n, x) = 0
\]

for all \( n \geq n_0 \). That is
\[
\xi \left( d(u_n, Tu_n), d(u_n, x) \right) \leq 0
\] (2.10)

for all \( n \geq n_0 \). Thus
\[
d(u_{n+1}, Tx) \leq H(Tu_n, Tx)
\]
\[
\leq r \text{M}r(u_n, x)
\]
\[
= r \max \left\{ d(u_n, x), d(u_n, Tu_n), d(x, Tx), d(u_n, Tx), d(x, Tu_n) \right\}
\]
\[
\leq r \max \left\{ d(u_n, x), d(u_n, u_{n+1}), d(x, Tx), d(u_n, Tx), d(x, u_{n+1}) \right\}
\]

for all \( n \geq n_0 \). Now, by taking limit as \( n \to \infty \) on both sides of the above inequality, it follows that
\[
d(z, Tx) \leq r \max \left\{ d(z, x), d(x, Tx), d(z, Tx) \right\}
\]

If \( \max \left\{ d(z, x), d(x, Tx), d(z, Tx) \right\} = d(z, Tx) \), then we obtain that
\[
d(z, Tx) \leq rd(z, Tx) < \beta d(z, Tx) < d(z, Tx),
\]
a contradiction and hence (2.9) holds for all \( x \neq z \). Now we show that \( z \in Tz \).
Assume on contrary that \( z \notin Tz \). Clearly, \( r < \frac{1}{b^2 + b} \) implies that \( 2rb < 1 \). We now choose \( a \in Tz \) such that \( a \neq z \) and \( d(z, a) < d(z, Tz) + \left( \frac{1}{2rb} - 1 \right) d(z, Tz) \).
That is
\[
2brd(z, a) < d(z, Tz).
\] (2.11)
Note that
\[ \xi \left( d(z, Tz), d(z, a) \right) \leq \frac{1}{b} \left( d(z, Tz) - d(z, a) \right) \]
\[ \leq d(z, Tz) - d(z, a) \leq d(z, a) - d(z, a) = 0. \]

Hence
\[ H(Tz, Ta) \leq r M_T(z, a) \]
\[ \leq r \max \{d(z, a), d(z, Tz), d(a, Ta), d(z, Ta), d(a, Tz)\} \]
\[ \leq r \max \{d(z, a), d(z, a), d(a, Ta), d(z, Ta), d(a, a)\} \]
\[ = r \max \{d(z, a), d(a, Ta), d(z, Ta)\}. \]

If \( \max \{d(z, a), d(a, Ta), d(z, Ta)\} = d(a, Ta) \), then we have
\[ d(a, Ta) \leq H(Tz, Ta) \leq r d(a, Ta) \]
which implies either \( a \in Ta \) or \( d(a, Ta) < d(a, Ta) \), a contradiction. Hence
\[ H(Tz, Ta) \leq r \max \{d(z, a), d(z, Ta)\}. \]

If \( \max \{d(z, a), d(a, Ta), d(z, Ta)\} = d(z, Ta) \), then (2.9) gives that
\[ H(Tz, Ta) \leq rd(z, Ta) \]
\[ \leq r^2 \max\{d(z, a), d(a, Ta)\} \]
\[ \leq r \max\{d(z, a), d(a, Ta)\}. \]

As \( \max\{d(z, a), d(a, Ta)\} = d(a, Ta) \), is not possible, we have
\[ H(Tz, Ta) \leq rd(z, a). \] (2.12)

From (2.9) and (2.12), we obtain that
\[ d(z, Ta) \leq r \max\{d(z, a), d(a, Ta)\} \leq r \max\{d(z, a), H(Tz, Ta)\} \leq rd(z, a). \] (2.13)

Now, by (2.11), (2.12), and (2.13), we have
\[ d(z, Tz) \leq bd(z, Ta) + bH(Tz, Ta) \]
\[ \leq brd(z, a) + brd(z, a) \]
\[ = 2brd(z, a) < d(z, Tz), \]
a contradiction. Hence \( z \in Tz \).

\[ \square \]

**Remark 2.2.** We obtain Theorem 1.12 as a special case of Theorem 2.1.
Remark 2.3. Theorem 1.13 follows from 2.1. Indeed, define the mapping $\xi$ by $\xi(s, t) = \frac{\xi_2(r)}{b} s - t$, where $\xi_2(r) = \frac{1}{1 + br}$. Clearly, $\xi(s, t) \leq \frac{s}{b} - t$ as $\xi_2(r) \leq 1$. Take $s = d(x, Tx)$, $t = d(x, y)$ and
$$\max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} = d(x, y).
$$

Corollary 2.4. Let $(X, d)$ be a complete b-metric space and $T : X \to CB(X)$. If for any $x, y \in X$, $d(x, Tx) \leq bd(x, y)$ implies that $H(Tx, Ty) \leq r \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ for some $r \in \left[0, \frac{1}{b^2 + b} \right)$. Then $T$ is a MWP operator.

Example 2.5. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $d : X \times X \to \mathbb{R}^+$ be defined as $d(x_1, x_2) = d(x_1, x_3) = 3$, $d(x_1, x_4) = d(x_1, x_5) = 12$, $d(x_2, x_3) = d(x_3, x_4) = d(x_3, x_5) = 9$, $d(x_2, x_4) = 8$, $d(x_2, x_3) = 6$, $d(x_4, x_5) = 2$, $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. As $12 = d(x_1, x_4) \not\leq d(x_1, x_2) + d(x_2, x_4) = 11$, $d$ is not a metric on $X$. On the other hand, $(X, d)$ is a complete b-metric space with parameter $b \geq \frac{12}{11} > 1$. Suppose that $\xi(s, t) = \frac{s}{b} - t \in \Lambda$, $r = \frac{2}{5}$.

Then $r < \frac{121}{276} = \frac{1}{b^2 + b}$. Define the mapping $T : X \to CB(X)$ by

$$Tx = \begin{cases} 
\{x_1\} & \text{if } x = x_1, x_2, x_3, \\
\{x_2\} & \text{if } x = x_4, \\
\{x_3\} & \text{if } x = x_5.
\end{cases}$$

Note that $H(Tx, Ty) = 0 \leq rM_T(x, y)$ for all $x, y \in \{x_1, x_2, x_3\}$. If $x = x_1$ and $y \in \{x_4, x_5\}$, then $H(Tx, Ty) = d(x, y) = 3 \leq 4.8 = rd(x, y) \leq rM_T(x, y)$. If $x = x_2$ and $y = x_4$, then we have $H(Tx_2, Tx_4) = d(x_1, x_2) = 3 \leq 3.2 = rd(x_2, x_4) \leq rM_T(x_2, x_4)$. For $x \in \{x_2, x_3\}$ and $y \in \{x_4, x_5\}$, we have $H(Tx, Ty) = 3 \leq 3.6 = rd(x, y) \leq rM_T(x, y)$. Note that

$$\xi\left(d(x_4, Tx_4), d(x_4, x_5)\right) = \frac{11d(x_4, x_2)}{12} - d(x_4, x_5) = \frac{16}{3} > 0, \text{ and}$$

$$\xi\left(d(x_5, Tx_5), d(x_5, x_4)\right) = \frac{11d(x_5, x_3)}{12} - d(x_5, x_4) = \frac{25}{4} > 0.$$ 

Hence, for all $x, y \in X$, we have $\xi(d(x, Tx), d(x, y)) \leq 0$ implies that $H(Tx, Ty) \leq rM_T(x, y)$. Thus all the conditions of Theorem 2.1 are satisfied. On the other hand, if we take $x = x_4, y = x_5$, then we have

$$H(Tx_4, Tx_5) = d(x_2, x_3) = 6 \text{ and}$$

$$M_T(x_4, x_5) = \max\{d(x_4, x_3), d(x_4, Tx_4), d(x_5, Tx_5), d(x_4, Tx_5), d(x_5, Tx_4)\} = \max\{d(x_4, x_3), d(x_4, x_2), d(x_5, x_3), d(x_4, x_3), d(x_5, x_2)\} = 9.$$
Hence \( H(Tx_4, Tx_5) = 6 \not\geq 3.6 = 9r = rM_T(x_4, x_5) \) for any \( r < \frac{121}{276} = \frac{1}{b^2 + b} \).
Thus, Theorem 1.12 is not applicable in this case. Hence Theorem 2.1 is a proper generalization of Theorem 1.12 which in turn generalize Theorems 1.3, 1.4 and [12, Theorem 1].

Example 2.6. Let \( X = \{x_1, x_2, x_3\} \) and \( d : X \times X \to \mathbb{R}^+ \) be defined as \( d(x_1, x_2) = 4, d(x_1, x_3) = 1, d(x_2, x_3) = 2, d(x, x) = 0 \) and \( d(x, y) = d(y, x) \) for all \( x, y \in X \). As \( 4 = d(x_1, x_2) \not\leq d(x_1, x_3) + d(x_3, x_2) = 3 \), \( d \) is not a metric on \( X \). Indeed \((X, d)\) is a \( b \)-metric space with \( b \geq \frac{4}{3} > 1 \). Define the mapping \( T : X \to CB(X) \) by

\[
Tx = \begin{cases} 
\{x_1, x_3\} & \text{if } x = x_1, x_3, \\
\{x_1\} & \text{if } x = x_2.
\end{cases}
\]

Let \( \xi(s, t) = \frac{s}{b} - t \in \Lambda \) and \( r = \frac{3}{10} \). Clearly, \( r < \frac{9}{28} = \frac{1}{b^2 + b} \). If \( x, y \in \{x_1, x_3\} \), then \( H(Tx, Ty) = 0 \leq rM_T(x, y) \). If \( x \in \{x_1, x_3\} \) and \( y = x_2 \), then \( H(Tx, Ty) = 1 \leq 1.2 \leq rM_T(x, y) \). Hence for any \( x, y \in X \), \( \xi(d(x, Tx), d(x, y)) \leq 0 \) implies that \( H(Tx, Ty) \leq rM_T(x, y) \). Thus, all the conditions of Theorem 2.1 are satisfied. On the other hand, if \( x = x_2, y = x_3 \), then \( \xi_2(r)d(x_3, Tx_3) = 0 \leq bd(x_3, x_2) = 2 \), and \( H(Tx_3, Tx_2) = d(x_1, x_3) = 1 \).

So, \( H(Tx_3, Tx_2) = 1 \not\leq 0.6 = 2r = rd(x_3, x_2) \) for any \( r < \frac{9}{28} = \frac{1}{b^2 + b} \). Hence Theorem 1.13 is not applicable in this case. This implies that Theorem 2.1 is a proper generalization of Theorem 1.13 which itself is a generalization of Theorem 1.5, and Theorem 1.3.

Corollary 2.7. Let \( (X, d) \) be a complete \( b \)-metric space and \( f : X \to X \) a Ciric-Suzuki type quasi-contractive operator. Then \( F(f) = \{u\} \), and the sequence \( \{f^n x\} \) converges to \( u \) for any choice of an element \( x \in X \).

Proof. It follows from Theorem 2.1 that \( F(f) \) is nonempty and for all \( x \in X \), the sequence \( f^n x \to u \) as \( n \to \infty \). To prove the uniqueness of fixed point of \( f \); let \( u, v \in F(f) \) with \( u \neq v \). Note that \( \xi(d(u, fu), d(u, v)) \leq \frac{1}{b}d(u, fu) - d(u, v) = -d(u, v) \leq 0 \). Thus, we have

\[
d(u, v) = d(fu, fv) \leq rM_f(u, v)
\]

\[
= r \max\{d(u, v), d(u, fu), d(v, fv), d(u, fv), d(v, fu)\}
\]

\[
= rd(u, v) < d(u, v),
\]

a contradiction and hence \( F(f) \) is singleton. □

CHARACTERIZATION OF A B-MATRIC SPACE COMPLETENESS VIA THE EXISTENCE OF A FIXED POINT OF CIRIC-SUZUKI TYPE QUASI-CONTRACTIVE MULTIVALUED OPERATORS AND APPLICATIONS
Corollary 2.8. Let \((X, d)\) be a complete b-metric space and \(f : X \to X\). If for any \(x, y \in X\), \(d(x, fx) \leq bd(x, y)\) implies that \(d(fx, fy) \leq rd(x, y)\) for some \(r \in \left[0, \frac{1}{b^2 + b}\right]\). Then \(F(f) = \{u\}\) and the sequence \(\{f^nx\}\) converges to \(u\) for any choice of an element \(x \in X\).

Corollary 2.9. Let \((X, d)\) be a complete b-metric space and \(f : X \to X\) a mapping. If there exists a \(r \in [0, 1)\) with \(r < \frac{1}{b^2 + b}\) such that \(\xi(d(x, fx), d(x, y)) \leq 0\) implies that \(d(fx, fy) \leq rd(x, y)\) for all \(x, y \in X\), where \(\eta : [0, 1) \to (0, 1]\). Then \(F(f) = \{u\}\), and the sequence \(\{f^nx\}\) converges to \(u\) for any choice of an element \(x \in X\).

Proof. It follows from Corollary 2.7.

Corollary 2.10. Let \((X, d)\) be a complete b-metric space and \(f : X \to X\) a mapping. If there exists a \(r \in [0, 1)\) with \(r < \frac{1}{b^2 + b}\) such that \(\eta(r)d(x, fx) \leq bd(x, y)\) implies that \(d(fx, fy) \leq rd(x, y)\) for all \(x, y \in X\), where \(\eta : [0, 1) \to (0, 1]\). Then \(F(f) = \{u\}\), and the sequence \(\{f^nx\}\) converges to \(u\) for any choice of an element \(x \in X\).

Proof. Consider \(\xi(s, t) = \frac{\eta(r)}{b}s - t \leq \frac{r}{b} - t\). Hence \(\xi \in \Lambda\). If \(s = d(x, fx)\) and \(t = d(x, y)\) then \(\xi(d(x, fx), d(x, y)) = \frac{\eta(r)}{b}s - t \leq 0\). Hence result follows from Corollary 2.9.

Corollary 2.11. Let \((X, d)\) be a complete strong b-metric space and \(f : X \to X\) a mapping. If there exists a \(r \in [0, 1)\) with \(r < \frac{1}{b^2 + b}\) such that \(\eta(r)d(x, fx) \leq bd(x, y)\) implies that \(d(fx, fy) \leq rd(x, y)\) for all \(x, y \in X\), where \(\eta : [0, 1) \to (0, 1]\). Then \(F(f) = \{u\}\), and the sequence \(\{f^nx\}\) converges to \(u\) for any choice of an element \(x \in X\).

Proof. It follows from Corollary 2.10 as every strong b-metric is b-metric.

3 Characterization of a b-metric space completeness

Connell studied properties of fixed point sets and presented an example \([15, \text{Example 3}]\) of a separable and locally contractible incomplete metric space that has a fixed point property (FPP) for contraction mappings. This shows that BCP does not characterize metric completeness (see also \([20]\)). Kannan \([24, 25]\) proved a fixed point theorem which is independent of BCP. Subrahmanian \([38]\) proved that if underlying metric space \(X\) has FPP for Kannan type contractions, then \(X\) is complete. Suzuki \([39]\) presented a fixed point theorem that also characterize metric completeness of \(X\). For more details on FPP and completeness properties of metric spaces, see \([11]\).
In this section, we present some results about the strong b-metric and b-metric completeness characterizations via fixed point results obtained in section 2.

Jovanovic et al. [23] proved the following version of BCP in b-metric spaces.  

**Theorem 3.1.** Let \((X,d)\) be a complete b-metric space and \(T : X \to X\) a map such that \(d(Tx, Ty) \leq rd(x, y)\) for all \(x, y \in X\) and some \(r \in \left[0, \frac{1}{b}\right)\). Then \(F(T)\) is singleton.

Dung et al. [19] replaced the condition \(0 \leq r < \frac{1}{b}\) with \(0 \leq r < 1\) and proved that BCP can be transported in b-metric spaces without imposing any additional condition on a contraction constant \(r\).

They proved the following result.  

**Theorem 3.2.** Let \((X,d)\) be a complete b-metric space and \(T : X \to X\) a map such that \(d(Tx, Ty) \leq rd(x, y)\) for all \(x, y \in X\) and some \(r \in [0, 1)\). Then \(F(T)\) is singleton.

Park and Rhoads [32] commented on characterization of metric completeness.

We present analogous comments in b-metric spaces.

Let \((X,d)\) be a b-metric space and \(B\) a class of mappings of a b-metric space \(X\) such that if any map in \(B\) has a fixed point then \(X\) is complete. Let \(A\) be a class of mappings of a b-metric space \(X\) containing \(B\) such that completeness of \(X\) implies the existence of fixed point of any map in \(A\).

**Theorem 3.3.** (compare [32]) If \((X,d)\) is a b-metric space, then  

\[ X \text{ is complete if and only if any map in } A \text{ has a fixed point.} \]

*Proof.* If \(X\) is complete then, any map in \(A\) has a fixed point. Conversely, let any map in \(A\) has a fixed point, then any map in \(B\) has a fixed point. Then by assumption on \(B\), \(X\) is complete. \(\square\)

We present the following lemma that is needed to prove the main result in this section.

**Lemma 3.4.** Let \((X,d)\) be a strong b-metric space and \(\{x_n\}\) a Cauchy sequence in \(X\). Then \(d(x, x_n)\) is a Cauchy sequence in \(\mathbb{R}\) for all \(x\) in \(X\).

*Proof.* Note that  
\[ d(x, x_n) \leq d(x, x_m) + bd(x_m, x_n) \]
for each \( n, m \in \mathbb{N} \). Thus, we have

\[ |d(x, x_n) - d(x, x_m)| \leq bd(x_m, x_n) \]

for each \( n, m \in \mathbb{N} \). The result follows as \( \{x_n\} \) is a Cauchy sequence in \( X \). \( \square \)

The following result gives the characterization of completeness of a strong b-metric space.

**Theorem 3.5.** Let \((X, d)\) be a strong b-metric space. For \( r \in [0,1) \) with \( r < \frac{1}{b^2+b} \), let \( A_{r,\eta} \) be a class of mappings \( T \) on \( X \) which satisfies the following:

(a) For any \( x, y \in X \)

\[ \eta(r)d(x,Tx) \leq bd(x,y) \] implies that \( d(Tx,Ty) \leq rd(x,y) \quad (3.1) \]

where \( \eta : [0,1) \rightarrow (0,1] \).

Let \( B_{r,\eta} \) be the class of mappings \( T \) on \( X \) satisfying (a) and the following:

(b) \( T(X) \) is countably infinite.

(c) Every subset of \( T(X) \) is closed.

Then the following are equivalent:

(i) \( (X, d) \) is complete,

(ii) Every mapping \( T \in A_{r,\eta} \) has a fixed point for all \( r \in [0,1) \) with \( r < \frac{1}{b^2+b} \).

(iii) There exists an \( r \in (0,1) \) with \( r < \frac{1}{b^2+b} \) such that every mapping \( T \in B_{r,\eta} \) has a fixed point.

**Proof.** It follows from Corollary 2.11 that (i) implies (ii). As \( B_{r,\eta} \subseteq A_{r,\eta} \), so (ii) implies (iii). We now show that (iii) implies (i). Suppose that \( (X, d) \) is not complete. That is, there exists a Cauchy sequence \( \{u_n\} \) which does not converge. Define a function \( f : X \rightarrow [0, \infty) \) by \( f(x) = \lim_{n \to \infty} d(x, u_n) \) for \( x \in X \).

By Lemma 3.4, \( \{d(x, u_n)\} \) is a Cauchy sequence in \( \mathbb{R} \) for each \( x \in X \). Hence \( f \) is well defined. Note that \( f(x) > 0 \) for every \( x \in X \) and \( \lim_{n \to \infty} f(u_n) = 0 \). Consequently, for every \( x \in X \) there exists a \( v \in \mathbb{N} \) such that

\[ f(u_v) \leq \left( \frac{r\eta(r)}{3b^3 + r\eta(r)} \right) f(x). \quad (3.2) \]
Define \( T(x) = u_\nu \). Then

\[
f(Tx) \leq \left( \frac{r \eta(r)}{3b^3 + r \eta(r)} \right) f(x) \quad \text{and} \quad Tx \in \{u_n : n \in \mathbb{N} \} \quad (3.3)
\]

for all \( x \in X \). From (3.3), we have \( f(Tx) < f(x) \), and hence \( Tx \neq x \) for all \( x \in X \). That is, \( T \) has no fixed point. As \( T(X) \subseteq \{u_n : n \in \mathbb{N} \} \), so (b) holds.

It is easy to show that (c) holds. Note that, for all \( x, y \in X \)

\[
f(x) - f(y) \leq bd(x, y) \quad f(y) - f(x) \leq bd(x, y) \quad f(x) - f(Tx) \leq bd(x, Tx)
\]

and

\[
d(Tx, Ty) \leq f(Tx) + bf(Ty).
\]

Fix \( x, y \in X \) such that \( \eta(r)d(x, Tx) \leq bd(x, y) \). We now show that (3.1) holds.

Observe that

\[
\begin{align*}
\left\{ \begin{array}{l}
d(x, y) \geq \frac{\eta(r)}{b} d(x, Tx) \\
\geq \frac{\eta(r)}{b^2} \left( 1 - \frac{r \eta(r)}{3b^3 + r \eta(r)} \right) f(x) = \frac{3b \eta(r)}{3b^3 + r \eta(r)} f(x).
\end{array} \right.
\end{align*}
\]

(3.4)

We now divide the proof in two cases.

Case (1) Suppose that \( f(y) \geq 2b f(x) \). Then

\[
d(Tx, Ty) \leq f(Tx) + bf(Ty)
\]

\[
\leq \frac{r \eta(r)}{3b^3 + r \eta(r)} f x + \frac{b r \eta(r)}{3b^3 + r \eta(r)} f y
\]

\[
\leq \frac{r}{3b} (fx + fy) + \frac{2r}{3b} (fy - 2b fx) = \frac{r}{3b} \left( \frac{1}{b} fx + \frac{1}{b} fy + \frac{2}{b} fy - \frac{4}{b} fx \right)
\]

\[
\leq \frac{r}{3} \left( \frac{3}{b} fy - \frac{3}{b} fx \right) \leq r \left( \frac{3}{b} fy - \frac{4}{b} fx \right) \leq rd(x, y).
\]

Case (2) If \( f(y) < 2b f(x) \), then by (3.4) we have

\[
d(Tx, Ty) \leq bf(Tx) + f(Ty)
\]

\[
\leq \frac{b r \eta(r)}{3b^3 + r \eta(r)} f x + \frac{r \ eta(r)}{3b^3 + r \eta(r)} f y
\]

\[
\leq \frac{b r \eta(r)}{3b^3 + r \eta(r)} f x + \frac{2b r \eta(r)}{3b^3 + r \eta(r)} f x
\]

\[
= \frac{3b r \eta(r)}{3b^3 + r \eta(r)} f x = \frac{3b \eta(r)}{3b^3 + r \eta(r)} f x \leq rd(x, y).
\]
Hence $\eta(r)d(x,Tx) \leq bd(x,y)$ implies that
\[d(Tx,Ty) \leq rd(x,y)\]
for all $x, y \in X$. From (iii), a mapping $T$ has a fixed point which gives a contradiction. Hence $X$ is complete and consequently (iii) implies (i). \qed

**Remark 3.6.** Let $\{x_n\}$ be a Cauchy sequence in a $b-$ metric space $X$. If $\{x_n\}$ is convergent to some $u \in X$, then for any $x \in X$, $\{d(x,x_n)\}$ is convergent in $\mathbb{R}$ and hence Cauchy in $\mathbb{R}$. If $\{x_n\}$ is not convergent, then from triangular inequality of $b$-metric, it does not follow necessarily the Cauchyness of $d(x,x_n)$ in $\mathbb{R}$. Assume that $F$ is the class of $b$-metrics $d$ and for any Cauchy sequence $\{x_n\}$ in $X$ and for any $x \in X$, $\{d(x,x_n)\}$ is Cauchy in $\mathbb{R}$. Consider a metric space $(X,\rho)$ with $d(x,y) = (\rho(x,y))^p$ for $p > 1$. Then $d$ is a $b$-metric on $X$ (see [26]). Hence $F$ is nonempty.

Now we present the following result which deals with characterization of a completeness of $b$-metric space.

**Theorem 3.7.** Let $(X,d)$ be a $b$-metric space such that $d \in F$. For $r \in [0,1)$ with $r < \frac{1}{b^2+1}$, let $A_{r,\eta}$ be a class mappings $T$ on $X$ which satisfies the following:

(a) For $x, y \in X$
\[\eta(r)d(x,Tx) \leq bd(x,y)\]
implies that $d(Tx,Ty) \leq rd(x,y)$ \quad (3.5)
where $\eta : [0,1) \to (0,1]$.

Let $B_{r,\eta}$ be the class of mappings $T$ on $X$ satisfying (a) and the following conditions:

(b) $T(X)$ is countably infinite.

(c) Every subset of $T(X)$ is closed.

Then the following are equivalent:

(i) $(X,d)$ is complete,

(ii) Every mapping $T \in A_{r,\eta}$ has a fixed point for all $r \in [0,1)$ with $r < \frac{1}{b^2+1}$.

(iii) There exists an $r \in (0,1)$ with $r < \frac{1}{b^2+1}$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.
Proof. By Corollary 2.10 (i) implies (ii). As $B_{r,\eta} \subseteq A_{r,\eta}$, so we have (ii) implies (iii). Now we prove that (iii) implies (i). Assume that (iii) holds. Suppose that $(X,d)$ is not complete. Define the function $f : X \to [0,\infty)$ by $f(x) = \lim_{n \to \infty} d(x, u_n)$ for $x \in X$. By given assumption, $\{d(x, u_n)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in X$. Hence $f$ is well defined. Note that $f(x) > 0$ for every $x \in X$ and $\lim_{n \to \infty} f(u_n) = 0$. Consequently, for every $x \in X$, there exists a $v \in \mathbb{N}$ such that
\[ f(u_v) \leq \left( \frac{r\eta(r)}{3b^4 + rb\eta(r)} \right) f(x). \] (3.6)

Define $T(x) = u_v$, then we have
\[ f(Tx) \leq \left( \frac{r\eta(r)}{3b^4 + rb\eta(r)} \right) f(x) \text{ and } Tx \in \{u_n : n \in \mathbb{N}\} \] (3.7)
for all $x \in X$. The rest of the proof is obtained following similar arguments to those arguments similar to those in the proof of Theorem 3.7.

4 Coincidence and common fixed point of hybrid pair of Ciric-Suzuki type quasi-contractive operators

In this section, we apply Theorem 2.1 to obtain the existence of coincidence and common fixed point of hybrid pair of Ciric-Suzuki type quasi-contractive multivalued operators and single-valued self mappings in the setup of b-metric spaces.

Theorem 4.1. Let $(X,d)$ be a b-metric space and $(f,T)$ a Ciric-Suzuki type quasi-contractive hybrid pair with $T(X) \subseteq f(X)$ and $f(X)$ a complete subspace of $X$. Then $C(f,T)$ is nonempty. Furthermore, $F(f,T)$ is nonempty if any of the following conditions hold:

$C_1$- The hybrid pair $(f,T)$ is $w-$compatible, $\lim_{n \to \infty} f^n(x) = u$ for some $u \in X$ and $x \in C(f,T)$ and $f$ is continuous at $u$.

$C_2$- The mapping $f$ is $T-$weakly commuting at some $x \in C(f,T)$ and $f^2x = fx$.

$C_3$- The mapping $f$ is continuous at at some $x \in C(f,T)$ and $\lim_{n \to \infty} f^n(u) = x$ for some $u \in X$.

Proof. By Lemma 1.6, there is a set $E \subseteq X$ such that $f : E \to X$ is one-to-one and $f(E) = f(X)$. Define the mapping $\mathcal{J} : f(E) \to CB(X)$ by $\mathcal{J}fx = Tx$ for
all \( f(x) \in f(E) \). The mapping \( \mathcal{T} \) is well defined because \( f \) is one-to-one. As \((f, T)\) is Ciric-Suzuki type quasi-contractive hybrid pair, for any \( x, y \in X \)

\[
\xi (d(fx, Tx), d(fx, fy)) \leq 0
\]

implies that

\[
H(Tx, Ty) \leq r \max \{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}
\]

for some \( r \in \left[0, \frac{1}{b^2 + b}\right]\) and \( \xi \in \Lambda \). Thus for all \( fx, fy \in f(E) \),

\[
\begin{cases}
\xi (d(fx, \mathcal{T}fx), d(fx, fy)) \leq 0 \\
H(\mathcal{T}fx, \mathcal{T}fy) \leq r \max \{d(fx, fy), d(fx, \mathcal{T}fx), d(fy, \mathcal{T}fy), d(fx, \mathcal{T}fy), d(fy, \mathcal{T}fx)\}
\end{cases}
\]

for some \( r \in \left[0, \frac{1}{b^2 + b}\right]\) and \( \xi \in \Lambda \). As \( f(X) \) is complete so is \( f(E) \). It follows

from Theorem 2.1 that the mapping \( \mathcal{T} \) on \( f(E) \) is MWP operator. Thus we

may choose a point \( u \in f(E) \) such that \( u \in \mathcal{T}u \). Since \( u \in f(E) = f(X) \), there

exists \( x \in X \) such that \( fx = u \). Hence \( fx \in \mathcal{T}fx = Tx \), that is, \( x \in C(f, T) \). To

prove \( F(f, T) \neq \emptyset \) : Suppose that \( (C_1) \) holds. Now, \( \lim_{n \to \infty} f^n(x) = u \) for

some \( u \in X \) and the continuity of \( f \) at \( u \) imply that \( fu = u \) and hence \( \lim_{n \to \infty} f^n(x) = fu \). From \( w \)-compatibility of a pair \((f, T)\), we have \( f^n(x) \in T(f^n(x)) \), that is \( f^n(x) \in C(f, T) \) for all \( n \in \mathbb{N} \). Suppose that \( f^n(x) \neq f(u) \) for all \( n \). Indeed, if \( f^n(x) = f(u) \) for some \( n \), then we have \( u = fu = f^n(x) \in T(f^{n-1}(x)) = T(u) \) and hence the result. Note that

\[
\xi (d(f^n(x), T(f^{n-1}(x))), d(f f^{n-1}(x), fu)) \\
\leq \frac{1}{b} (d(f^n(x), T(f^{n-1}(x))) - d(f f^{n-1}(x), fu)) = 0 - d(f f^{n-1}(x), fu) < 0.
\]

Hence

\[
d(f^n x, Tu) \leq H(T f^{n-1} x, Tu)
\]

\[
\leq r \max \{d(f^n x, fu), d(f^n x, T f^{n-1} x), d(fu, Tu), d(f^n x, Tu), d(fu, T f^{n-1} x)\}
\]

\[
\leq r \max \{d(f^n x, fu), d(f^n x, f^n x), d(fu, Tu), d(f^n x, Tu), d(fu, f^n x)\}
\]

\[
\leq r \max \{d(f^n x, fu), d(f^n x, f^n x), d(fu, Tu), d(f^n x, Tu), d(fu, f^n x)\}.
\]

On taking limit as \( n \to \infty \) on both sides of the above inequality, we obtain that

\[
d(fu, Tu) \leq rd(fu, Tu).\]

Hence \( d(fu, Tu) = 0 \) implies that \( u = fu \in Tu \). That

is, \( F(f, T) \) is nonempty. If \( (C_2) \) holds, then \( f^2 x = fx \) for some \( x \in C(f, T) \).

Also, \( f \) is \( T \)-weakly commuting, \( fx = f^2 x = Tfx \). Hence \( fx \in F(f, T) \). If

\( (C_3) \) holds , then we have \( \lim_{n \to \infty} f^n(u) = x \) for some \( u \in X \) and \( x \in C(f, T) \).
By continuity of $f$, $x = fx \in Tx$. Hence in all the three cases, we have $F(f, T) \neq \emptyset$.

**Corollary 4.2.** Let $(X, d)$ be a b-metric space, $f : X \to X$, $T : X \to CB(X)$ with $T(X) \subseteq f(X)$ and $f(X)$ a complete subspace of $X$. If for any $x, y \in X$
\[ \xi(d(fx, Tx), d(fx, fy)) \leq 0 \] implies that $H(Tx, Ty) \leq rd(fx, fy)$
where $r < \frac{1}{b^2 + b}$ and $\xi \in \Lambda$. Then $C(f, T)$ is nonempty. Furthermore, $F(f, T)$ is nonempty if any of the following conditions hold:

**C4** - The hybrid pair $(f, T)$ is $w$-compatible, $\lim_{n \to \infty} f^n(x) = u$ for some $u \in X$ and $x \in C(f, T)$ and $f$ is continuous at $u$.

**C5** - The mapping $f$ is $T$-weakly commuting at some $x \in C(f, T)$ and $f^2x = fx$.

**C6** - The mapping $f$ is continuous at at some $x \in C(f, T)$ and $\lim_{n \to \infty} f^n(u) = x$ for some $u \in X$.

### 5 Stability and uniform convergence results

In this section, we find an upper bound of Hausdorff distance between the fixed point sets of two Ciric-Suzuki type quasi-contractive multivalued operators and then study the uniform convergence of such sets in the setup of b-metric spaces.

**Theorem 5.1.** Let $(X, d)$ be a complete b-metric space and $T_1, T_2 : X \to P(X)$. Suppose that $T_i$ is Ciric-Suzuki type quasi-contractive multivalued operator for each $i \in \{1, 2\}$. If there exists $\lambda > 0$ such that
\[ H(T_1x, T_2x) \leq \lambda \] for all $x \in X$. Then $F(T_i)$ is closed subset of $X$ and $T_i$ is a MWP operator for each $i \in \{1, 2\}$. Also, the following holds:
\[ H(F(T_1), F(T_2)) \leq \frac{\lambda}{1 - b \max_{i \in \{1, 2\}} \gamma_i} \] where
\[ \gamma_i = \frac{b\beta_i}{1 - b\beta_i}, \quad \beta_i = r_i + \alpha_i, \quad \text{and} \quad \alpha_i = \frac{1}{2} \left( \frac{1}{b^2 + b} - r_i \right) \] for $i \in \{1, 2\}$. 
Proof. By Theorem 2.1, \( F(T_i) \) is nonempty for each \( i \in \{1, 2\} \). Let \( \{x_n\} \) be a sequence in \( F(T_1) \) such that \( x_n \to z \) as \( n \to \infty \). Note that

\[
\xi (d(x_n, T_1 x_n), d(z, x_n)) \leq \frac{1}{b} d(x_n, T_1 x_n) - d(z, x_n) \\
\leq d(x_n, T_1 x_n) - d(z, x_n) \\
\leq d(x_n, x_n) - d(z, x_n) = -d(z, x_n) \leq 0.
\]

Hence, we have

\[
d(z, T_1 z) \leq bd(z, x_n) + bd(x_n, T_1 z) \\
\leq bd(z, x_n) + bH(T_1 z, T_1 x_n) \\
\leq bd(z, x_n) + b r_1 \max \{d(z, x_n), d(z, T_1 z), d(T_1 x_n, x_n), d(x_n, T_1 z), d(z, T_1 x_n)\} \\
\leq bd(z, x_n) + b r_1 \max \{d(z, x_n), d(z, T_1 z), d(x_n, T_1 z)\}.
\]

On taking the limit as \( n \to \infty \) we obtain that

\[
d(z, T_1 z) \leq b r_1 d(z, T_1 z) \leq \frac{1}{b + 1} d(z, T_1 z).
\]

As \( b \geq 1 \), so \( d(z, T_1 z) = 0 \), that is, \( z \in T_1 z \). Hence \( F(T_1) \) is closed. Similarly, \( F(T_2) \) is a closed subset of \( X \). Following arguments similar to those in the proof of Theorem 2.1, we conclude that \( T_i \) is MWP operator for each \( i \in \{1, 2\} \).

We now show that (5.2) holds for all \( x \) in \( X \). As \( r_i < \frac{1}{b^2 + b} < 1 \), there exist \( \alpha_i \in \mathbb{R}^+ \) such that \( \frac{r_i}{2} + \alpha_i = \frac{1}{2} \left( \frac{1}{b^2 + b} \right) \) which gives that

\[
r_i + \alpha = \frac{1}{2} \left( \frac{1}{b^2 + b} + r_i \right).
\]

We set \( \beta_i = r_i + \alpha_i \). Note that \( 0 < \beta_i < 1 \) and \( \alpha_i > 0 \). Following arguments similar to those in the proof of Theorem 2.1 with \( x_0 \in F(T_1) \) and \( x_1 \in T_2 x_0 \), we obtain a Cauchy sequence \( \{x_n\} \) in \( X \) such that \( x_{n+1} \in T_2 x_n \) for all \( n \geq 1 \) and it satisfies:

\[
d(x_n, x_{n+1}) \leq \gamma_2 d(x_{n-1}, x_n)
\]

and

\[
d(x_n, x_{n+1}) \leq \gamma_2 d(x_{n-1}, x_n) \leq (\gamma_2)^2 d(x_{n-2}, x_{n-1}) \leq ... \leq (\gamma_2)^n d(x_0, x_1).
\] (5.3)
where $\gamma_2 = \frac{b\beta_2}{1 - b\beta_2}$. We choose an element $u$ in $X$ such that $x_n \to u$ as $n \to \infty$ and $u \in T_2u$. From (5.3), we obtain that

$$
d(x_n, x_{n+p}) \leq bd(x_n, x_{n+1}) + \ldots + b^{p-1}d(x_{n+p-2}, x_{n+p-1}) + b^{p-1}d(x_{n+p-1}, x_{n+p})
\leq b\gamma_2^n d(x_0, x_1) + \ldots + b^{p-1}\gamma_2^{n+p-2}d(x_0, x_1) + b^{p-1}\gamma_2^n d(x_0, x_1)
\leq (b\gamma_2^n)\left(1 + b\gamma_2 + \ldots + (b\gamma_2)^{p-2} + \frac{1}{b}\right) d(x_0, x_1).
$$

Thus, we have

$$
d(x_n, x_{n+p}) \leq \frac{(b\gamma_2^n)\left(1 - (b\gamma_2)^{p}\right)}{1 - b\gamma_2} d(x_0, x_1). \quad (5.4)
$$

On taking limit as $p \to \infty$ on both sides of the above inequality, we have

$$
d(x_n, u) \leq \frac{(b\gamma_2^n)\left(1 - (b\gamma_2)^{p}\right)}{1 - b\gamma_2} d(x_0, x_1). \quad (5.5)
$$

Also, from (5.1) and (5.5), we have

$$
d(x_0, u) \leq \frac{1}{1 - b\gamma_2} d(x_0, x_1) \leq \frac{\lambda}{1 - b\gamma_2}. \quad (5.6)
$$

Similarly, for each $z_0 \in T_2z_0$, we get $v \in T_1v$ such that

$$
d(z_0, v) \leq \frac{1}{1 - b\gamma_1} d(z_0, z_1) \leq \frac{\lambda}{1 - b\gamma_1}. \quad (5.7)
$$

It follows from (5.6), (5.7) and Lemma 1.11 that

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1 - \max\{b\gamma_1, b\gamma_2\}} = \frac{\lambda}{1 - b \max\{\gamma_i\} i \in \{1,2\}}.
$$

The following theorem generalizes the results in [30, 37] for a sequence of Ciric-Suzuki type quasi-contractive multivalued operators in b-metric spaces.

**Theorem 5.2.** Let $(X, d)$ be a complete b-metric space and $T_n : X \to P(X)$, a sequence of Ciric-Suzuki type quasi-contractive multivalued operator for each $n \in \mathbb{N}$. If $\{T_n\}$ converges to $T_0$ uniformly on $X$, then $\lim_{n \to \infty} H(F(T_n), F(T_0)) = 0$. 


Proof. Let $\gamma_i$ for each $i \in \mathbb{N}^*$ be as given in the proof of Theorem 5.1. Then $\gamma_i > 0$ for $i \in \mathbb{N}^*$ and $b \max_{i \in \mathbb{N}^*} \gamma_i < 1$. As $\{T_n\}$ converges to $T_0$ uniformly on $X$, so for any $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that
\[
\sup_{x \in X} H(T_n(x), T_0(x)) < \left(1 - b \max_{i \in \mathbb{N}^*} \gamma_i\right) \varepsilon
\]
for all $n \geq n_0$. If we set, $\lambda = \left(1 - b \max_{i \in \mathbb{N}^*} \gamma_i\right) \varepsilon$, then $H(T_n(x), T_0(x)) < \lambda$ for all $n \geq n_0$ and $x \in X$. By Theorem 5.1, we have
\[
H(F(T_n), F(T_0)) \leq \frac{\lambda}{1 - b \max_{i \in \mathbb{N}^*} \gamma_i} = \varepsilon
\]
for all $n \geq n_0$. \qed

6 Multivalued fractals in b-metric spaces

Let $(X, d)$ be a b-metric space and $T_i : X \rightarrow K(X)$, where $K(X)$ a collection of nonempty compact subsets of $X$.

The system $T = (T_1, T_2, \ldots, T_k)$ is called an iterated multifunction system (briefly IMS). If $T_i$ is upper semicontinuous for each $i = 1, 2, \ldots, k$, then the single valued operator $T_T : K(X) \rightarrow K(X)$ defined by $T_T(A) = \bigcup_{i=1}^k T_i(A)$ is called mult fractal generated by the IMS $T = (T_1, T_2, \ldots, T_k)$. Since the image of a compact set under an upper semicontinuous multivalued mapping is compact, therefore operator $T_T$ is well defined ([8, 10, 14]).

A set $\hat{A} \in K(X)$ is called multivalued fractal with respect to IMS $T = (T_1, T_2, \ldots, T_k)$ if and only if $\hat{A} \in F(T_T)$.

Theorem 6.1. Let $(X, d)$ be a b-metric space and $T_i : X \rightarrow K(X)$ upper semicontinuous multivalued operators for each $i \in \{1, 2, \ldots, k\}$. Suppose that for any $x, y \in X$,
\[
\xi(d(x, T_i(x)), d(x, y)) \leq 0 \text{ implies that } \\
H(T_i(x), T_i(y)) \leq r_i \max\{d(x, y), d(x, T_i(y)), d(y, T_i(x))\}
\]
where $r_i < \frac{1}{b^2 + b}$ for each $i \in \{1, 2, \ldots, k\}$ and $\xi \in \Lambda$. If $\frac{1}{b}d(x, T_i(x)) \leq d(x, y)$ for all $x \in A, y \in B$ and $i \in \{1, 2, \ldots, k\}$. Then $T_T : (K(X), H) \rightarrow (K(X), H)$ is a Ciric-Suzuki type quasi-contractive operator, that is
\[
\xi(H(A, T_T A), H(A, B)) \leq 0 \text{ implies that } \\
H(T_T A, T_T B) \leq r \max\{H(A, B), H(A, T_T A), H(B, T_T B), H(A, T_T B), H(B, T_T A)\}
\]
(6.1)
for all \(A, B \in K(X)\). Also, there exists a unique multivalued fractal \(\hat{A} \in K(X)\) such that \(\lim_{n \to \infty} H(T^n_\hat{A}, A) = 0\) for every \(A \in K(X)\).

**Proof.** For each \(i \in \{1, 2, ..., k\}\), we have \(\frac{1}{b}d(x, T_i x) \leq d(x, y)\) for all \(x \in A, y \in B\). Thus \(\xi(d(x, T_i x), d(x, y)) \leq 0\) for all \(x \in A, y \in B\). Hence, for each \(i \in \{1, 2, ..., k\}\)

\[
H(T_i x, T_i y) \leq r_i \max \{d(x, y), d(x, T_i x), d(y, T_i y), d(x, T_i y), d(y, T_i x)\}
\]

(6.2) for all \(x, y \in B\). By (6.2), we have

\[
\delta(T_i A, T_i B) = \sup_{x \in A} \left( \inf_{y \in B} \delta(T_i x, T_i y) \right)
\]

\[
= \sup_{x \in A} \inf_{y \in B} \delta(T_i x, T_i y) \leq \sup_{x \in A} \inf_{y \in B} H(T_i x, T_i y)
\]

\[
\leq \sup_{x \in A} \inf_{y \in B} r_i \max \{d(x, y), d(x, T_i y), d(y, T_i x)\}
\]

\[
\leq r_i \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{x \in A} \inf_{y \in B} d(x, T_i y), \sup_{x \in A} \inf_{y \in B} d(y, T_i x) \right\}
\]

\[
= r_i \max \{\delta(A, B), \delta(A, T_i B), \delta(B, T_i A)\}
\]

\[
= r_i \max \{H(A, B), H(A, T_i B), H(B, T_i A)\}
\]

(6.3)

for all \(A, B \in K(X)\), for each \(i \in \{1, 2, ..., k\}\). That is,

\[
\delta(T_i A, T_i B) \leq r_i \max \{H(A, B), H(A, T_i A), H(B, T_i B), H(A, T_i B), H(B, T_i A)\}
\]

(6.4) for all \(A, B \in K(X)\), for each \(i \in \{1, 2, ..., k\}\). Similarly,

\[
\delta(T_i B, T_i A) \leq r_i \max \{H(A, B), H(A, T_i A), H(B, T_i B), H(A, T_i B), H(B, T_i A)\}
\]

(6.5) for all \(A, B \in K(X)\), for each \(i \in \{1, 2, ..., k\}\). Also, from (6.3) and (6.4) we obtain that

\[
H(T_i A, T_i B) \leq r_i \max \{H(A, B), H(A, T_i A), H(B, T_i B), H(A, T_i B), H(B, T_i A)\}
\]

(6.6) for all \(A, B \in K(X)\), for each \(i \in \{1, 2, ..., k\}\). Note that

\[
H \left( \bigcup_{i=1}^{k} T_i A, \bigcup_{i=1}^{k} T_i B \right) \leq \max_{i=1}^{\infty} \left\{ H(T_i A, T_i B) \right\}
\]

\[
\leq \max_{i=1}^{\infty} \left( r_i \max \{H(A, B), H(A, T_i A), H(B, T_i B), H(A, T_i B), H(B, T_i A)\} \right)
\]

\[
\leq \left( \max_{i=1}^{\infty} r_i \right) \max \{H(A, B), H(A, T_i A), H(B, T_i B), H(A, T_i B), H(B, T_i A)\}.
\]
Hence
\[ H(T_T A, T_T B) \leq r \max \{ H(A, B), H(A, T_T A), H(B, T_T B), H(A, T_T B), H(B, T_T A) \}, \]
where, \( r = \max_{i \in \{1,2,\ldots,k\}} r_i. \) Consequently, \( \xi (H(A, T_T A), H(A, B)) \leq 0 \) implies that
\[ H(T_T A, T_T B) \leq r \max \{ H(A, B), H(A, T_T A), H(B, T_T B), H(A, T_T B), H(B, T_T A) \} \]
for all \( A, B \in K(X). \) It now follows from Corollary 2.7 that \( F(T_T) = \{ \hat{A} \} \) and \( \lim_{n \to \infty} H(T^n_T A, \hat{A}) = 0 \) for every \( A \in K(X). \)

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CHARACTERIZATION OF A B-MATRIC SPACE Completeness Via THE
EXISTENCE OF A FIXED POINT OF CIRIC-SUZUKI TYPE
QUASI-CONTRACTION MULTIVALUED OPERATORS AND APPLICATIONS