On the generators of a generalized numerical semigroup

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Abstract

We give a characterization on the sets \( A \subseteq \mathbb{N}^d \) such that the monoid generated by \( A \) is a generalized numerical semigroup (GNS) in \( \mathbb{N}^d \). Furthermore we give a procedure to compute the hole set \( \mathbb{N}^d \setminus S \), where \( S \) is a GNS, if a finite set of generators of \( S \) is known.

1 Introduction

Let \( \mathbb{N} \) be the set of non negative integers. A numerical semigroup is a submonoid \( S \) of \( \mathbb{N} \) such that \( \mathbb{N} \setminus S \) is a finite set. The elements of \( H(S) = \mathbb{N} \setminus S \) are called the holes of \( S \) (or gaps) and the largest element in \( H(S) \) is known as the Frobenius number of \( S \), denoted by \( F(S) \). The number \( g = |H(S)| \) is named the genus of \( S \). It has been proved that every numerical semigroup \( S \) has a unique minimal set of generators \( G(S) \), that is in \( S \) every element is a linear combination of elements in \( G(S) \) with coefficients in \( \mathbb{N} \). Furthermore the set of minimal generators of a numerical semigroup is characterized by the following: the set \( \{ a_1, a_2, \ldots, a_n \} \) generates a numerical semigroup if and only if the greatest common divisor of the elements \( a_1, a_2, \ldots, a_n \) is 1. For the background on this subject, a very good reference is [9]. In [3] it is provided a straightforward generalization of numerical semigroups in \( \mathbb{N} \) for submonoids of \( \mathbb{N}^d \): a monoid \( S \subseteq \mathbb{N}^d \) is called a generalized numerical semigroup (GNS) if \( H(S) = \mathbb{N}^d \setminus S \), the set of holes of \( S \), is a finite set. Also

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in this case the cardinality of $\mathbb{N}^d \setminus S$ is called the genus of $S$. In [3] the tree of
generalized numerical semigroups is efficiently calculated up to a given genus
and asymptotic properties of the number of generalized numerical semigroups
of a given genus are discussed. In this paper we want to extend some ideas and
results for classical numerical semigroups to generalized numerical semigroups.

We study basic properties of a GNS in order to characterize its minimal sys-
tem of generators. More precisely, in Section 2 we prove first that every GNS
in $\mathbb{N}^d$ has a unique minimal system of generators. Then we investigate under
which conditions a finite set $A \subseteq \mathbb{N}^d$ generates a GNS. In Section 3, by using a
connection between submonoids of $\mathbb{N}^d$ and power series expansions of rational
functions, we deduce an algorithm to compute the set of holes of a GNS, if a
finite set of generators of $S$ is given.

2 Minimal generators

Throughout the paper we denote by $e_1, e_2, \ldots, e_d$ the standard basis vectors
in $\mathbb{R}^d$ (that is, for $i = 1, \ldots, d$, $e_i$ is the vector whose $i$-th component is 1 and
the other components are zero). Furthermore, if $A \subseteq \mathbb{N}^d$, we denote $\langle A \rangle =
\{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}, a_1, \ldots, a_n \in A\}$, that is the submonoid of
$\mathbb{N}^d$ generated by the set $A$. Moreover if $t \in \mathbb{N}^d$, its $i$-th component is denoted
by $t^{(i)}$.

Lemma 2.1. [9, Lemma 2.3] Let $S$ be a submonoid of $\mathbb{N}^d$. Then $S^* \setminus (S^* + S^*)$
is a system of generators for $S$. Moreover, every system of generators of $S$
contains $S^* \setminus (S^* + S^*)$.

Lemma 2.2. Let $S$ be a GNS of genus $g$ with $H(S) = \{h_1, h_2, \ldots, h_{g-1}, h\}$. Let $h$
be a maximal element in $H(S)$ with respect to the natural partial order in
$\mathbb{N}^d$. Then $S' = S \cup \{h\}$ is a GNS, in particular $H(S') = \{h_1, h_2, \ldots, h_{g-1}\}$
and $S'$ has genus $g - 1$.

Proof. Let $S' = \langle S \cup \{h\} \rangle$. $S'$ is a GNS since $S \subseteq S' = \langle S \cup \{h\} \rangle$, in particular
$H(S) \supseteq H(S')$. Let us prove that $S'$ has genus $g - 1$. We suppose there
exists $h_j \in H(S)$, $j \in \{1, \ldots, g - 1\}$, such that $h_j \in S' = \langle S \cup \{h\} \rangle$. Then
$h_j = \sum_k \mu_k g_k + \lambda h$, with $g_k \in S$. If $\lambda = 0$ then $h_j \in S$, contradiction. If
$\lambda \neq 0$ then $h_j \geq h$ against the maximality of $h$ in $H(S)$. So $h_j \not\in S'$ for
$j \in \{1, \ldots, g - 1\}$, hence $H(S') = \{h_1, h_2, \ldots, h_{g-1}\}$. □

Proposition 2.3. Every GNS admits a finite system of generators.

Proof. Let $S \subseteq \mathbb{N}^d$ be a GNS. We prove the statement by induction on the
genus $g$ of $S$. If $g = 0$ then $S = \mathbb{N}^d$, that is generated by the standard basis vectors
$\{e_1, e_2, \ldots, e_d\}$. Let $S \subseteq \mathbb{N}^d$ be a GNS of genus $g + 1$ and let $h$ be a
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maximal element in $H(S)$ with respect to the natural partial order in $\mathbb{N}^d$. By Lemma 2.2 $S' = S \cup \{h\}$ is a GNS in $\mathbb{N}^d$ of genus $g$, that is finitely generated by induction hypothesis. Hence let $G(S')$ be a finite system of generators for $S'$. We have $h \in G(S')$ because $h$ cannot belong to $S$. So $G(S') \subset S \cup \{h\}$ and we can denote $G(S') = \{g_1, g_2, \ldots, g_s, h\}$ with $g_i \in S$ for every $i = 1, 2, \ldots, s$. Let $\mathcal{B} = \{g_1, \ldots, g_s, h + g_1, h + g_2, \ldots, h + g_s, 2h, 3h\}$. By the maximality of $h$ in $H(S)$ we have $\mathcal{B} \subseteq S$ and furthermore it is easy to prove that $\mathcal{B}$ is a system of generators for $S$. Hence $S$ is finitely generated. 

**Corollary 2.4.** Every GNS admits a unique finite system of minimal generators.

**Proof.** By Lemma 2.1 every GNS admits a unique system of minimal generators, that is $S^* \setminus (S^* + S^*)$, which is contained in every system of generators. By Proposition 2.3 such a system of generators is finite. 

**Definition 2.5.** Let $t \in \mathbb{N}^d$, we define the set $\pi(t) = \{n \in \mathbb{N}^d \mid n \leq t\}$ where $\leq$ is the natural partial order defined in $\mathbb{N}^d$.

**Remark 2.6.** Notice that for every $t \in \mathbb{N}^d$ the set $\pi(t)$ is finite and it represents the set of integer points of the hyper-rectangle whose vertices are $t$, its projections on the coordinate planes, the origin of axes, and the points in the coordinate axes $(t^{(1)}, 0, \ldots, 0), (0, t^{(2)}, 0, \ldots, 0), \ldots, (0, \ldots, 0, t^{(d)})$. If $s \notin \pi(t)$ then $s$ has at least one component larger than the respective of $t$.

**Lemma 2.7.** Let $S \subseteq \mathbb{N}^d$ be a monoid. Then $S$ is a GNS if and only if there exists $t \in \mathbb{N}^d$ such that for all elements $s \notin \pi(t)$ then $s \in S$.

**Proof.** Let $S$ be a GNS in $\mathbb{N}^d$ whose hole set is $H(S) = \{h_1, h_2, \ldots, h_g\}$. Let $t^{(i)} \in \mathbb{N}$ be the largest number appearing in the $i$-th coordinate of elements in $H(S)$ for $i \in \{1, \ldots, d\}$, in other words $t^{(i)} = \max\{h_1^{(i)}, h_2^{(i)}, \ldots, h_g^{(i)}\}$. It is easy to see that $t = (t^{(1)}, t^{(2)}, \ldots, t^{(d)}) \in \mathbb{N}^d$ fulfills the thesis. Conversely, let $t \in \mathbb{N}^d$ be an element such that for every $s \notin \pi(t)$ it is $s \in S$. Therefore if $h \in \mathbb{N}^d \setminus S$ then $h \in \pi(t)$, that is $(\mathbb{N}^d \setminus S) \subseteq \pi(t)$ and since $\pi(t)$ is a finite set then $S$ is a GNS. 

For the proof of the next theorem, that is the main result of this paper, we consider that the Frobenius Number of $\mathbb{N}$ (the trivial numerical semigroup) is 0, although it is usually defined to be $-1$ in the existing literature.

**Theorem 2.8.** Let $d \geq 2$ and let $S = \langle A \rangle$ be the monoid generated by a set $A \subseteq \mathbb{N}^d$. Then $S$ is a GNS if and only if the set $A$ fulfills each one of the following conditions:

1. For all \( j = 1, 2, \ldots, d \) there exist \( a_1^{(j)}e_j, a_2^{(j)}e_j, \ldots, a_r^{(j)}e_j \in A \), \( r_j \in \mathbb{N} \setminus \{0\} \), such that \( \gcd(a_1^{(j)}, a_2^{(j)}, \ldots, a_r^{(j)}) = 1 \) (that is, the elements \( a_i^{(j)} \), \( 1 \leq i \leq r_j \), generate a numerical semigroup).

2. For every \( i, k, 1 \leq i < k \leq d \) there exist \( x_{ik}, x_{ki} \in A \) such that \( x_{ik} = e_i + n_{i}^{(k)}e_k \) and \( x_{ki} = e_k + n_{k}^{(i)}e_i \) with \( n_{i}^{(k)}, n_{k}^{(i)} \in \mathbb{N} \).

**Proof.** \( \Rightarrow \) If \( A \) does not satisfy the first condition for some \( j \) then there exist infinite elements \( ae_j, a \in \mathbb{N} \setminus \{0\} \), which do not belong to \( S \). If \( A \) does not satisfy the second condition for some \( i \neq j \), then there are infinite elements \( e_i + ne_k \) with \( n \in \mathbb{N} \setminus \{0\} \) which do not belong to \( S \).

\( \Leftarrow \) For every \( j = 1, 2, \ldots, d \), let \( S_j \) be the numerical semigroup generated by \( \{a_1^{(j)}, a_2^{(j)}, \ldots, a_r^{(j)}\} \). We denote with \( F^{(j)} \) the Frobenius number of \( S_j \). It is easy to verify that for all \( n \in \mathbb{N} \setminus \{0\} \), the element \( (F^{(j)} + n)e_j \in \mathbb{N}^d \) belong to \( S \). Let \( v = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{N}^d \) be the element defined by

\[
v^{(j)} = \sum_{i=1}^{d} F^{(i)}n_i^{(j)} + F^{(j)}\]

for any \( j = 1, 2, \ldots, d \). Let us prove that \( \mathbf{x} \in S \) for all \( \mathbf{x} \notin \pi(v) \) so, by Lemma 2.7, \( S \) is a GNS.

Let \( \mathbf{x} = (x^{(1)}, x^{(2)}, \ldots, x^{(d)}) \in \mathbb{N}^d \) such that \( x^{(j)} > v^{(j)} \) for some \( j \in \{1, \ldots, d\} \).

Then there exists \( m_j \in \mathbb{N} \setminus \{0\} \) such that \( x^{(j)} = v^{(j)} + m_j \).

If \( k_1, k_2, \ldots, k_r \in \{1, 2, \ldots, d\} \setminus \{j\} \) are such that \( x^{(k_i)} \leq F^{(k_i)} \) for every \( i \in \{1, 2, \ldots, r\} \), so \( x^{(k_i)}n_{k_i}^{(j)} \leq F^{(k_i)}n_{k_i}^{(j)} \) for every \( i = 1, \ldots, r \), then for every \( i \) there exists \( p_i \in \mathbb{N} \) such that \( F^{(k_i)}n_{k_i}^{(j)} = x^{(k_i)}n_{k_i}^{(j)} + p_i \).

Moreover let \( h_1, \ldots, h_s \in \{1, \ldots, d\} \setminus \{j\} \) be the components of \( \mathbf{x} \) such that \( x^{(h_i)} > F^{(h_i)} \) for every \( i \in \{1, \ldots, s\} \), hence \( x^{(h_i)}e_{h_i} \in S \), for all \( i \).
Then we consider the following equalities:

\[ x = \sum_{i=1}^{d} x^{(i)} e_i = \sum_{i=1}^{r} x^{(k_i)} e_{k_i} + \sum_{i=1}^{s} x^{(h_i)} e_{h_i} + x^{(j)} e_j \]

\[ = \sum_{i=1}^{r} x^{(k_i)} e_{k_i} + \sum_{i=1}^{s} x^{(h_i)} e_{h_i} + \left( \sum_{i \neq j}^{d} F^{(i)} n^{(j)}_i + F^{(j)} + m_j \right) e_j \]

\[ = \sum_{i=1}^{r} x^{(k_i)} e_{k_i} + \left( x^{(k_i)} n^{(j)}_{k_i} \right) e_j + \sum_{i=1}^{s} x^{(h_i)} e_{h_i} + \left( \sum_{i=1}^{s} F^{(h_i)} n^{(j)}_{h_i} + F^{(j)} + m_j \right) e_j \]

\[ = \sum_{i=1}^{r} x^{(k_i)} \left( e_{k_i} + n^{(j)}_{k_i} e_j \right) + \sum_{i=1}^{s} x^{(h_i)} e_{h_i} + \left( \sum_{i=1}^{s} F^{(h_i)} n^{(j)}_{h_i} + \sum_{i=1}^{r} p_i + F^{(j)} + m_j \right) e_j. \]

Therefore \( x \) is a sum of elements in \( S \) (note that the first sum is a linear combination of elements in \( A \), whose coefficients are non-negative integers). So \( S \) is a GNS.

**Corollary 2.9.** Let \( S \subseteq \mathbb{N}^d \) be a GNS and let \( A \) be a finite system of generators of \( S \). With the notation of the previous theorem for the elements in \( A \), let \( S_j \) be the numerical semigroup generated by \( \{a^{(j)}_1, a^{(j)}_2, \ldots, a^{(j)}_r\} \) and \( F^{(j)} \) the Frobenius number of \( S_j \), for \( j = 1, \ldots, d \). Let \( v = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{N}^d \) defined by:

\[ v^{(j)} = \sum_{i \neq j}^{d} F^{(i)} n^{(j)}_i + F^{(j)}. \]

Then \( H(S) \subseteq \pi(v) \).

**Proof.** It easily follows from the proof of Theorem 2.8.

**Example 2.10.** Let \( S \subseteq \mathbb{N}^4 \) be the GNS generated by \( A = \{(1, 0, 0, 0), (1, 0, 0, 1), (0, 1, 0, 0), (0, 1, 0, 1), (0, 0, 2, 1), (0, 0, 0, 2), (0, 0, 1, 3), (0, 0, 0, 5)\} \). Actually \( S \) is a GNS and its hole set is \( H(S) = \{(0, 0, 0, 1), (0, 0, 0, 3), (0, 0, 1, 1)\} \). Let us verify that the conditions of Theorem 2.8 are satisfied. The generators described in condition 1) of the previous theorem are \( \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 2), (0, 0, 0, 5)\} \). About the condition 2) we have to verify that \( A \) contains at least one element of the following shapes:
The generators described in condition 2) of the previous theorem are \(\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}\). Observe that the set \(A' = A \setminus \{(0, 0, 1, 3)\}\) is a set of generators of a GNS \(S'\), different from \(S\), with a greater number of holes.

**Example 2.11.** Let \(S \subseteq \mathbb{N}^2\) be the GNS whose hole set is \(H(S) = \{(1, 0), (2, 0), (2, 1)\}\). The set of minimal generators of \(S\) is \(\{(0, 1), (1, 1), (3, 0), (4, 0), (5, 0)\}\). We can identify \(F^{(1)} = 2, F^{(2)} = 0, n^{(1)}_2 = 0, n^{(2)}_1 = 1\) so \(v = (F^{(2)}n^{(1)}_2 + F^{(1)}n^{(2)}_1, F^{(1)}n^{(2)}_1 + F^{(2)}) = (2, 2)\). In Figure 1 the point \(v\) is marked in red, the couples of nonnegative integers in the red area represent the elements in \(\pi(v)\).

![Figure 1](image-url)

The holes of \(S\) are marked in black and we can see that they are all in the red area, that is \(\pi(v)\). Moreover all the points overside the red area are in \(S\). Indeed \(v' = (2, 1)\) satisfies Lemma 2.7 too and \(|\pi(v')| < |\pi(v)|\). Anyway this fact does not always occur, as we will see in the next example.

**Example 2.12.** Let \(S \subseteq \mathbb{N}^2\) be the monoid generated by \(G(S) = \{(2, 0), (0, 2), (3, 0), (0, 3), (1, 4), (4, 1)\}\). By Theorem 2.8 \(S\) is a GNS. Actually the hole set of \(S\) is \(H(S) = \{(0, 1), (1, 0), (1, 1), (1, 2), (1, 3), (1, 5), (2, 1), (3, 1), (5, 1)\}\). We have \(F^{(1)} = 1, F^{(2)} = 1, \ldots\)
\( n_1^{(2)} = 4, n_2^{(1)} = 4 \), so we consider \( v = (F(2)n_2^{(1)} + F(1), F(1)n_1^{(2)} + F(2)) = (5, 5) \).

The set \( H(S) \) is contained in \( \pi(v) \):

In this case we can argue that it does not exist an element \( w \in \mathbb{N}^2 \) such that \( \pi(w) \) contains every hole of \( S \) and \( |\pi(w)| < |\pi(v)| \).

**Remark 2.13.** Let \( S = \langle A \rangle \) be a monoid generated by \( A \subseteq \mathbb{N}^d \). For every \( j = 1, 2, \ldots, n \), we denote with \( A_j \subseteq \mathbb{N}^d - 1 \) the set of the elements in \( \mathbb{N}^d - 1 \), obtained from the elements in \( A \) removing the \( j \)-th component. Then the condition 2) of Theorem 2.8 is equivalent to the following statement: for every \( j = 1, 2, \ldots, d \), \( \langle A_j \rangle = \mathbb{N}^d - 1 \).

### 3 Linear combinations in \( \mathbb{N}^d \) with coefficients in \( \mathbb{N} \)

Let \( S \subseteq \mathbb{N}^d \) be a finitely generated monoid and \( \{a_1, a_2, \ldots, a_n\} \) be a system of generators for \( S \). We denote by \( M \) the \( d \times n \) matrix whose \( i \)-th column is the vector \( a_i \in \mathbb{N}^d \) for \( i = 1, \ldots, n \). It is easy to see that an element \( b \in S \) if and only if the system \( Mx = b \) admits solutions in \( \mathbb{N}^n \). In fact this statement is equivalent to say that \( b \) is a linear combination of \( \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{N}^d \) with nonnegative integer coefficients.

**Definition 3.1.** Let \( A \subseteq \mathbb{N}^d \) be a finite set. We define the polynomial:

\[
F_A = \sum_{v \in A} x^v,
\]

where \( x^v = x_1^{v(1)} x_2^{v(2)} \cdots x_d^{v(d)} \) is the monomial in \( K[X_1, \ldots, X_d] \) associated to \( v = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \). We consider the power series expansion of \( 1/(1 - F_A) \) the following formal series:

\[
P(F_A) = \sum_{k=0}^{\infty} (F_A)^k.
\]

The following lemma ([5, Lemma 2.2] for \( d = 1 \)) is obtained by applying *Leibnitz’s rule*:

\[
(a_1 + a_2 + \cdots + a_m)^n = \sum_{h_1+h_2+\cdots+h_m=n} \frac{n!}{h_1!h_2!\cdots h_m!} a_1^{h_1} a_2^{h_2} \cdots a_m^{h_m}.
\]

**Lemma 3.2.** Let \( A = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{N}^d \) and \( b \in \mathbb{N}^d \). Then \( b \) is a linear combination of \( a_1, a_2, \ldots, a_n \) with nonnegative integer coefficients if and only if the coefficient of \( x^b \) in \( P(F_A) \) is nonzero.
Proof. By Leibnitz’s rule we obtain:

\[(F_A)^t = (x_1^{a_1^{(1)}} x_2^{a_2^{(1)}} \cdots x_d^{a_d^{(1)}} + x_1^{a_1^{(2)}} x_2^{a_2^{(2)}} \cdots x_d^{a_d^{(2)}} + \cdots + x_1^{a_1^{(d)}} x_2^{a_2^{(d)}} \cdots x_d^{a_d^{(d)}})^t =
\]

\[= \sum_h K_h x_1^{a_1^{(1)}h_1+a_2^{(1)}h_2+\cdots+a_n^{(1)}h_n} x_2^{a_1^{(2)}h_1+a_2^{(2)}h_2+\cdots+a_n^{(2)}h_n} \cdots x_d^{a_1^{(d)}h_1+a_2^{(d)}h_2+\cdots+a_n^{(d)}h_n},
\]

where the sum is extended to \(h_1, \ldots, h_n \in \mathbb{N}\) with \(h_1 + \cdots + h_n = t\) and \(K\) is a nonzero coefficient.

If \(b = \sum_{i=1}^n \lambda_i a_i\), set \(t = \sum_{i=1}^n \lambda_i\), then \(x^b\) is a monomial in \((F_A)^t\). Conversely, if \(x^b\) has nonzero coefficient in \(P(F_A)\) then

\[x^b = x_1^{a_1^{(1)}h_1+a_2^{(1)}h_2+\cdots+a_n^{(1)}h_n} x_2^{a_1^{(2)}h_1+a_2^{(2)}h_2+\cdots+a_n^{(2)}h_n} \cdots x_d^{a_1^{(d)}h_1+a_2^{(d)}h_2+\cdots+a_n^{(d)}h_n}
\]

with \(h_i \in \mathbb{N}\) for \(i = 1, \ldots, n\) that is \(b = \sum_{i=1}^n h_i a_i\).

Definition 3.3. Let \(A = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{N}^d\) with \(a_i = (a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(d)})\) for \(i = 1, 2, \ldots, n\), and \(b \in \mathbb{N}^d\). Let \(t = \min\{\sum_{j=1}^d a_i^{(j)} \mid i = 1, 2, \ldots, n\}\). We define the positive integer

\[N_b := \left\lfloor \frac{\sum_{j=1}^d b^{(j)}}{t} \right\rfloor.
\]

Proposition 3.4. Let \(A = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{N}^d\) and \(b \in \mathbb{N}^d\). Then \(b \in \langle A \rangle\) if and only if the coefficient of \(x^b\) is nonzero in the polynomial:

\[F(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{N_b} (F_A)^k.
\]

Proof. By lemma 3.2 it is enough to show that the coefficient of \(x^b\) is zero in \(F(x_1, \ldots, x_d)\) if and only if it is zero also in \(P(F_A)\), that is \(\sum_{k=0}^{\infty} \sum_{j=0}^{N_b} (F_A)^k = 0\).

We suppose that the coefficient of \(x^b\) is nonzero in \(P(F_A)\). Then there exists \(r \in \mathbb{N}\) such that \(x^b\) is a monomial in \((F_A)^r\). By Leibnitz’s rule we obtain:

\[(F_A)^r = (x_1^{a_1^{(1)}} x_2^{a_2^{(1)}} \cdots x_d^{a_d^{(1)}} + x_1^{a_1^{(2)}} x_2^{a_2^{(2)}} \cdots x_d^{a_d^{(2)}} + \cdots + x_1^{a_1^{(d)}} x_2^{a_2^{(d)}} \cdots x_d^{a_d^{(d)}})^r
\]

\[= \sum_h K_h x_1^{a_1^{(1)}h_1+a_2^{(1)}h_2+\cdots+a_n^{(1)}h_n} x_2^{a_1^{(2)}h_1+a_2^{(2)}h_2+\cdots+a_n^{(2)}h_n} \cdots x_d^{a_1^{(d)}h_1+a_2^{(d)}h_2+\cdots+a_n^{(d)}h_n},
\]
where $h = (h_1, \ldots, h_n)$ with $h_1 + h_2 + \cdots + h_n = r$ and $K$ is the correspondent coefficient, but we do not need its exact value.

If $x_1^{(1)} x_2^{(2)} \cdots x_d^{(d)}$ appears in the sum, then there exist $h_1, h_2, \ldots, h_n$ with $h_1 + h_2 + \cdots + h_n = r$, such that the following equalities are satisfied:

$$a_1^{(1)} h_1 + a_2^{(2)} h_2 + \cdots + a_n^{(n)} h_n = b^{(1)}$$
$$a_2^{(2)} h_2 + \cdots + a_n^{(n)} h_n = b^{(2)}$$
$$\vdots$$
$$a_d^{(d)} h_1 + a_2^{(2)} h_2 + \cdots + a_n^{(n)} h_n = b^{(d)}.$$

We sum the right-hand side and the left-hand side of all equalities, obtaining that:

$$r = h_1 + h_2 + \cdots + h_n \leq (a_1^{(1)} + a_2^{(2)} + \cdots + a_1^{(1)}) h_1 + (a_2^{(2)} + \cdots + a_2^{(2)}) h_2 + \cdots + (a_n^{(n)} + a_n^{(n)} + \cdots + a_n^{(n)}) h_n = b^{(1)} + b^{(2)} + \cdots + b^{(d)}.$$

Eventually, if $t = \min\{\sum_{j=1}^d a_{i,j} \mid i = 1, 2, \ldots, n\}$ then $\frac{\sum_{j=1}^d a_{i,j}}{t} \geq 1$ for $i = 1, 2, \ldots, d$. So we can divide the right-hand side of inequality by $t$ and we obtain:

$$r = h_1 + h_2 + \cdots + h_n \leq \frac{\sum_{j=1}^d a_{1,j}}{t} h_1 + \frac{\sum_{j=1}^d a_{2,j}}{t} h_2 + \cdots + \frac{\sum_{j=1}^d a_{n,j}}{t} h_n = \frac{b^{(1)} + b^{(2)} + \cdots + b^{(d)}}{t}.$$

It follows that $r \leq N_b$. So, if the coefficient of $x^b$ in $P(F_A)$ is nonzero then the greatest power in which it is obtained is at last $N_b$, for greater powers we are sure that monomial does not appear.

An application of the previous proposition is the following criterion for the existence of $\mathbb{N}$-solutions in a linear system with nonnegative integer coefficients.

**Corollary 3.5.** Let $M$ be a $d \times n$ matrix with entries in $\mathbb{N}$ whose columns are the vectors of the set $A = \{a_1, a_2, \ldots, a_n\}$ and let $b \in \mathbb{N}^d$. Then the linear system $Mx = b$ admits solutions $x \in \mathbb{N}^n$ if and only if the coefficient of $x^b$ is nonzero in the polynomial:

$$F(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{N_b} (F_A)^k.$$
The previous arguments suggest the following results.

**Corollary 3.6.** Let $S \subseteq \mathbb{N}^d$ be a GNS, $A = \{a_1, a_2, \ldots, a_n\}$ be a finite system of generators for $S$ and $v \in \mathbb{N}^d$. Then $v \in S$ if and only if the coefficient of $x^v$ is nonzero in the polynomial:

$$F(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{N_v} (F_A)^k.$$

If $S$ is a GNS and a finite system of generators for $S$ is known, then Corollary 3.6 provides a way to establish whether an element $v \in S$. Furthermore it can be done with a finite computation, that is the building of a polynomial.

**Remark 3.7.** Recall that if $S \subseteq \mathbb{N}^d$ is a GNS and $A$ a finite system of generators for $S$, by Theorem 2.8 $A$ satisfies the following conditions:

1. For all $j = 1, 2, \ldots, d$, there exist $a_1^{(j)} e_j, a_2^{(j)} e_j, \ldots, a_n^{(j)} e_j \in A$ such that $GCD(a_1^{(j)}, a_2^{(j)}, \ldots, a_n^{(j)}) = 1$

2. For every $i, k \in \{1, 2, \ldots, d\}$ with $i < k$ there exist $x, y \in A$ such that $x = e_i + n_i^{(k)} e_k$ and $y = e_k + n_k^{(i)} e_i$ with $n_i^{(k)}, n_k^{(i)} \in \mathbb{N}$.

For every $j = 1, 2, \ldots, d$, let $S_j$ be the numerical semigroup generated by $\{a_1^{(j)}, a_2^{(j)}, \ldots, a_n^{(j)}\}$. We denote by $F^{(j)}$ the Frobenius number of $S_j$. Let $v = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{N}^d$ be the element defined by

$$v^{(j)} = \sum_{i \neq j} F^{(i)} n_i^{(j)} + F^{(j)}.$$

It is proved that $H(S) \subseteq \pi(v)$ (Corollary 2.9), and $\pi(v)$ is a finite set.

We conclude giving a simple algorithm to compute the set of holes of $S$, that is $H(S)$, if a finite system of generators for $S$ is known.

**Algorithm.**

Let $S \subseteq \mathbb{N}^d$ be a GNS and $A = \{a_1, a_2, \ldots, a_n\}$ be a finite system of generators of $S$. To compute $H(S)$ we have to do the following steps:

1. Compute the element $v$ of the Remark 3.7.

2. For all $x \in \pi(v)$ we verify: if $x$ is not a $\mathbb{N}$-linear combination of elements in $A$ then $x \in H(S)$. This check can be done by Corollary 3.6.

At the end of the second step the set $H(S)$ is computed.
References


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