Bounds on the Third Order Hankel Determinant for Certain Subclasses of Analytic Functions

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Abstract

Let $A$ be the class of analytic functions $f(z)$ in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with the Taylor series expansion about the origin given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta.$$ 

The focus of this paper is on deriving upper bounds for the third order Hankel determinant $H_3(1)$ for two new subclasses of $A$.

1 Introduction

Let $A$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in A$ is respectively said to be with bounded turning, starlike or convex if and only if for $z \in \Delta$, $\text{Re} f'(z) > 0$, $\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > 0$ or $\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0$. The classes of these functions are respectively denoted by $R, S^*$ and $C$. For $n \geq 0$ and $q \geq 1$, key words: Analytic functions, Hankel determinant, Functions with positive real part.

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the $q^{th}$ Hankel determinant is defined as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots \\ \vdots & \vdots \\ a_{n+q-1} & \cdots & a_{n+2(q-1)} \end{vmatrix}$$ (2)

This determinant has been considered by several authors (see, for example, [1, 2, 3, 10, 17, 18, 22]). In fact Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f$ given by (1) with bounded boundary. In particular, upper bounds for the second Hankel determinant were obtained by several authors [6, 9, 12, 20, 21] for different classes of analytic functions. Upper bound on the third Hankel determinant for different classes of functions has been studied recently [1, 2, 3, 10, 16, 22]. In the present investigation, the focus is on the third order Hankel determinant $H_3(1)$ for the classes $R_\beta^\alpha$ and $S_\beta^\alpha$ in $\Delta$ defined as follows:

**Definition 1.1.** Let $f$ be given by (1). Then $f \in R_\beta^\alpha$ if and only if for any $z \in \Delta, 0 \leq \beta < 1, 0 \leq \alpha \leq 1$,

$$Re\{f'(z) + \alpha zf''(z)\} > \beta.$$ (3)

The choice $\alpha = 0, \beta = 0$ yields $Re\ f'(z) > 0, z \in \Delta$, defining the class $R$ of bounded turning [15] while the choice $\alpha = 0, \beta = 0$ yields $Re\ f'(z) > \beta$ [5].

**Definition 1.2.** Let $f$ be given by (1). Then $f \in S_\beta^\alpha$ if and only if for any $z \in \Delta, 0 \leq \beta < 1, 0 \leq \alpha \leq 1$,

$$Re\ \left\{zf'(z) + \alpha zf''(z) f'(z) \right\} > \beta.$$ (4)

The choice $\alpha = 0, \beta = 0$ yields $Re\ \frac{zf'(z)}{f(z)} > 0, z \in \Delta$, defining the class $S^*$ of starlike functions [19] and the choice of $\alpha = 0$ yields $Re\ \frac{zf'(z)}{f(z)} > \beta, z \in \Delta$, defining the class $S^*(\beta)$ starlike functions of order $\beta$ [19]. Setting $n = 1$ in (2), $H_3(1)$ is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

and for $f \in A$,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$
Using the triangle inequality, we have

\[ |H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \]  

(4)

In obtaining an upper bound for \(|H_3(1)|\), the approach used is to first determine upper bounds for the functionals \(|a_2a_3 - a_4|, |a_2a_4 - a_3^2|\) and \(|a_3 - a_2^2|\). Furthermore, techniques employed in [13, 14] are useful in establishing the results (see, for example [6, 9, 12, 21]).

2 Preliminary Results

Some preliminary results required in the following sections are now listed. Let \(P\) denote the class of functions

\[ p(z) = 1 + c_1z + c_2z^2 + \cdots \]  

(5)

which are regular in \(\Delta\) and satisfy \(Re\ p(z) > 0, \ z \in \Delta\). Throughout this paper, we assume that \(p(z)\) is given by (5) and \(f(z)\) is given by (1). To prove the main results, the following known Lemmas are required.

**Lemma 2.1.** [4] Let \(p \in P\). Then \(|c_k| \leq 2, k = 1, 2, \ldots\) and the inequality is sharp.

**Lemma 2.2.** [13, 14] Let \(p \in P\). Then

\[ 2c_2 = c_1^2 + x(4 - c_1^2) \]  

(6)

and

\[ 4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \]  

(7)

for some \(x, y\) such that \(|x| \leq 1\) and \(|y| \leq 1\).

3 Main Results

For functions \(f \in R^\beta\), Lemma 3.1- Lemma 3.3 give the upper bounds for the three functionals mentioned earlier while Theorem 3.1 presents an estimate for \(|H_3(1)|\).

**Lemma 3.1.** Let \(f \in R^\beta\). Then

\[ |a_2a_3 - a_4| \leq \frac{(1 - \beta)}{2(1 + 3\alpha)} \]  

(8)

**Proof.** Let \(f \in R^\beta\). Then there exists a \(p\) such that
\[ f'(z) + \alpha zf'''(z) = (1 - \beta)p(z) + \beta, \quad p(0) = 1, \quad Re\ p(z) > 0. \]

Equating the coefficients, we find that

\[ a_2 = \frac{c_1(1 - \beta)}{2(1 + \alpha)}, \quad a_3 = \frac{c_2(1 - \beta)}{3(1 + 2\alpha)}, \quad a_4 = \frac{c_3(1 - \beta)}{4(1 + 3\alpha)}, \quad a_5 = \frac{c_4(1 - \beta)}{5(1 + 4\alpha)}. \]

The functional \(|a_2a_3 - a_4|\) is given by

\begin{align*}
|a_2a_3 - a_4| &= \left| \frac{c_1c_2(1 - \beta)^2}{6(1 + \alpha)(1 + 2\alpha)} - \frac{c_3(1 - \beta)}{4(1 + 3\alpha)} \right|, \quad (9)
\end{align*}

Substituting for \(c_2\) and \(c_3\) from (6) and (7) of Lemma 2.2, we obtain

\[ |a_2a_3 - a_4| = \frac{(1 - \beta)}{48(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)} \left| 4(1 + 3\alpha)(1 - \beta)c_1(c_1^2 + x(4 - c_1^2)) 
- 3(1 + \alpha)(1 + 2\alpha)(c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2)) \right| 
\]

\[ = A(\alpha, \beta) \left| c_1^3(2a - 3b) - 2c_1x(4 - c_1^2)(3b - a) - 3bx^2c_1(4 - c_1^2) 
- 2y \times 3b(1 - |x|^2)(4 - c_1^2) \right| 
\]

where \(A(\alpha, \beta) = \frac{(1 - \beta)}{48(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)}, \quad a = 2(1 + 3\alpha)(1 - \beta), \quad b = (1 + \alpha)(1 + 2\alpha).\)

Suppose now that \(c_1 = c\). Since \(|c| = |c_1| \leq 2\), using the Lemma 2.1, we may assume without restriction that \(c \in [0, 2]\) and on applying the triangle inequality with \(\rho = |x| \leq 1\), we get

\[ |a_2a_3 - a_4| \leq A(\alpha, \beta) \left\{ c^3|2a - 3b| + 2c(4 - c^2)(3b - a) + 3b\rho^2c(4 - c^2) 
+ 2 \times 3b(1 - \rho^2)(4 - c^2) \right\} 
\]

\[ = A(\alpha, \beta) \left\{ c^3|2a - 3b| + 2c(4 - c^2)(3b - a) + 3b\rho^2(4 - c^2)(c - 2) 
+ 6b(4 - c^2) \right\} = F(\rho). \]

Next we maximize the function \(F(\rho)\).

\[ F'(\rho) = A(\alpha, \beta) \left\{ 2c(4 - c^2)(3b - a) + 6b\rho(4 - c^2)(c - 2) \right\} \quad (10) \]

\(F'(\rho) = 0\) implies \(\rho = \frac{c(3b - a)}{6b^2 - c^2}\). Set \(\rho^* = \frac{c(3b - a)}{6b^2 - c^2}\). Now \(0 \leq \rho^* \leq 1\). Also we have \(F''(\rho) = A(\alpha, \beta)\{6b(4 - c^2)(c - 2)\} < 0\), for \(c < 2\). Thus \(\rho^*\) is the only
value in \([0, 1]\) at which \(F(\rho)\) attains a maximum. Hence \(F(\rho) \leq F(\rho^*)\).

Thus

\[
F(\rho) \leq A(\alpha, \beta) \left\{ c^3|2a - 3b| + \frac{c^2}{3b}(3b-a)^2(2+c) + 6b(4-c^2) \right\}
\]

\[
= A(\alpha, \beta) \left\{ c^3|2a - 3b| + (3b-a)^2 - c^2\left\{6b - \frac{2(3b-a)^2}{3b}\right\} + 24b \right\}
\]

\[
F(\rho) \leq A(\alpha, \beta) \left\{ c^3\gamma - c^2\delta + 24b \right\} = G(c),
\]

where \(\gamma = |2a - 3b| + \frac{(3b-a)^2}{3b}, \delta = [6b - \frac{2(3b-a)^2}{3b}]\). \(G'(c) = 0\) implies \(c = 0\) and at \(c = 0, G''(c) < 0\). Thus, the upper bound of \(F(\rho)\) corresponds to \(\rho = \rho^*\) and \(c = 0\). Hence \(|a_2a_3 - a_4| \leq \frac{(1-\beta)}{2(1+3\alpha)}\).

\[\square\]

**Corollary 3.1.** Choosing \(\alpha = 0, \beta = 0\) in (8), we get \(|a_2a_3 - a_4| \leq \frac{1}{2}\).

This result coincides with the corresponding result in [3].

**Lemma 3.2.** Let \(f \in R_\beta^\alpha\). Then

\[
|a_2a_4 - a_3^2| \leq \frac{4}{9} \frac{(1-\beta)^2}{(1+2\alpha)^2}
\]

(11)

**Proof.** Let \(f \in R_\beta^\alpha\). In a manner similar to the proof of Lemma 3.1, we can derive

\[
|a_2a_4 - a_3^2| = \left| \frac{c_1c_3(1-\beta)^2}{8(1+\alpha)(1+3\alpha)} - \frac{c_2^2(1-\beta)^2}{9(1+2\alpha)^2} \right|
\]

(12)

Substituting for \(c_2\) and \(c_3\) from (6) and (7) of Lemma 2.2, we obtain

\[
= \left| \frac{c_1(1-\beta)^2}{32(1+\alpha)(1+3\alpha)}\left[c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2)\right]
\]

\[
+ 2y(1-|x|^2)(4-c_1^2) - \frac{(1-\beta)^2}{36(1+2\alpha)^2}\left[c_1^2 + x(4-c_1^2)^2\right]\right|.
\]

\[
= \frac{(1-\beta)^2}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\left[9c_1(1+2\alpha)^2[c_1^3 + 2xc_1(4-c_1^2)]
\]

\[
- x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2)\right] - 8(1+\alpha)(1+3\alpha)[c_1^2 + x^2(4-c_1^2)^2 + 2xc_1^5(4-c_1^2)]\right|.
\]
Let $N = \frac{(1-\beta)^2}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$, $a = 9(1 + 2\alpha)^2$, $b = 8(1 + \alpha)(1 + 3\alpha)$ and $a - b = 9(1 + 2\alpha)^2 - 8(1 + \alpha)(1 + 3\alpha) = 1 + 12\alpha^2 + 4\alpha \geq 0$, since $\alpha \geq 0$.

$$|a_2a_4 - a_3^2| = N \left[ a_1 \left[ c_4^1 + 2x c_1 (4 - c_1^2) - x^2 c_1 (4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \right] 
- b \left[ c_4^1 + x^2 (4 - c_1^2) + 2x c_1 (4 - c_1^2) \right] \right]
= N \left[ c_4^1 (a - b) + 2x c_1 (4 - c_1^2)(a - b) - x^2 (4 - c_1^2)[a c_1^2 + b (4 - c_1^2)] \right]
+ 2y a c_1 (1 - |x|^2)(4 - c_1^2) \right].$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0, 2]$ and on applying the triangle inequality with $\rho = |x| \leq 1$, we get

$$|a_2a_4 - a_3^2| \leq N \left\{ c_4^1 (a - b) + 2\rho c_2^1 (4 - c_2^1)(a - b) 
+ \rho^2 (4 - c_2^1)[c_2^1 (a - b) - 2a c_1 + 4b + 2a c_2 (4 - c_2^1)] \right\}
= N \left\{ c_4^1 (a - b) + 2\rho c_2^1 (4 - c_2^1)(a - b) 
+ \rho^2 (4 - c_2^1)(a - b)(c - 2)(c - \frac{2b}{a - b}) + 2a c_2 (4 - c_2^1) \right\} = F(\rho).$$

Differentiating $F(\rho)$, we get

$$F'(\rho) = N[2c_2^1 (4 - c_2^1)(a - b) + 2\rho (4 - c_2^1)(a - b)(c - 2)(c - \frac{2b}{a - b})] \geq 0,$$

since $a - b > 0$, $2b/(a - b) > 2$ so that $c - 2b/(a - b) < c - 2 < 0$ and $(c - 2)(c - \frac{2b}{a - b}) > 0$ for all $c \in [0, 2]$. This implies that $F(\rho)$ is an increasing function of $\rho$ on the closed interval $[0, 1]$. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0, 1]$. That is,

$$|a_2a_4 - a_3^2| \leq N \left\{ c_4^1 (a - b) + 2c_2^1 (4 - c_2^1)(a - b) 
+ (4 - c_2^1)(a - b)(c - 2)(c - \frac{2b}{a - b}) + 2a c_2 (4 - c_2^1) \right\}
= N \left\{ -2c_2^1 (a - b) - 4c_2^1 (4b - 3a) + 16b \right\} = G(c).$$

$G'(c) = 0$ implies $c = 0$ so that at $c = 0, G''(c) < 0$. Therefore $c = 0$ is a point of maximum for $G(c)$. Thus, the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence, $|a_2a_4 - a_3^2| \leq \frac{4}{9} (1-\beta)^2$.

**Corollary 3.2.** Choosing $\alpha = 0$, $\beta = 0$ in (11), we get $|a_2a_4 - a_3^2| \leq \frac{4}{9}$.

This result coincides with [7].
Corollary 3.3. Choosing $\alpha = 0$ in (11), we get $|a_2 a_4 - a_3^2| \leq \frac{1}{4}(1 - \beta)^2$. This result coincides with [11].

Lemma 3.3. Let $f \in R_\alpha^\beta$. Then for $1/3 \leq \beta < 1$, 
\[
|a_3 - a_2^2| = \frac{2(1 - \beta)}{3(1 + 2\alpha)} \tag{13}
\]

Proof. Let $f \in R_\alpha^\beta$. Then by proceeding as in Lemma 3.1, we have 
\[
|(a_3 - a_2^2)| = \left| \frac{c_2(1 - \beta)}{3(1 + 2\alpha)} - \frac{c_2^2(1 - \beta)^2}{4(1 + \alpha)^2} \right| \tag{14}
\]

Substituting for $c_2$ from (6) of Lemma 2.2, we obtain 
\[
|(a_3 - a_2^2)| = \frac{(1 - \beta)}{12(1 + 2\alpha)(1 + \alpha)^2} \left| 2(1 + \alpha)^2[c_1^2 + x(4 - c_1^2)] - 3(1 + 2\alpha)(1 - \beta)c_1^2 \right|
\]

where $M = \frac{(1 - \beta)}{12(1 + 2\alpha)(1 + \alpha)^2}$, $k_1 = 2(1 + \alpha)^2$, $k_2 = 3(1 + 2\alpha)(1 - \beta)$. Set $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0, 2]$ and on applying the triangle inequality with $\rho = |x| \leq 1$, we get 
\[
|a_3 - a_2^2| \leq M[c^2 |k_1 - k_2| + k_1 \rho(4 - c^2)] = F(\rho).
\]

Differentiating $F(\rho)$, we get $F'(\rho) = M[k_1(4 - c^2)] \geq 0$, implying that $F(\rho)$ is an increasing function of $\rho$ on a closed interval [0, 1]. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0, 1]$. That is, 
\[
|(a_3 - a_2^2)| \leq M[c^2 |k_1 - k_2| + k_1(4 - c^2)] = G(c).
\]

By hypothesis, $\beta \geq 1/3$ and hence $k_1 - k_2 = 2\alpha^2 - 2\alpha - 1 + 3\beta(1 + 2\alpha) \geq 2\alpha^2$. Hence $G(c) = M[4k_1 - c^2 k_2]$, $G'(c) = -2Mk_2 c$ and $G''(c) = -2Mk_2$. Since $c \in [0, 2]$, it follows that $G(c)$ attains the maximum at $c = 0$. Thus, the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence $|a_3 - a_2^2| \leq M[4k_1] = \frac{2(1 - \beta)}{3(1 + 2\alpha)}$. \qed
**Corollary 3.4.** Choosing $\alpha = 0$ in (13), we get $|a_3 - a_2^2| \leq \frac{2}{3}(1 - \beta)$. This result coincides with [10], for $1/3 \leq \beta < 1$.

**Remark 3.1.** Let $f \in R^3_\alpha$. By Lemma 2.1, we have

\[
\begin{align*}
|a_3| &= \left| \frac{c_2(1 - \beta)}{3(1 + 2\alpha)} \right| \leq \frac{2(1 - \beta)}{3(1 + 2\alpha)}, \\
|a_4| &= \left| \frac{c_3(1 - \beta)}{4(1 + 3\alpha)} \right| \leq \frac{(1 - \beta)}{2(1 + 3\alpha)}, \\
|a_5| &= \left| \frac{c_4(1 - \beta)}{5(1 + 4\alpha)} \right| \leq \frac{2(1 - \beta)}{5(1 + 4\alpha)}.
\end{align*}
\]

Using the above results, the upper bound for $|H_3(1)|$, $f \in R^3_\alpha$ is immediately obtained.

**Theorem 3.1.** Let $f \in R^3_\alpha$. Then for $1/3 \leq \beta < 1$,

\[
|H_3(1)| \leq \frac{8(1 - \beta)^3}{27(1 + 2\alpha)^3} + \frac{(1 - \beta)^2}{4(1 + 3\alpha)^2} + \frac{4(1 - \beta)^2}{15(1 + 2\alpha)(1 + 4\alpha)}.
\]

In the following results, with similar approach and technique, an upper bound for $|H_3(1)|$ is attained for $f \in S^3_\alpha$. As before, we first derive estimates for the functionals $|a_2a_3 - a_4|$, $|a_2a_4 - a_3^2|$ and $|a_3 - a_3^2|$. Their estimates are given in Lemmas 3.4, 3.5, and 3.6.

**Lemma 3.4.** Let $f \in S^3_\alpha$. Then

\[
|a_2a_3 - a_4| \leq \frac{2(1 - \beta)}{3(1 + 4\alpha)}.
\]

**Proof.** Let $f \in S^3_\alpha$. Then there exists a $p \in P$ such that

\[
z f'(z) + az^2f''(z) = [(1 - \beta)p(z) + \beta]f(z),
\]

for some $z \in \Delta$. Equating the coefficients, we have

\[
\begin{align*}
a_2 &= c_1(1 - \beta) \frac{1}{1 + 2\alpha}, \\
a_3 &= c_2(1 - \beta) \frac{2}{2(1 + 3\alpha)} + \frac{c_1^2(1 - \beta)^2}{2(1 + 2\alpha)(1 + 3\alpha)}, \\
a_4 &= c_3(1 - \beta) \frac{3}{3(1 + 4\alpha)} + \frac{c_1c_2(3 + 8\alpha)(1 - \beta)^2}{6(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)} + \frac{c_1^3(1 - \beta)^3}{6(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)}.
\end{align*}
\]
and

\[ a_5 = \frac{c_1^4(1 - \beta)^4}{24(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)(1 + 5\alpha)} + \frac{c_2^2(1 - \beta)^2}{8(1 + 3\alpha)(1 + 5\alpha)} + \frac{c_1^2c_2(1 - \beta)^3(20\alpha + 6)}{24(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)(1 + 5\alpha)} + \frac{c_1c_3(1 - \beta)^2(4 + 14\alpha)}{12(1 + 2\alpha)(1 + 4\alpha)(1 + 5\alpha)} + \frac{c_4(1 - \beta)}{4(1 + 5\alpha)}. \]

Thus, we have

\[ \rho \left| a_2a_3 - a_4 \right| = \frac{(1 - \beta)}{6(1 + 2\alpha)^2(1 + 3\alpha)(1 + 4\alpha)} |c_1c_2(1 - \beta)4\alpha(1 + 2\alpha) + 2c_1^2(1 - \beta)^2(1 + 5\alpha) - 2c_3(1 + 2\alpha)^2(1 + 3\alpha)| \]

Substituting for \( c_2 \) and \( c_3 \) from (6) and (7) of Lemma 2.2, we have

\[ |a_2a_3 - a_4| = B(\alpha, \beta) \left| \frac{4\alpha(1 + 2\alpha)(1 - \beta)c_1(c_1^2 + x(4 - c_1^2))}{2} + c_1^2(1 - \beta)^2(1 + 5\alpha) - \frac{1}{4} |c_1^3 + 2xc_1(4 - c_1^2)| - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \right|, \]

where

\[ B(\alpha, \beta) = \frac{(1 - \beta)}{6(1 + 2\alpha)^2(1 + 3\alpha)(1 + 4\alpha)}, \]

\[ r_1 = 2\alpha(1 + 2\alpha)(1 - \beta) + 2(1 - \beta)^2(1 + 5\alpha) - \frac{(1 + 2\alpha)^2(1 + 3\alpha)}{2}, \]

\[ r_2 = 2\alpha(1 + 2\alpha)(1 - \beta) - (1 + 2\alpha)^2(1 + 3\alpha), r_3 = (1 + 2\alpha)^2(1 + 3\alpha). \]

Suppose now that \( c_1 = c \). Since \( |c| = |c_1| \leq 2 \), using the Lemma 2.1, we may assume without restriction that \( c \in [0, 2] \) and on applying the triangle inequality with \( \rho = |x| \leq 1 \), we get,

\[ |a_2a_3 - a_4| \leq \beta(\alpha, \beta) \{ |r_1|c^3 + |r_2|\rho c(4 - c^2) + \frac{r_1^3}{2} \rho^2 c(4 - c^2) + r_3(4 - c^2) - r_3\rho(4 - c^2) \} = \beta(\alpha, \beta) \{ |r_1|c^3 + |r_2|\rho c(4 - c^2) + \frac{r_1^3}{2} \rho^2 (c - 2)(4 - c^2) + r_3(4 - c^2) \} = F(\rho). \]
Next we maximize the function $F(\rho)$. Differentiating $F(\rho)$, we get
\[
F'(\rho) = B(\alpha, \beta) [r_2 c (4 - c^2) + r_3 \rho (4 - c^2) (c - 2)].
\]

$F'(\rho) = 0$ implies $\rho = \frac{|r_2 c|}{r_3 (2 - c^2)}$. Set $\rho^* = \frac{|r_2 c|}{r_3 (2 - c^2)}$. Now, $0 \leq \rho^* \leq 1$. Also we have $F''(\rho) = B(\alpha, \beta) r_3 (4 - c^2) (c - 2) \leq 0$. Thus $\rho^*$ is the only value in $[0, 1]$ at which $F(\rho)$ attains maximum. Hence $F(\rho) \leq F(\rho^*)$. Thus
\[
F(\rho) \leq B(\alpha, \beta) [r_2 c (2 + c) + 4 r_3 - r_3 c^2]
\]
\[
= B(\alpha, \beta) [c^3 \gamma - c^2 \delta + 4 r_3] = G(c),
\]
where $\gamma = |r_1| + \frac{r_2^2}{r_3^2}$, $\delta = r_3 - \frac{r_2^2}{r_3^2} \geq 0$, $G'(c) = 0$ implies $c = 0$ and at $c = 0$, $G''(c) < 0$. Therefore $c = 0$ is a point of maximum of $G(c)$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = \rho^*$ and $c = 0$. Hence $|a_2 a_3 - a_4| \leq \frac{2 (1 - \beta)}{3(1 + 4 \alpha)}$.

**Corollary 3.5.** Choosing $\alpha = 0$, $\beta = 0$ in (15), we get $|a_2 a_3 - a_4| \leq \frac{2}{3}$.

**Corollary 3.6.** Choosing $\alpha = 0$, in (15), we get
\[
|a_2 a_3 - a_4| \leq \frac{2 (1 - \beta)}{3}.
\]

**Lemma 3.5.** Let $f \in S^\beta_\alpha$. Then
\[
|a_2 a_4 - a_3^2| \leq \frac{(1 - \beta)^2}{(1 + 3 \alpha)^2}
\]
(17)

**Proof.** Let $f \in S^\beta_\alpha$. Then by proceeding as in Lemma 3.4, we have
\[
|a_2 a_4 - a_3^2| = 
\frac{c_1 c_3 (1 - \beta)^2}{3(1 + 2 \alpha)(1 + 4 \alpha)} - \frac{c_2 (1 - \beta)^2}{4(1 + 3 \alpha)^2} - \frac{c_4 (1 - \beta)^4 (1 + 6 \alpha)}{12(1 + 2 \alpha)^2 (1 + 3 \alpha)^2 (1 + 4 \alpha)}
\]
\[
- \frac{c_1^2 c_2 (1 - \beta)^3 (2 \alpha)}{12(1 + 2 \alpha)^2 (1 + 3 \alpha)^2 (1 + 4 \alpha)}
\]
(18)
\[
= \frac{(1 - \beta)^2}{48(1 + 2 \alpha)^2 (1 + 3 \alpha)^2 (1 + 4 \alpha)} \left| 16(1 + 2 \alpha)(1 + 3 \alpha)^2 c_1 c_3 
- 12 c_2^2 (1 + 2 \alpha)^2 (1 + 4 \alpha) - 4 c_1^2 (1 - \beta)^2 (1 + 6 \alpha) - 4 (1 - \beta) 2 \alpha c_1^2 c_2 \right|.
\]
Substituting for \(c_2\) and \(c_3\) from (6) and (7) of Lemma 2.2, we obtain
\[
|a_2a_4 - a_3^2| = M |k_1c_1|c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2)|
\]
\[-k_2[c_1^2 + x^2(4 - c_1^2)^2 + 2xc_1^2(4 - c_1^2)] - k_3c_1^3 - k_4c_1^2[c_1^2 + x(4 - c_1^2)],
\]
where
\[
M = \frac{(1-\beta)^2}{48(1+2\alpha)(1+3\alpha)(1+4\alpha)},
\]
\[k_1 = 4(1 + 2\alpha)(1 + 3\alpha)^2, k_2 = 3(1 + 2\alpha)^2(1 + 4\alpha), k_3 = 4(1 - \beta)^2(1 + 6\alpha) \text{ and } k_4 = 8\alpha(1 - \beta).
\]
\[
|a_2a_4 - a_3^2| = M |c_1^2|k_1 - k_2 - k_3 - k_4| + x|c_1^2(4 - c_1^2)|k_2 - k_3 - k_4| - x^2|c_1^3(4 - c_1^2)k_3
\]
\[-x^2(4 - c_1^2)^2k_2 + 2yc_1k_1(1 - |x|^2)(4 - c_1^2)|.
\]
Suppose now that \(c_1 = c\). Since \(|c| = |c_1| \leq 2\), using the Lemma 2.1, we may assume without restriction that \(c \in [0, 2]\) and on applying triangle inequality with \(\rho = |x| \leq 1\), we obtain
\[
|a_2a_4 - a_3^2| \leq M \{ c^4(k_1 - k_2 - k_3 - k_4) + \rho c^2(4 - c^2)|k_2 - k_2 - k_4| + \rho^2(4 - c^2)(c^2(k_1 - k_2) - 2ck_1 + 4k_2) + 2ck_1(4 - c^2) \}
\]
\[= M \{ c^4(k_1 - k_2 - k_3 - k_4) + \rho c^2(4 - c^2)|k_2 - k_2 - k_4| + \rho^2(4 - c^2)(k_1 - k_2)(c - \frac{2k_2}{k_1 - k_2}) + 2ck_1(4 - c^2) \} = F(\rho).
\]
Differentiating \(F(\rho)\), we get
\[
F'(\rho) = M[c^2(4 - c^2)|k_1 - k_2| - k_2 - k_4| + 2\rho(4 - c^2)(k_1 - k_2)(c - \frac{2k_2}{k_1 - k_2}) \geq 0,
\]
since \(2k_2/(k_1 - k_2) > 2\) so that \(c - 2k_2/(k_1 - k_2) < c - 2 < 0\) and \(k_1 - k_2 = (1 + 2\alpha)/36\alpha^2 + 12\alpha + 1 > 0\) as \(\alpha > 0\), and so \((c - 2)(c - 2k_2/(k_1 - k_2)) 0\) for all \(c \in [0, 2]\). This implies that \(F(\rho)\) is an increasing function of \(\rho\) on a closed interval \([0, 1]\). Hence \(F(\rho) \leq F(1)\) for all \(\rho \in [0, 1]\). That is,
\[
F(\rho) \leq M \{ c^4|k_1 - k_2 - k_3 - k_4| + (4 - c^2)|k_2 - k_2 - k_4| + (c^2(k_1 - k_2) + 4k_2) \*
\]
\[= M \{ [c^4|k_1 - k_2 - k_3 - k_4| - (2k_1 - 2k_2 - k_4)] - (k_1 - k_2)]
\[-c^2(4k_2 - 4[(2k_1 - 2k_2 - k_4) - 4(k_1 - k_2)] + 16k_2 = G(c).
\]
\(G'(c) = 0\) implies \(c = 0\) so that at \(c = 0\). \(G''(c) < 0\). Therefore \(c = 0\) is a point of maximum for \(G(c)\). Thus the upper bound of \(F(\rho)\) corresponds to \(\rho = 1\) and \(c = 0\). Hence
\[
|a_2a_4 - a_3^2| \leq \frac{(1-\beta)^2}{1 + 3\alpha^2},
\]
\(\square\).
Corollary 3.7. Choosing $\alpha = 0, \beta = 0$ in (17), we get $|a_2a_4 - a_3^2| \leq 1$. This result coincides with [8].

Corollary 3.8. Choosing $\alpha = 0$, in (17), we get $|a_2a_4 - a_3^2| \leq (1 - \beta)^2$.

Lemma 3.6. Let $f \in S_\beta^0$. Then for $1/2 \leq \beta < 1$,

$$|a_3 - a_2^2| \leq \frac{1 - \beta}{1 + 3\alpha}$$

(19)

Proof. Let $f \in S_\beta^0$. Then by proceeding as in Lemma 3.4, we have

$$|a_3 - a_2^2| = \frac{c_2(1 - \beta)}{2(1 + 3\alpha)} \cdot \frac{c_1^2(1 - \beta)^2(1 + 4\alpha)}{2(1 + 2\alpha)^2(1 + 3\alpha)}$$

(20)

Substituting for $c_2$ from Lemma 2.2 we obtain

$$|a_3 - a_2^2| = \left| \frac{(1 - \beta)}{2(1 + 3\alpha)} \cdot \frac{c_1^2 + x(4 - c_1^2)}{2(1 + 2\alpha)^2(1 + 3\alpha)} - \frac{c_1^2(1 - \beta)^2(1 + 4\alpha)}{2(1 + 2\alpha)^2(1 + 3\alpha)} \right|$$

$$= \frac{(1 - \beta)}{4(1 + 2\alpha)^2(1 + 3\alpha)} \cdot \left| \frac{c_1^2 + x(4 - c_1^2)}{2(1 + 2\alpha)^2(1 + 3\alpha)} - \frac{c_1^2(1 - \beta)^2(1 + 4\alpha)}{2(1 + 2\alpha)^2(1 + 3\alpha)} \right|$$

$$= M \left| b_1(c_1^2 + x(4 - c_1^2)) - b_2c_1^2 \right|$$,

where $M = \frac{(1 - \beta)}{4(1 + 2\alpha)^2(1 + 3\alpha)}$, $b_1 = (1 + 2\alpha)^2$, $b_2 = 2(1 - \beta)(1 + 4\alpha)$. Therefore

$$|a_3 - a_2^2| = M \left| b_1c_1^2 + b_1x(4 - c_1^2) - b_2c_1^2 \right| = M \left| (b_1 - b_2)c_1^2 + b_1x(4 - c_1^2) \right|.$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0, 2]$ and on applying triangle inequality with $\rho = |x| \leq 1$, we obtain

$$|a_3 - a_2^2| \leq M[c^2|b_1 - b_2| + b_1\rho(4 - c^2)] = F(\rho).$$

Differentiating $F(\rho)$, we get $F'(\rho) = Mb_1(4 - c^2) > 0$, implying that $F(\rho)$ is an increasing function of $\rho$ on a closed interval $[0, 1]$. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0, 1]$. That is

$$|(a_3 - a_2^2)| \leq M[c^2|b_1 - b_2| + b_1(4 - c^2)] = G(c).$$

By hypothesis, $\beta \geq 1/2$ and hence $b_1 - b_2 = 4\alpha^2 - 4\alpha - 1 + 2\beta(1 + 4\alpha) \geq 4\alpha^2$. Hence $G(c) = M[b_1 - b_2c^2]$, $G'(c) = -2b_2Mc$ and $G''(c) = -2b_2M$. Since $c \in [0, 2]$, it follows that $G(c)$ attains a maximum at $c = 0$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence $|(a_3 - a_2^2)| \leq \frac{(1 - \beta)}{(1 + 3\alpha)}$. \qed
Corollary 3.9. Choosing $\alpha = 0$ in (19), we get $|a_3 - a_2^2| \leq (1 - \beta)$.

Using Lemma 2.1, the following estimates can be deduced.

Remark 3.2. Let $f \in S^{\beta}_\alpha$. By Lemma 2.1, we have

$$|a_3| = \left| \frac{c_2(1 - \beta)}{2(1 + 3\alpha)} + \frac{c_1^2(1 - \beta)^2}{2(1 + 2\alpha)(1 + 3\alpha)} \right|,$$

$$\leq \frac{(1 - \beta)(3 + 2\alpha - 2\beta)}{(1 + 2\alpha)(1 + 3\alpha)},$$

$$|a_4| = \left| \frac{c_3(1 - \beta)}{3(1 + 4\alpha)} + \frac{c_1c_2(3 + 8\alpha)(1 - \beta)^2}{6(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)} + \frac{c_3^2(1 - \beta)^3}{6(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)} \right|,$$

$$\leq \frac{(1 - \beta)[12 + 12\alpha^2 + 4\beta^2 - 16\alpha \beta - 14\beta + 26\alpha]}{3(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)}$$

and

$$|a_5| = \left| \frac{c_4(1 - \beta)^4}{24(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)(1 + 5\alpha)} + \frac{c_2^2(1 - \beta)^2}{8(1 + 3\alpha)(1 + 5\alpha)} + \frac{c_1c_2(1 - \beta)^2(20\alpha + 6)}{24(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)(1 + 5\alpha)} + \frac{c_4(1 - \beta)}{12(1 + 2\alpha)(1 + 4\alpha)(1 + 5\alpha)} \right|,$$

$$\leq \frac{(1 - \beta) \left\{ 120 + 288\alpha^2 - 16\beta^3 + 744\alpha^2 + 548\alpha - 188\beta \right\}}{24(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)(1 + 5\alpha)}.$$

Finally, using the above results, an upper bound for $|H_3(1)|$, $f \in S^{\beta}_\alpha$ is immediately obtained. 3.2.

Theorem 3.2. Let $f \in S^{\beta}_\alpha$. Then for $1/2 \leq \beta < 1$,

$$|H_3(1)| \leq \frac{(1 - \beta)^3(3 + 2\alpha - 2\beta)}{(1 + 2\alpha)(1 + 3\alpha)^3} + \frac{2(1 - \beta)[12 + 12\alpha^2 + 4\beta^2 - 16\alpha \beta - 14\beta + 26\alpha]}{9(1 + 2\alpha)(1 + 3\alpha)(1 + 4\alpha)^2} + \frac{(1 - \beta)^2 \left\{ 120 + 288\alpha^2 - 16\beta^3 + 744\alpha^2 + 548\alpha - 188\beta \right\}}{24(1 + 2\alpha)(1 + 3\alpha)^2(1 + 4\alpha)(1 + 5\alpha)}.$$

Remark 3.3. The determination of the sharp estimates for $|H_3(1)|$ for functions belonging to the classes $R^{\beta}_\alpha$ and $S^{\beta}_\alpha$ remain to be explored.
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