Positive bounded solutions for nonlinear polyharmonic problems in the unit ball

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Abstract

In this paper, we study the existence of positive solutions for the following nonlinear polyharmonic equation \((-\Delta)^m u + \lambda f(x, u) = 0\) in \(B\), subject to some boundary conditions, where \(m\) is a positive integer, \(\lambda\) is a nonnegative constant and \(B\) is the unit ball of \(\mathbb{R}^n\) \((n \geq 2)\). Under some appropriate assumptions on the nonnegative nonlinearity term \(f(x, u)\) and by using the Schauder fixed point theorem, the existence of positive solutions is obtained. At last, examples are given for illustration.

1 Introduction

The goal of the paper is to study the existence of positive continuous bounded solutions for the following nonlinear elliptic higher order problem:

\[
\begin{cases}
(-\Delta)^m u + \lambda f(x, u) = 0 \text{ in } B, \\
u > 0 \text{ in } B, \\
\lim_{x \to \xi \in \partial B} \frac{u(x)}{(1-|x|^2)^{m-1}} = \varphi(\xi),
\end{cases}
\]

where \(m\) is a positive integer, \(B = \{x \in \mathbb{R}^n : |x| < 1\}\) is the unit ball of \(\mathbb{R}^n\) \((n \geq 2)\), \(\partial B = \{x \in \mathbb{R}^n : |x| = 1\}\) is the boundary of \(B\), \(\lambda\) is a nonnegative constant, \(\varphi\) is a nontrivial nonnegative continuous function on \(\partial B\) and \(f : B \times [0, \infty) \to [0, \infty)\) is continuous.

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The polyharmonic operator \((-\Delta)^m\), \(m \in \mathbb{N}^*\), has been studied several years later. Indeed, in [5], Boggio showed that the Green function \(G_{m,n}\) of the operator \((-\Delta)^m\) on \(B\) with Dirichlet boundary conditions \(u = \frac{\partial}{\partial \nu} u = \cdots = \frac{\partial^{m-1}}{\partial \nu^{m-1}} u = 0\) on \(\partial B\), is given by:

\[
G_{m,n}(x,y) = k_{m,n} |x - y|^{2m-n} \int_1^{\frac{|x-y|}{2}} \left( \frac{\nu^2 - 1}{\nu^{m-1}} \right)^{n-1} d\nu,
\]

where \(k_{m,n}\) is a positive constant, \(\frac{\partial}{\partial \nu}\) is the outward normal derivative and for \(x, y \in B\), \(|x,y|^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2)\).

In [2], the estimates on the Green function \(G_{m,n}\) of \((-\Delta)^m\) on \(B\) and particularly the 3G-theorem (see [2], Theorem 2.8), allowed the authors to introduce a large functional class called Kato class denoted by \(K_{m,n}\) (see Definition 1 below). This class plays a key role in the study of some nonlinear polyharmonic equations (see [2, 3, 6, 10]). For related results we refer to the recent monograph [12] and the papers [7, 9, 11, 13, 14, 15].

**Definition 1.** (See [2]) A Borel measurable function \(q\) on \(B\) belongs to the Kato class \(K_{m,n}\) if \(q\) satisfies the following condition:

\[
\lim_{\alpha \to 0} \left( \sup_{x \in B} \int_{B \cap B(x,\alpha)} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) |q(y)| \, dy \right) = 0.
\]

Here and always \(\delta(x) = 1 - |x|\) is the Euclidian distance between \(x\) and \(\partial B\).

As typical example of functions belonging to the class \(K_{m,n}\), we quote

**Example 1.** (See [3]) The function \(q\) defined on \(B\) by

\[
q(x) = \frac{1}{(\delta(x))^\lambda \left( \log \frac{2}{\delta(x)} \right)^\mu},
\]

is in \(K_{m,n}\) if and only if \(\lambda < 2m\) and \(\mu \in \mathbb{R}\) or \(\lambda = 2m\) and \(\mu > 1\).

Before presenting our main result, we lay out a number of potential theory tools and some notations which will be used throughout the paper. Let \(\varphi\) be a nontrivial nonnegative continuous function on \(\partial B\), we denote by \(H\varphi\) the bounded continuous solution of the Dirichlet problem

\[
\begin{align*}
\Delta u & = 0 \quad \text{in} \; B, \\
\frac{\partial u}{\partial \nu} & = \varphi.
\end{align*}
\]

We set \(\omega\) the function defined on \(B\) by

\[
\omega : x \to (1 - |x|^2)^{m-1} H\varphi(x).
\]
We remark that the function \( \omega \) is a bounded continuous solution of the problem
\[
\begin{aligned}
(-\Delta)^m \omega &= 0 \quad \text{in } B, \\
\lim_{x \to \xi \in \partial B} \frac{\omega(x)}{(1-|x|^2)^{m-1}} &= \varphi(\xi).
\end{aligned}
\] (1.2)

For simplicity, we denote by
\[
C_0(B) = \{ v \text{ continuous on } B \text{ and } \lim_{x \to \xi \in \partial B} v(x) = 0 \}.
\]

We also refer to \( V_{m,n} g \) the \( m \)-potentiel of a nonnegative measurable function \( g \) on \( B \) by
\[
V_{m,n} g(x) = \int_B G_{m,n}(x,y) g(y) dy, \quad x \in B.
\]

Recall that for each nonnegative measurable function \( g \) on \( B \) such that \( g \) and \( V_{m,n} g \) are in \( L^1_{loc}(B) \), we have
\[
(-\Delta)^m(V_{m,n} g) = g \quad \text{(in the distributional sense)}.
\]

We assume that the function \( f \) satisfies the following assumptions:

- \( (H_1) \) \( f : B \times [0, \infty) \to [0, \infty) \) is continuous and nondecreasing with respect to the second variable.
- \( (H_2) \) The function \( q = \frac{f(\cdot, \omega)}{\omega} \) belongs to the Kato class \( K_{m,n} \).

**Theorem 1.** Assume \( (H_1) - (H_2) \). Then there exists \( \lambda_0 > 0 \) such that for each \( \lambda \in [0, \lambda_0] \), problem (1.1) has a positive continuous solution \( u \) satisfying for each \( x \in B \)
\[
\left( 1 - \frac{\lambda}{\lambda_0} \right) \omega(x) \leq u(x) \leq \omega(x). \tag{1.3}
\]

We remark that for \( m = 1 \), we find again the result of [1] which was our original motivation for deriving our study.

**Remark 1.** Note that problem (1.1) is a perturbation of problem (1.2). In view of (1.3), we see that the behavior of the obtained solution is not affected by the perturbation term.

The outline of this paper is as follows. In Section 2, we state some already known results on the Green function \( G_{m,n} \) and the functions in the class \( K_{m,n} \) that will be used in our study. Section 3 is devoted to the proof of Theorem 1. The last section is reserved for some examples.

Finally, we mention that the letter \( c \) will be a positive generic constant which may vary from line to line.
2 Properties of the Green function $G_{m,n}$ and the class $K_{m,n}$

To make the paper self contained, this section is devoted to recall some results established in [2, 4, 8, 10] that will be useful for our study.

**Proposition 1.** If $x, y \in B$ such that $|x - y| \geq r > 0$, then there exists $c > 0$ such that

$$G_{m,n}(x, y) \leq c \frac{(\delta(x)\delta(y))^m}{r^n}.$$ 

**Proposition 2.** Let $q$ be a function in $K_{m,n}$, then

(i) The constant $\alpha_q = \sup_{x,y \in B} \int_B G_{m,n}(x,z)G_{m,n}(z,y) |q(z)| dz$ is finite.

(ii) The function $x \mapsto (\delta(x))^2 q(x)$ is in $L^1(B)$.

(iii) For each nonnegative harmonic function $h$ in $B$, we have for $x \in B$

$$\int_B G_{m,n}(x,y)(1 - |y|^2)^{m-1} h(y) |q(y)| dy \leq \alpha_q (1 - |x|^2)^{m-1} h(x).$$

(iv) For each $x_0 \in \overline{B}$, we have

$$\lim_{\alpha \to 0} \left( \sup_{x \in B, x \in B(x_0, \alpha)} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) |q(y)| dy \right) = 0.$$

(v) The function $x \mapsto \int_B \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) |q(y)| dy$ is in $C_0(B)$.

3 Proof of Theorem 1

We begin this section with the following lemma which plays a key role in the proof of Theorem 1.

**Lemma 1.** If $f$ satisfies $(H_2)$, then

$$\lambda_0 := \inf_{x \in B} \frac{\omega(x)}{V_{m,n}(f(\cdot, \omega))(x)} > 0. \quad (3.1)$$

**Proof.** Since $f$ satisfies $(H_2)$, then the function $q = \frac{f(\cdot, \omega)}{\omega}$ belongs to the Kato class $K_{m,n}$. It follows from Proposition 2 (iii) that for each $x \in B$, $V_{m,n}(f(\cdot, \omega))(x) = V_{m,n}(q \omega)(x) \leq \alpha_q \omega(x)$. 

This gives that for each \( x \in B \),
\[
\frac{\omega(x)}{V_{m,n}(f(\cdot,\omega))(x)} \geq \frac{1}{\alpha q}.
\]
This implies that
\[
\lambda_0 \geq \frac{1}{\alpha q} > 0.
\]

Now, we are ready to prove our main result.

**Proof of Theorem 1.** Let \( \Lambda \) be the non-empty closed convex set given by
\[
\Lambda = \{ v \in C_0(B) : \left( 1 - \frac{\lambda}{\lambda_0} \right) \omega \leq v \leq \omega \}.
\]
We define the operator \( T \) on \( \Lambda \) by
\[
Tv = \omega - \lambda V_{m,n}(f(\cdot,v)).
\]
We aim to prove that \( T \) has a fixed point in \( \Lambda \). First, we shall prove that \( TA \) is relatively compact in \( C_0(B) \). Since \( \omega \in C_0(B) \), it is enough to show that the family
\[
\{ V_{m,n}(f(\cdot,v)) : v \in \Lambda \}
\]
is relatively compact in \( C_0(B) \).
Let \( v \in \Lambda \), then by hypothesis \( (H_1) \) we obtain that
\[
0 \leq V_{m,n}(f(\cdot,v)) \leq V_{m,n}(f(\cdot,\omega)) = V_{m,n}(q \omega).
\]
Applying Proposition 2 \( (iii) \), we get
\[
0 \leq V_{m,n}(f(\cdot,v)) \leq \alpha_q \| \omega \|_{\infty}.
\]
Thus the family \( \{ V_{m,n}(f(\cdot,v)) : v \in \Lambda \} \) is uniformly bounded.
Now, we aim at proving that \( \{ V_{m,n}(f(\cdot,v)) : v \in \Lambda \} \) is equicontinuous on \( B \).
Let \( x_0 \in B \) and \( \varepsilon > 0 \). By Proposition 2 \( (iv) \), there exists \( \alpha > 0 \) such that
\[
0 \leq \sup_{z \in B} \int_{B \cap B(x_0,2\alpha)} \left( \frac{\delta(y)}{\delta(\xi)} \right)^{m-1} G_{m,n}(z,y)q(y)dy \leq \frac{\varepsilon}{2^m \| H \|_{\infty}}.
\]
Let \( x, x' \in B \cap B(x_0, \alpha) \), then for each \( v \in \Lambda \), we have

\[
|V_{m,n} (f(., v)) (x) - V_{m,n} (f(., v)) (x')| \\
\leq \int_B |G_{m,n}(x,y) - G_{m,n}(x',y)| q(y)\omega(y)dy \\
\leq 2^{m-1} \|H\varphi\|_\infty \int_B |G_{m,n}(x,y) - G_{m,n}(x',y)| (\delta(y))^{m-1} q(y)dy \\
\leq 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B(x_0,2\alpha)} |G_{m,n}(x,y) - G_{m,n}(x',y)| (\delta(y))^{m-1} q(y)dy \\
+ 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B^c(x_0,2\alpha)} |G_{m,n}(x,y) - G_{m,n}(x',y)| (\delta(y))^{m-1} q(y)dy \\
:= I_1 + I_2.
\]

From (3.3), we get that

\[
I_1 \leq 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B(x_0,2\alpha)} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y)q(y)dy \\
+ 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B(x_0,2\alpha)} \left( \frac{\delta(y)}{\delta(x')} \right)^m G_{m,n}(x',y)q(y)dy \\
\leq 2^m \|H\varphi\|_\infty \sup_{z \in B} \int_{B \cap B(x_0,2\alpha)} \left( \frac{\delta(y)}{\delta(z)} \right)^m G_{m,n}(z,y)q(y)dy \\
\leq \varepsilon
\]

On the other hand, if \( |y - x_0| \geq 2\alpha \) then \( |y - x| \geq \alpha \) and \( |y - x'| \geq \alpha \). So by applying Proposition 1, we reach

\[
|G_{m,n}(x,y) - G_{m,n}(x',y)| (\delta(y))^{m-1} q(y) \\
\leq (G_{m,n}(x,y) + G_{m,n}(x',y)) (\delta(y))^{m-1} q(y) \\
\leq \frac{c}{\alpha^m} ((\delta(x)\delta(y))^m + (\delta(x')\delta(y))^m) (\delta(y))^{m-1} q(y) \\
\leq c(\delta(y))^{2m-1} q(y).
\]

Now, since for \( y \in B \cap B^c(x_0,2\alpha) \), \( x \mapsto G_{m,n}(x,y) \) is continuous in \( B \cap B(x_0, \alpha) \) and from Proposition 2 \((ii)\) the function \( y \mapsto (\delta(y))^{2m-1} q(y) \) is in \( L^1(B) \) then we deduce by the dominated convergence theorem that

\[
I_2 \to 0 \text{ as } |x - x'| \to 0.
\]

Thus \( \{V_{m,n} (f(., v)), v \in \Lambda\} \) is equicontinuous on \( B \).
Next, we claim that \( V_{m,n}(f, v)(x) \to 0 \) as \( x \to \xi \in \partial B \) uniformly in \( v \in \Lambda \). Let \( \xi \in \partial B \) and \( x \in B \cap B(\xi, \alpha) \). Then for each \( v \in \Lambda \), we have from (3.2)

\[
V_{m,n}(f, v)(x) \leq 2^{m-1} \|H\varphi\|_\infty \int_B G_{m,n}(x, y) (\delta(y))^{m-1} q(y)dy \\
\leq 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B(\xi, 2\alpha)} G_{m,n}(x, y) (\delta(y))^{m-1} q(y)dy \\
+ 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B^c(\xi, 2\alpha)} G_{m,n}(x, y) (\delta(y))^{m-1} q(y)dy \\
\leq 2^{m-1} \|H\varphi\|_\infty \sup_{\delta} \int_{B \cap B(\xi, 2\alpha)} \left( \frac{\delta(y)}{\delta(z)} \right)^{m-1} G_{m,n}(z, y) q(y)dy \\
+ 2^{m-1} \|H\varphi\|_\infty \int_{B \cap B^c(\xi, 2\alpha)} G_{m,n}(x, y) (\delta(y))^{m-1} q(y)dy.
\]

By applying Proposition 2 (iv), we obtain that

\[
2^{m-1} \|H\varphi\|_\infty \sup_{\delta} \int_{B \cap B(\xi, 2\alpha)} \left( \frac{\delta(y)}{\delta(z)} \right)^{m-1} G_{m,n}(z, y) q(y)dy \to 0 \ \text{as} \ \alpha \to 0.
\]

For \( y \in B \cap B^c(\xi, 2\alpha) \), we have \(|x - y| \geq \alpha\). Hence, it follows from Proposition 1 and Proposition 2 (ii) that

\[
2^{m-1} \|H\varphi\|_\infty \int_{B \cap B^c(\xi, 2\alpha)} G_{m,n}(x, y) (\delta(y))^{m-1} q(y)dy \\
\leq c \left( \int_B (\delta(y))^{2m-1} q(y)dy \right) (\delta(x))^m \to 0 \ \text{as} \ x \to \xi.
\]

Therefore by Ascoli’s theorem, we conclude that the family \{\( V_{m,n}(f, v) \), \( v \in \Lambda \)\} is relatively compact in \( C_0(B) \).

For \( v \in \Lambda \), we have from (3.2)

\[
\omega - \lambda V_{m,n}(f, v) \leq Tv \leq \omega.
\]

This implies from (3.1) that

\[
\left( 1 - \frac{\lambda}{\lambda_0} \right) \omega \leq Tv \leq \omega.
\]

Combining this with the fact that \( Tv \in C_0(B) \), we deduce that \( T\Lambda \subset \Lambda \).

Now, we prove the continuity of \( T \) in \( \Lambda \) in the supremum norm. Let \( (v_k)_k \) be a sequence in \( \Lambda \) which converges uniformly to a function \( v \) in \( \Lambda \). Then for \( k \in \mathbb{N} \) and each \( x \in B \),

\[
|Tv_k(x) - Tv(x)| \leq \int_B G_{m,n}(x, y) |f(y, v_k(y)) - f(y, v(y))| dy.
\]
On the other hand, from the monotonicity of the function $f$, we have for $k \in \mathbb{N}$ and $(x, y) \in B^2$,

$$G_{m,n}(x, y) |f(y, v_k(y)) - f(y, v(y))| \leq 2G_{m,n}(x, y)f(y, \omega(y)) \leq 2G_{m,n}(x, y)q(y)\omega(y).$$

Since by Proposition 2 (iii), $\int_B G_{m,n}(x, y)q(y)\omega(y)dy \leq \alpha_q \|\omega\|_{\infty} < \infty$, we conclude by the continuity of $f$ with respect to the second variable and the dominated convergence theorem that

$$|Tv_k(x) - Tv(x)| \to 0 \text{ as } k \to \infty.$$

Consequently, since $TA$ is relatively compact in $C_0(B)$, we deduce that the pointwise convergence implies the uniform convergence, namely

$$\|Tv_k - Tv\|_{\infty} \to 0 \text{ as } k \to \infty.$$

Thus we have proved that $T$ is a compact mapping from $\Lambda$ to itself. Hence the Schauder’s fixed point theorem implies the existence of $u \in \Lambda$ such that

$$u = Tu,$$

that is

$$u = \omega - \lambda V_{m,n}(f(., u)). \quad (3.4)$$

It is clear that $u$ is continuous and satisfies (1.3) and it remains to verify that $u$ is a solution of (1.1).

Since $0 \leq f(., u) \leq 2^{m-1}\|H\varphi\|_{\infty} (\delta(\cdot))^{m-1}q$ then by Proposition 2 (ii), we obtain that $f(., u) \in L^1_{loc}(B)$ and from (3.4), we have $V_{m,n}(f(., u)) \in L^1_{loc}(B)$. Hence, we have in the distributional sense

$$(-\Delta)^m V_{m,n}(f(., u)) = f(., u) \text{ in } B.$$

Now, applying $(-\Delta)^m$ on both sides of (3.4), we obtain that

$$(-\Delta)^m u = -\lambda f(., u) \text{ in } B.$$

Finally, we have

$$\lim_{x \to \xi \in \partial B} \frac{u(x)}{(1 - |x|^2)^{m-1}} = \varphi(\xi) - \lambda \lim_{x \to \xi \in \partial B} \frac{V_{m,n}(f(., u))(x)}{(1 - |x|^2)^{m-1}}.$$

Since for $x \in B$, we have

$$0 \leq \frac{V_{m,n}(f(., u))(x)}{(1 - |x|^2)^{m-1}} \leq 2^{m-1}\|H\varphi\|_{\infty} \int_B \left(\frac{\delta(y)}{\delta(x)}\right)^{m-1} G_{m,n}(x, y)q(y)dy,$$
we deduce by Proposition 2 that
\[
\lim_{x \to \xi \in \partial B} \frac{V_{m,n}(f(u))(x)}{(1 - |x|^2)^{m-1}} = 0.
\]
Hence
\[
\lim_{x \to \xi \in \partial B} \frac{u(x)}{(1 - |x|^2)^{m-1}} = \varphi(\xi).
\]
This ends the proof. \(\square\)

4 Examples

We take up in this section some examples illustrating our main result.

**Example 2.** Let \(\varphi\) be a positive continuous function on \(\partial B\). Let \(p\) be a nonnegative measurable function satisfying for each \(x \in B\),
\[
p(x) \leq \frac{c}{(\delta(x))^\lambda \left(\log\left(\frac{1}{\delta(x)}\right)\right)\mu}
\]
with \(\lambda < 2m\) and \(\mu \in \mathbb{R}\) or \(\lambda = 2m\) and \(\mu > 1\). We consider \(g : [0, \infty) \to [0, \infty)\) nondecreasing and continuous function such that for each \(c > 0\), there exists \(\eta > 0\) satisfying
\[
g(t) \leq \eta t, \quad \forall t \in [0, c].
\]
Then by Theorem 1, there exists \(\lambda_0 > 0\) such that for each \(\lambda \in [0, \lambda_0)\), the problem
\[
\begin{cases}
-\Delta^m u + \lambda p(x) g(u) = 0 \text{ in } B, \\
u > 0 \text{ in } B, \\
\lim_{x \to \xi \in \partial B} \frac{u(x)}{(1 - |x|^2)^{m-1}} = \varphi(\xi),
\end{cases}
\]
has a positive continuous solution \(u\) satisfying (1.3).

**Example 3.** Let \(\varphi\) be a positive continuous function on \(\partial B\). Let \(p\) be a nonnegative measurable function satisfying for each \(x \in B\),
\[
p(x) \leq \frac{1}{(\delta(x))^{2m}}
\]
with \(\lambda < 2m\) and \(f\) be the function defined on \(B \times [0, \infty)\) by \(f(x,u) = p(x)u^\sigma\) with \(\sigma \geq 1\). Therefore by Theorem 1, there exists \(\lambda_0 > 0\) such that for each \(\lambda \in [0, \lambda_0)\), the problem
\[
\begin{cases}
-\Delta^m u + \lambda p(x) u^\sigma = 0 \text{ in } B, \\
u > 0 \text{ in } B, \\
\lim_{x \to \xi \in \partial B} \frac{u(x)}{(1 - |x|^2)^{m-1}} = \varphi(\xi),
\end{cases}
\]
has a positive continuous solution \(u\) satisfying (1.3).

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